2019/4/10

Deformation of Einstein metrics and almost complex structures on strictly pseudoconvex domains

AH

conformal class γ (<u>conformal infinity</u>)

$(h \sim \hat{h} \iff \exists f > 0 \quad \hat{h} = fh)$

$$g \sim \frac{dx^2 + h}{x^2} \quad \text{for some } h \in \gamma$$

ACH

$\dim_{\mathbb{R}} X = 2n \geq 4$

contact distrib H

$\left(\begin{array}{l} H \subset T\partial X \quad \text{corank } 1 \\ \text{(if } \ker\theta = H, \ \theta \wedge (d\theta)^{n-1} \text{ nowhere vanishing} \end{array} \right)$

"compatible" almost CR structure γ

$\left(\begin{array}{l} \gamma \in \Gamma(\mathrm{End}(H)), \quad \gamma^2 = -\mathrm{id}_H \\ \underbrace{d\theta(\cdot, \gamma\cdot) \text{ on } H}_{h_{\theta,\gamma}} \text{ is symmetric, } > 0 \\ \qquad\qquad h_{\theta,\gamma} \quad \text{the } \underline{\text{Levi form}} \end{array} \right)$

$$g \sim \frac{1}{2}\left(4\frac{dx^2}{x^2} + \frac{\theta^2}{x^4} + \frac{h_{\theta,\gamma}}{x^2}\right) \quad \text{for some } \theta$$

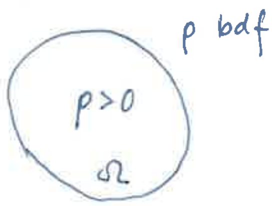
<u>Thm.</u> (Cheng-Yau 1980) $n \geq 2$

$\Omega \subset \mathbb{C}^n$ bdd strictly pseudoconvex domain, $\partial\Omega$ C^∞

$\exists!$ complete Kähler, $\mathrm{Ric}(g) = -(n+1)g$

<u>Def</u>. Ω spc $\iff$ $-\partial\bar{\partial}\rho > 0$ as a Herm. form on $\ker \partial\rho \subset T^{1,0}\mathbb{C}^n|_{\partial\Omega}$

(This is the Levi form for $\theta = \frac{i}{2}(\partial\rho - \bar{\partial}\rho)|_{T\partial\Omega}$.)



Fact Cheng-Yau's g is ACH(-Einstein)
with conf. infinity given by the natural (H_0, γ_0).

Einstein deformations.

AH On $B^n \subset \mathbb{R}^n$, from $g_{\text{Poincaré}} = \dfrac{4}{(1-|x|^2)^2} \sum_{i=1}^{n} (dx^i)^2$
Graham-Lee (1991)

ACH On $B^{2n} \subset \mathbb{C}^n$, from $g_{\mathbb{C}H^n}$, $\omega_{\mathbb{C}H^n} = -i\partial\bar\partial \log(1-|z|^2)$
Roth (1999), Biquard (2000)

Thm. 1 (M.) Let $n \geq 3$. Ω as before.
If $\gamma \in \mathrm{End}(H_0)$ is a compatible almost CR str sufficiently close to γ_0 in $C^{2,\alpha}$-top., then
$\boxed{\exists g \text{ ACHE with conf. infinity } (H_0, \gamma)}$.

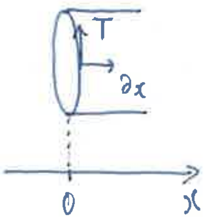
Def. (g, J) ACH almost Hermitian str if
1) g Riem, J almost cpx, $g(J\cdot, J\cdot) = g(\cdot,\cdot)$
2) For some (H, γ),

$$g \sim \frac{1}{2}\left(4\frac{dx^2}{x^2} + \frac{\theta^2}{x^4} + \frac{h_{\theta,\gamma}}{x^2}\right) =: g_{\theta,\gamma}$$

$J \sim J_{\theta,\gamma}$ ← corresponds to $T^{1,0}_{J_{\theta,\gamma}}$ spanned by

$$\underbrace{Z_1, \dots, Z_{n-1}}_{\substack{\text{loc. frame of} \\ T^{1,0}_\gamma \subset \mathbb{C}H}},\ \partial_x + i\underset{\substack{\uparrow \\ \text{Reeb v.f. assoc. w/ } \theta}}{T}$$

$$\begin{pmatrix} \theta(T) = 1 \\ d\theta(T, \cdot) = 0 \end{pmatrix}$$



$$\mathcal{E}_g[J] = \int \left(|N|^2 + \frac{1}{2}|\tau|^2 \right) dV_g \qquad \text{for } (g, J) \text{ ACH a.H.}$$

$N^{\bar{k}}{}_{ij}$ Nijenhuis tensor $\quad [Z_i, Z_j] = N^{\bar{k}}{}_{ij} Z_{\bar{k}} \mod T^{1,0}$

$\tau = \mathrm{tr}_g T$ where $\omega = g(J\cdot, \cdot)$,

$$d\omega = -i\left(N_{\bar{i}jk}\, \theta^{i} \wedge \theta^{j} \wedge \theta^{k} - T_{\bar{i}jk}\, \theta^{\bar{i}} \wedge \theta^{j} \wedge \theta^{k} \right) + (\text{cpx conj})$$

$\mathcal{E}_g[J]$ can be ∞, but rel. change for compactly supp. var. makes sense.

E-L equation: $\quad S_{\bar{i}\bar{j}} = 0, \quad S_{\bar{i}\bar{j}} = i\left(\nabla^{\bar{k}} N_{[\bar{i}\bar{j}]\bar{k}} + \frac{1}{2} \nabla_{[\bar{i}}\, \tau_{\bar{j}]} + \cdots \right)$

skew-sym (0,2) tensor $\qquad$ 2nd order eq of J

Thm. 2 (M.)

$\boxed{}$ can be replaced with:

$\exists (g, J)$ ACH almost Herm., with conf. infinity (H_0, γ), s.t. g is Einstein and $S_{\bar{i}\bar{j}} = 0$.

Possible appl.

Burns-Epstein inv. (1990)

$\Omega \subset \mathbb{C}^n$ $\quad g$ Cheng-Yau $\quad$ Define ren. Chern forms $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n$

$$\int_\Omega \tilde{c}_n = \chi(\Omega) + \int_{\partial\Omega} (\text{curv \& torsion expression})$$

Generalizes to ACHE? (Biquard-Herzlich 2005 for $n = 2$)

Ideas of proof

Step 1 Use $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ acting on $\Lambda^{0,1}(T^{1,0})$.
(w.r.t. g_{CY})

Approximate solution + modification
$(g_{\theta,\gamma}, J_{\theta,\gamma})$ by implicit function thm

$$Q(g) = \mathrm{Ric}(g) + (n+1)g + \delta_g^*\left(\delta_{g_{\theta,\gamma}} g + \frac{1}{2} d\, \mathrm{tr}_{g_{\theta,\gamma}} g\right)$$

Need to know that the linearization (at (g_{CY}, J_0)) of

$$(Q,S)^\bullet : C^{2,\alpha}_\delta(S^2T^*\Omega) \oplus C^{2,\alpha}_\delta(\Lambda^{0,2}\Omega) \to C^{0,\alpha}_\delta(S^2T^*\Omega) \oplus C^{0,\alpha}_\delta(\Lambda^{0,2}\Omega) \quad (\delta > 0)$$

order of decay

is isomorphic. Suffices to show that

$$\dot Q : \cancel{C^{2,\alpha}_\delta}\,L^2(S^2T^*\Omega) \longrightarrow \cancel{C^{0,\alpha}_\delta}\,L^2(S^2T^*\Omega)$$

$$\dot S : \cancel{C^{2,\alpha}_\delta}\,L^2(\Lambda^{0,2}\Omega) \longrightarrow \cancel{C^{0,\alpha}_\delta}\,L^2(\Lambda^{0,2}\Omega)$$

have trivial kernels.

For Kähler-Einstein $\mathrm{Ric}(g) = \lambda g$,

$$(S^2T^*\Omega)_{\mathbb{C}} = S^{2,0} \oplus \Lambda^{1,1} \oplus \boxed{\begin{matrix} S^{0,2} \\ \oplus \\ \Lambda^{0,2}\end{matrix}} = \Lambda^{0,1} \otimes \Lambda^{0,1} \cong \Lambda^{0,1} \otimes T^{1,0}$$

$\dot Q$ acts as $\nabla^*\nabla - 2\lambda$

(cf. Koiso 1983)

$(\dot Q, \dot S)$ acts as $\Delta_{\bar\partial}$

So: Reduction #1 Show $\ker_{(2)} \Delta_{\bar\partial} = 0$ for g_{CY}.

Step 2 Use cohomology (to show near-bdry analysis suffices).

$$\ker_{(2)} \Delta_{\bar\partial} \cong H^{0,1}_{(2),\mathrm{red}}(\Omega, T^{1,0}) := \frac{\ker \bar\partial}{\overline{\operatorname{im} \bar\partial}} \qquad \text{"} = 0 \text{"}$$

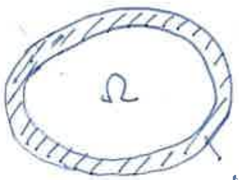
$$\Uparrow$$

$$H^{0,1}_{(2)}(\Omega, T^{1,0}) := \frac{\ker \bar\partial}{\operatorname{im} \bar\partial} \qquad \text{"} = 0 \text{"}$$

Exact sequence (Ohsawa ?) :

$$\cdots \to \underset{\substack{\| \\ 0 \text{ by Oka-Cartan}}}{H^{0,1}_c(\Omega, T^{1,0})} \to H^{0,1}_{(2)}(\Omega, T^{1,0}) \to \varinjlim_{\substack{K \subset \Omega \\ \text{compact}}} H^{0,1}_{(2)}(\Omega \setminus K, T^{1,0}) \to \cdots$$

Reduction # 2



ρ fixed bdf

$U_\delta = \{0 < \rho < \delta\}$

Show $H^{0,1}_{(2)}(U_\delta, T^{1,0}) = 0$ for small $\delta > 0$.

Step 3 Establish $\|\alpha\|^2 \le C\left(\|\bar\partial\alpha\|^2 + \|\bar\partial^*\alpha\|^2\right)$

for $\alpha \in \operatorname{dom} \bar\partial \cap \operatorname{dom} \bar\partial^* \subset L^2(\Omega, \Lambda^{0,1} \otimes T^{1,0})$.

Kodaira-Nakano $\quad \|\bar\partial\alpha\|^2 + \|\bar\partial^*\alpha\|^2 = \|\nabla^{1,0}\alpha\|^2 + (\mathrm{curv}) + \underset{\substack{\uparrow \\ \times}}{\int_{\partial_{in} U_\delta}}$

Morrey-Kohn-Hörmander $\quad \|\bar\partial\alpha\|^2 + \|\bar\partial^*\alpha\|^2 = \|\nabla^{0,1}\alpha\| + \cdots$
(for str. ψ-convex)

——— " ——— $\quad \|\bar\partial\alpha\|^2 + \|\bar\partial^*\alpha\|^2 = \|\nabla_b^{0,1}\alpha\| + \|\nabla_\xi^{1,0}\alpha\|^2$
(for str. ψ-concave, geometrically interpreted) $\qquad + \cdots$

The last one gives the desired estimate <u>when $n \ge 4$</u> (!)

$n = 3$: Discuss weighted cohomology.