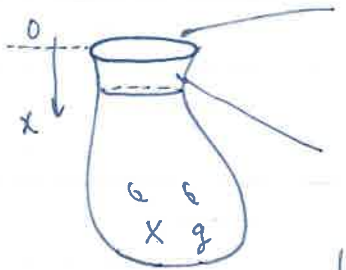
Canonical almost complex structures on ACH Einstein manifolds

2019/2/20

AH



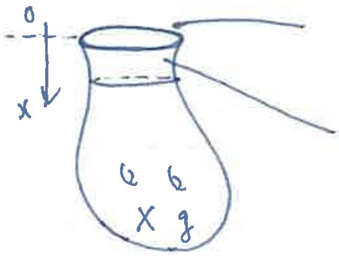
∂X : conf class γ — conformal infinity of g

$(h \sim \hat{h} \Leftrightarrow \exists f>0$ s.t. $\hat{h} = fh)$

$g \sim \dfrac{dx^2 + h}{x^2}$ for some $h \in \gamma$

$\left| g - \dfrac{dx^2+h}{x^2} \right|_{\frac{dx^2+h}{x^2}} = o(1)$ as $x \to 0$

ACH



∂X : contact distribution H + "CR structure" γ on H — conf. infinity

$g \sim \dfrac{1}{2}\left(4\dfrac{dx^2}{x^2} + \dfrac{\theta^2}{x^4} + \dfrac{h_{\theta,\gamma}}{x^2} \right)$ for some θ

Assumption on conf. infinity in complex case

$\dim X = 2n$

- H is a contact distribution over ∂X, i.e., if $\ker\theta = H$, then $\theta \wedge (d\theta)^{n-1}$ nowhere vanishing
- γ is a <u>compatible almost CR structure</u>,
  i.e., $\gamma \in \Gamma(\mathrm{End}\, H)$ s.t. $\gamma^2 = -\mathrm{id}_H$ and
  $d\theta(\cdot, \gamma\,\cdot)$ defines a positive sym. form of H (upon $\theta \rightsquigarrow -\theta$ if needed)
  $\stackrel{!!}{}$ $h_{\theta,\gamma}$ the <u>Levi form</u>

<u>Rem</u>. (1) $\hat{\theta} = f\theta \Rightarrow h_{\hat{\theta},\gamma} = f h_{\theta,\gamma}$

(2) Integrable + strictly pseudoconvex $\Rightarrow$ compatible.

Existence of Einstein metrics

Deformation

AH Model $\mathbb{H}^n$ ⇝ (Graham-Lee '91)

$B^n \subset \mathbb{R}^n$ $g = \dfrac{4\, \mathrm{Euc}}{(1-|x|^2)^2}$

γ conf. class on S^{n-1} close to γ_{std} (in $C^{2,\alpha}$)

$\Rightarrow \exists g$ AHE with conf. inf. γ

ACH Model $\mathbb{C}H^n$ ⇝ (Roth '99, Biquard '00)

$B^{2n} \subset \mathbb{C}^n$

$g = \sum_{i,j=1}^{n} \partial_i \partial_{\bar{j}} \left(\log \dfrac{1}{1-|z|^2} \right) dz^i d\bar{z}^j$

γ compatible almost CR on $H_{std} \subset TS^{2n-1}$ close to γ_{std} (in $C^{2,\alpha}$)

$\Rightarrow \exists g$ ACHE with conf. inf. (H_{std}, γ)

(Cheng-Yau '80) ⇝ (M. preprint '16)

$\Omega \subset\subset \mathbb{C}^n$ bdd str. ψ-convex domain, $\partial\Omega$ C^∞

$\Rightarrow \exists !$ complete KE g with $\mathrm{Ric}(g) = -(n+1) g$

g is ACHE, conf inf given by the natural CR str. (inherited)

Same result under assump $n \geq 3$

(? if $n = 2$)

Today Enhance the meaning of deformation of interior geometry in complex setting by considering "deformed almost cpx structures."

Possible appl.

Burns-Epstein inv. ('90) (See also Marugame '16)

$\Omega \subset\subset \mathbb{C}^n$ g Cheng-Yau Define ren. Chern forms $\tilde{c}_2, \dots, \tilde{c}_n$

$$\int_\Omega \tilde{c}_n = \chi(\Omega) + \int_{\partial\Omega} (\text{curv. \& torsion expr})$$

Generalize to ACHE? (Biquard-Herzlich '05 for $n=2$)

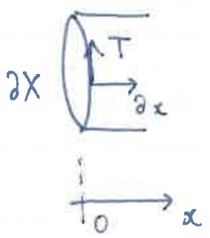
Def. (g, J) is an <u>ACH almost Hermitian str</u> if

1) g Riem., J almost cpx, $g(J\cdot, J\cdot) = g(\cdot, \cdot)$

2) For some H and γ,

$$g \sim \frac{1}{2}\left(4\frac{dx^2}{x^2} + \frac{\theta^2}{x^4} + \frac{h_{\theta,\gamma}}{x^2}\right)$$

$$J \sim J_{\theta,\gamma}$$



$J_{\theta,\gamma}$ corresponds to hol. tang. bundle $T^{1,0}$ spanned by

$$\underbrace{Z_1, \dots, Z_{n-1}}_{\text{loc. frame of } T_\gamma^{1,0} \subset \mathbb{C}H}, \quad \partial_x + iT$$

T: Reeb v.f. assoc. w/ θ $\left(\begin{array}{l}\theta(T) = 1 \\ d\theta(T, \cdot) = 0\end{array}\right)$

In $\mathcal{J}_g = \{ J \mid (g, J) \text{ is ACH almost Hermitian} \}$,

consider $\mathcal{E}_g[J] = \displaystyle\int \left(|N|^2 + \frac{1}{2}|\tau|^2\right) dV_g$

$N = N^{\bar{k}}{}_{ij}$ the Nijenhuis tensor $\quad [Z_i, Z_j] = N^{\bar{k}}{}_{ij} \overline{Z_k} \mod T^{1,0}$

$\tau = \operatorname{tr}_g T$ where $\omega = g(J\cdot, \cdot)$ and

$$d\omega = -i\left(N_{ijk}\, \theta^i \wedge \theta^j \wedge \theta^k - T_{\bar{i}jk}\, \theta^{\bar{i}} \wedge \theta^j \wedge \theta^k\right) + (\text{complex conj.})$$

Rel. change of $\mathcal{E}_g$ for compactly supp. variation makes sense.

E-L equation:

$$S_{ij} = i\left((\nabla^k + \tau^k) N_{[ij]k} + \frac{1}{2}\nabla_{[i}\tau_{j]} + \frac{1}{2} N_{[i|kl} T_{|j]}{}^{kl} - \frac{1}{4} N_{kij}\tau^k + \frac{1}{4} T^k{}_{ij}\tau_k\right) = 0$$

∇: the Ehresmann-Libermann connection

Lemma

If (g, J) is Kähler-Einstein with $\mathrm{Ric}(g) = \lambda g$, ($\lambda = -(n+1)$ for ACHE)
the linearization of $J \longmapsto S$ is

$$\dot{S}_{ij} = -\frac{1}{2}\left(\nabla_k \nabla^k \dot{J}_{ij} + \lambda \dot{J}_{ij}\right).$$

(Note $\dot{J}$ skew-sym. (2,0)-form
skew-sym. ← $g(\dot{J}\cdot, \cdot) + g(\cdot, \dot{J}\cdot) = 0$; (2,0)-form ← $\dot{J}J + J\dot{J} = 0$)

If $\dot{J}$ and $\dot{S}$ are regarded as (0,1)-forms w/ values in $T^{1,0}$,
this is equivalent to

$$\dot{S} = \frac{1}{2}\Delta_{\bar\partial}\dot{J} \quad \left(= \frac{1}{2}\left(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*\right)\dot{J}\right).$$

Rem. Kodaira's infinitesimal deformation of cpx str
on closed cpx manifolds:
$A \in \Omega^{0,1}(T^{1,0})$ satisfying $\Delta_{\bar\partial} A = 0$.

Theorem (M.)

$\Omega \subset\subset \mathbb{C}^n$ str. ψ-convex, $\partial\Omega$ C^∞. Assume $n \geq 3$.
γ compatible almost CR str. on $H_{\mathrm{nat}} \subset T\partial\Omega$
suff. close to γ_{nat}

$\Longrightarrow$ $\exists (g, J)$ ACH almost Herm str w/ conf inf γ
s.t. g is Einstein & J satisfies $S = 0$.

Key of the proof

One eventually wants to show that, for small $\delta > 0$,

$$C^{2,\alpha}_\delta(\Omega, \Lambda^{2,0}) \longrightarrow C^{0,\alpha}_\delta(\Omega, \Lambda^{2,0}), \quad \dot{J} \longmapsto \dot{S}$$

(some decay at ∂X)

is an isom. This is true because of the vanishing
of the L^2-kernel.