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CR structures, ACH-Einstein fillings, and almost complex structures

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Problem

Given a CR structure on ∂ of a domain, extend it to some geometric structure on the interior —— preferably in a canonical way.

Def. M^{2n-1} ($n \geq 2$) smooth mfd, $H \subset TM$ rank $2n-2$

(1) $\gamma \in \Gamma(\operatorname{End} H)$, $\gamma^2 = -\mathrm{id}_H$

an almost CR (Cauchy-Riemann) str.

(2) γ is integrable if $[\Gamma(H^{1,0}), \Gamma(H^{1,0})] \subset \Gamma(H^{1,0})$.

Model situation

$\Omega \Subset \mathbb{C}^n$ strictly pseudoconvex, $\partial\Omega$ smooth

$\Omega = \{\varphi < 0\}$, $\varphi = 0$ and $d\varphi$ nowhere vanishing on $\partial\Omega$,

$$\left(\frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) \Bigg|_{\ker \partial\varphi \, (\subset T^{1,0}_p \Omega)} > 0 \quad \text{at } \forall p \in \partial\Omega.$$

$$T\mathbb{C}^n \circlearrowleft J \rightsquigarrow \underset{\underset{T\partial\Omega}{\cap}}{H} = \ker d^c\varphi \circlearrowleft \gamma = J|_H$$

the natural CR str on $\partial\Omega$

$\boxed{\partial\Omega}$ $\qquad\qquad\qquad$ $\boxed{\Omega}$

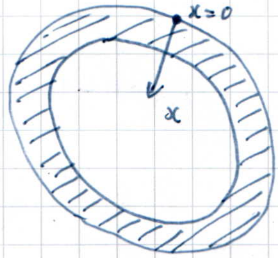
the natural CR str. γ $\xrightarrow{\text{extension}}$ integrable almost cpx str J (restr of that of $\mathbb{C}^n$)

the natural CR str. γ $\xrightarrow{\text{"extension"}}$ complete K-E metric g

$\mathrm{Ric}(g) = \lambda g, \quad \lambda < 0$

(Cheng-Yau '80)

$$g \sim 4\frac{dx^2}{x^2} + \frac{\theta^2}{x^4} + \frac{h}{x^2}$$



θ contact 1-form, $\ker\theta = H$,

$h = d\theta(\cdot, \gamma\,\cdot)$

Def. $\overline{X}^{2n}$ smooth cpt manifold-w-bdry

$X = \mathrm{int}(\overline{X})$, $\quad x \in C^\infty(\overline{X})$ bdf

A Riem. metric g on X is asymp. cpx. hyperbolic if

$$g = 4\frac{dx^2}{x^2} + \frac{\theta^2}{x^4} + \frac{h}{x^2} + O(x^\delta) \quad \text{for some } \delta > 0,$$

where θ is some contact 1-form on ∂X and

$$h = d\theta(\cdot, \gamma\,\cdot) \quad \text{on} \quad H = \ker\theta$$

for some almost CR str γ.

Rem.

(1) $d\theta(\cdot, \gamma\cdot)$ symm $\overset{\Longleftarrow}{\not\Longrightarrow}$ γ integrable

(2) We call an almost CR str γ compatible if $d\theta(\cdot, \gamma\cdot)$ is symm and > 0.

Def. (cont.) The comp. almost CR str γ is called the conformal infinity of (X, g).

Prob. (refined)

Given a compatible almost CR str. γ on ∂X,

① $\exists$ ACH metric g, with γ as conf. infinity, satisfying the Einstein eqn?

② How should we extend γ into an almost complex structure of X?

(Probably it should be compatible with g in ①. And then?)

Rem. Considering nonint. CR str. is crucial when $n \geq 3$ because of the works of Kiremidjian ('76, '77) and Hamilton ('77, '79).

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Thm. for ①

(1) (Roth '99, Biquard '00)

(X, g_0) ACH-Einstein, conf. inf. $= \gamma_0$.

Assume (X, g_0) has no nontrivial L^2 infinitesimal Einstein deformation.

$\Longrightarrow$ For γ close to γ_0 (in $C^{k,\alpha}$), one can deform g_0 and get g ACHE, conf inf $= \gamma$.

"Locally" uniquely determined.

Satisfied if $K < 0$ (hence for $\mathbb{CH}^n = B^{2n}$)

(2) (M.) $\Omega \Subset \mathbb{C}^n$ str. ψ-convex, $n \geq 3$.

Then g_{CY} satisfies the assumption.

To deform J, I propose using

$$\mathcal{E}_g[J] = \int \left(|N|^2 + \frac{1}{2} |\tau|^2 \right) dV_g .$$

Here, N = Nijenhuis ;

∇ Ehresmann-Libermann (Chern in aH setting)

(2,0)-part of the torsion $\begin{cases} N^{\bar{k}}{}_{ij} \\ T^k{}_{ij} \end{cases}$ $\quad \tau_i := T^k{}_{ik}$

Thm. for 2 $(M,)$

$\Omega \subset\subset \mathbb{C}^n$, str. ψ-convex, $n \geq 3$

γ_0 the natural CR on $\partial\Omega$, γ close to γ_0.

Then, by deformation, we obtain

g ACHE w/ conf inf γ,

J almost cpx, extending γ, comp w/ g,

satisfying the Euler-Lagrange eqn of $\mathcal{E}_g$.

$$(\mathrm{EL}) \quad (\nabla^k + \tau^k) N_{[ij]k} + \frac{1}{2} \nabla_{[i} T_{j]} + \frac{1}{2} N_{[i|kl} T_{|j]}{}^{kl} - \frac{1}{4} N_{kij} \tau^k + \frac{1}{4} T^k{}_{ij} \tau_k = 0.$$

Q How to prove | Linearization of (EL)

Why this functional | at g_{CY}

$= \Delta_{\bar\partial}$ acting on skew-sym. $T^{1,0}$-valued $(0,1)$-forms

Recall: $J^2 = -\mathrm{id}$ $\rightsquigarrow$ $J\dot{J} + \dot{J}J = 0$

$g(JX, JY) = g(X, Y)$ $\rightsquigarrow$ $g(\dot{J}X, Y) + g(\dot{J}Y, X) = 0$

	infinitesimal change		Linearizations of gauged Einstein/(EL)
g	sym. 2-tensor σ	$= \begin{cases} \sigma_{i\bar{j}} & \text{Herm. sym.} \\ \oplus & \\ \sigma_{\bar{i}\bar{j}} & \text{sym. of type } (0,2) \end{cases}$	$\Delta_{\bar\partial} - 2\lambda$ $(\lambda < 0)$ acting on $(1,1)$-forms
J	skew-sym. $T^{1,0}$-valued $(0,1)$-forms		$\Delta_{\bar\partial}$ acting on $T^{1,0}$-valued $(0,1)$-forms

$\Uparrow$

We write the combined (linear) operator P

<u>Prop.</u> $P : C^{k,\alpha}_{\delta} \longrightarrow C^{k-2,\alpha}_{\delta}$ is an isomorphism for $k \geq 2$, $0 < \alpha < 1$, and small $\delta > 0$.

$\rightsquigarrow$ (implicit function theorem works.

<u>Proof of Prop.</u>

<u>Step 1</u> Prove $\ker_{L^2}(\Delta_{\bar\partial}$ on $\Lambda^{0,1} \otimes T^{1,0}) = 0$. (M.)

<u>Step 2</u> Elements of $\ker(P : C^{k,\alpha}_{\delta} \to C^{k-2,\alpha}_{\delta})$ are actually L^2. (Roth, Biquard; see also Lee '06.)