5. AH Einstein metrics

5.1 AH Einstein equation

$$\text{Einstein} \overset{\text{def}}{\Longleftrightarrow} R_{ij} = \lambda g_{ij}, \ \lambda \in \mathbb{R}$$

If g is AH, then

$$R_{ijkl} \to -(g_{ik} g_{jl} - g_{il} g_{jk})$$

at ∂X

$$\begin{aligned}
\therefore \quad R_{ij} &= R_{ki}{}^{k}{}_{j} \\
&\to -(g_k{}^k g_{ij} - g_{kj} g_i{}^k) \\
&= -(\delta_k{}^k g_{ij} - g_{kj} \delta_i{}^k) \\
&= -((n+1) g_{ij} - g_{ij}) \\
&= -n g_{ij}.
\end{aligned}$$

Hence λ is forced to be $\lambda = -n$.

Prob 5.1

Given $\bar{X}$ and a conf class $[h]$ on ∂X,

$\exists ?$ g AH metric on X

s.t.

(i) $R_{ij} = -n g_{ij}$.

(ii) conf infinity $= [h]$.

Features:

(I) Nonlinearity

Ric(g) nonlinear

Even if $g = g_{ref} + u$,

Ric$(g_{ref} + u)$ is nonlinear in u.

$$\mathrm{Ric} \longleftarrow R_{ij}{}^{k}{}_{l} = X_i \Gamma^{k}{}_{jl} - X_j \Gamma^{k}{}_{il} + \underbrace{\Gamma^{k}{}_{ip}\Gamma^{p}{}_{jl} - \Gamma^{k}{}_{jp}\Gamma^{p}{}_{il}}_{\text{Nonlinearity}}$$

Use linearized problem. Banach sp implicit function theorem (, continuity method, ...)

Lem 5.2 (Implicit function theorem)

$\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}$ Banach spaces

$U_1 \times U_2 \subset \mathbb{B}_1 \times \mathbb{B}_2$ open nbhd of $(0,0)$

$F : U_1 \times U_2 \longrightarrow \mathbb{B}$ continuously Fréchet differentiable

Suppose

(i) $F(0,0) = 0$.

(ii) $(dF)_{(0,0)}(0, \cdot) : \mathbb{B}_2 \longrightarrow \mathbb{B}$ isomorphism

$\Rightarrow$ $(0,0) \in \exists W_1 \times \exists W_2 \subset U_1 \times U_2$ open

$\exists f : W_1 \longrightarrow W_2$ cont F differentiable

s.t. $F(x,y) = 0 \iff y = f(x)$ for $x \in W_1$, $y \in W_2$.

(II) <u>Non-unique nature</u>

$$\mathrm{Ric}(g) = -ng$$

$$\Rightarrow \quad \mathrm{Ric}(\varphi^* g) = -n\,\varphi^* g$$

for $\varphi : X \to X$ diffeo.

(infinitesimally, $(d\,\mathrm{Ric})_g(\mathcal{L}_\xi g) = -n\,\mathcal{L}_\xi g$

$(d\,\mathrm{Ric})_g + n$ has inevitably nontriv kernel (even in C_0^∞ !). Doesn't proceed as § 2.

DeTurck trick of gauge fixing.

5.2 Gauge-fixing condition

Choices:

① Harmonic coordinates;

or, assume $(X, g) \xrightarrow[\text{identity}]{} (X, g_{ref})$

is a harmonic map.

(Graham-Lee 1991, Lee 2006)

② "Bianchi gauge" (Biquard 2000)

$g = g_{ref} + u$

$$B_{g_{ref}}(u) := \delta_{g_{ref}} u + \frac{1}{2} d \operatorname{tr}_{g_{ref}} u = 0$$

$$\left((\delta_{g_{ref}} u)_i := -(\nabla^{g_{ref}})^j u_{ij} \right)$$

Cor 1.7: $B_g(\operatorname{Ric}(g)) = 0$

③ Fefferman-Graham gauge (1985)

Works better for explicit computation of approx solution

(see Graham-[illegible] 2005)

Consider

$$\begin{cases} \mathrm{Ric}(g_{ref}+u) = -n(g_{ref}+u) \\ B_{g_{ref}}(u) = 0 \end{cases} \qquad (5.1)$$

(5.1) implies

$$\Phi_{g_{ref}}(u) := \mathrm{Ric}(g_{ref}+u) + n(g_{ref}+u) + \delta^*_{g_{ref}+u} B_{g_{ref}}(u)$$

$$= 0.$$

$$(\delta^*\alpha)_{ij} = \frac{1}{2}(\nabla_i \alpha_j + \nabla_j \alpha_i)$$

Prop 5.3 g_{ref} AH

$u \in C^{2,0}_{\delta}(X, \mathrm{Sym}^2 T^*X)$ for some $\delta > 0$,

$\Phi_{g_{ref}}(u) = 0$, $\mathrm{Ric}(g_{ref} + u) < 0$

(eigenvalues wrt $g_{ref} + u$ are negative)

$\Longrightarrow$ u satisfies (5.1).

[Proof] ~~May assume $\delta < n$.~~

$$\Phi_{g_{ref}}(u) = \underset{\substack{\uparrow \\ \text{geom ell LDO} \\ \text{(Lem 5.4)}}}{(\text{linear in } u)} + \underset{\substack{\cap \\ C^{1,0}_{\delta}}}{\left(\begin{array}{c}\text{quadratic} \\ \text{or more} \\ \text{in } u\end{array}\right)}$$

By Cor 3.9, $\Phi_{g_{ref}}(u) = 0 \Rightarrow u \in C^{m,\alpha}_{\delta}$ $(\forall m\ \forall \alpha)$.

$$\Phi_{g_{ref}}(u) = 0$$

$$\mathrm{Ric}(g_{ref}+u) + n(g_{ref}+u) + \delta^*_{g_{ref}+u} B_{g_{ref}}(u) = 0.$$

Apply $B_{g_{ref}+u}$.

$$\left(\delta_{g_{ref}+u} + \frac{1}{2} d\, \mathrm{Tr}_{g_{ref}+u}\right) \delta^*_{g_{ref}+u} \underbrace{B_{g_{ref}}(u)}_{\gamma} = 0$$

Let $\nabla = \nabla^{g_{ref}}$ L-C.

$$-\nabla^j\left(\frac{1}{2}(\nabla_i \gamma_j + \nabla_j \gamma_i)\right) + \frac{1}{2}\nabla_i \nabla^j \gamma_j = 0$$

$$\underbrace{-\nabla^j \nabla_i \gamma_j} - \nabla^j \nabla_j \gamma_i + \underbrace{\nabla_i \nabla^j \gamma_j} = 0$$

$$R_i{}^j{}_j{}^k \gamma_k = -R_i{}^k \gamma_k$$

$$-\nabla^j \nabla_j \gamma_i - R_i{}^j \gamma_j = 0$$

$$
\begin{aligned}
\Delta |\gamma|^2 &= -\nabla^*\nabla \langle \gamma, \gamma \rangle \\
&= -2 \langle \nabla^*\nabla \gamma, \gamma \rangle + \underbrace{2 \langle \nabla \gamma, \nabla \gamma \rangle}_{\geq 0} \\
&\geq -2 \langle \underbrace{\nabla^*\nabla \gamma}_{= -\nabla^j \nabla_j \gamma_i}, \gamma \rangle \\
&\geq -2 \langle R_i{}^j \gamma_j, \gamma_i \rangle \geq 0.
\end{aligned}
$$

$|\gamma|^2$ subharmonic. $|\gamma|^2 \equiv 0$ by maximum principle. $\square$

5.3 Linearization of Φ, coercivity estimate.

Lem 5.4

Lichnerowicz Laplacian

$$(d\Phi_g)_0(u) = \frac{1}{2}\Delta_L u + nu$$

$$= \frac{1}{2}\left(\underset{\substack{\uparrow \\ -\nabla^k\nabla_k u_{ij}}}{\nabla^*\nabla u} - 2\tilde{R}u + 2\widetilde{Ric}\,u\right) + nu$$

where $(\tilde{R}u)_{ij} := R_i{}^k{}_j{}^l u_{kl}$

$(\widetilde{Ric}\,u)_{ij} := \frac{1}{2}\left(R_i{}^k u_{jk} + R_j{}^k u_{ik}\right)$

$$P := (d\Phi_g)_0 : \Gamma(\mathrm{Sym}^2 T^*X) \to \Gamma(\mathrm{Sym}^2 T^*X)$$

Cor 5.5

(1) If g satisfies $Ric = -ng$.

then

$$P = \frac{1}{2}\nabla^*\nabla - \tilde{R}.$$

(2) $\underbrace{\text{If } g}_{\text{moreover}}$ has negative sect curvature,

then $\ker_{L^2} P = \ker\left(P : H^2 \to L^2\right)$

$= 0.$

Prop 5.6 On $\mathbb{H}^{n+1}$,

$$(u, Pu) \geq C\|u\|^2, \quad u \in C_0^\infty.$$

[Proof]

① If u is trace-free,

$$2Pu = D^*Du - \tilde{R}u + nu$$

$\uparrow$ cov exterior derivative for $T\mathbb{H}^{n+1}$-valued diff forms

$$-(\tilde{R}u, u) = -\int R_{ikjl}\, u^{ij} u^{kl}\, dV$$

$$= \int (g_{ij} g_{kl} - g_{il} g_{jk}) u^{ij} u^{kl}\, dV$$

$$= \int (0 - |u|^2)\, dV$$

$$= -\|u\|^2.$$

$$\therefore \quad (u, Pu) \geq \frac{n-1}{2} \|u\|^2.$$

② If $u_{ij} = f g_{ij}$, then

$$Pu = \frac{1}{2}(-\Delta + 2n) f \cdot g_{ij}$$

$$\therefore \quad (u, Pu) \geq n \|u\|^2. \qquad \square$$

5.4 Characteristic roots & approx solution

On the upper-half sp model of $\mathbb{H}^{n+1}$,

$$(Pu)_{00} = -(x\partial_x)^2 u_{00} + n\, x\partial_x u_{00} + 2n u_{00} + O(x)$$

$$(Pu)_{0a} = -(x\partial_x)^2 u_{0a} + n\, x\partial_x u_{0a} + (n+1) u_{0a} + O(x)$$

$$(Pu)_{ab} = -(x\partial_x)^2 u_{ab} + n\, x\partial_x u_{ab} + 2(\operatorname{tr} u_{ab})\delta_{ab} + O(x)$$

The last component decomposes:

$$(\operatorname{tf} Pu)_{ab} = -(x\partial_x)^2(\operatorname{tf} u_{ab}) + n\, x\partial_x(\operatorname{tf} u_{ab}) + O(x),$$

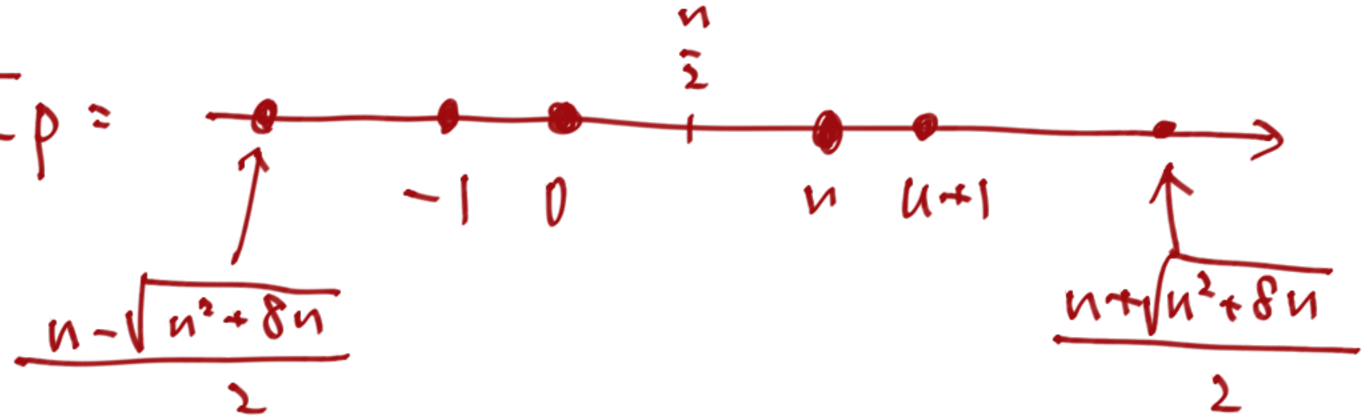
$$\operatorname{tr}(Pu)_{ab} = -(x\partial_x)^2(\operatorname{tr} u_{ab}) + n\, x\partial_x(\operatorname{tr} u_{ab}) + 2n(\operatorname{tr} u_{ab}) + O(x).$$

So

Prop 5.7

$$I_p(s) = \mathrm{diag}\Big(-s^2+ns+2n,\ -s^2+ns+(n+1),\ -s^2+ns,\ -s^2+ns+2n \Big).$$

$\Sigma_p =$



This computation also shows:

if $u, v \in \Gamma(\mathrm{Sym}^2 T^*\mathbb{H}^{n+1})$ are smooth on $[0,\infty) \times \mathbb{R}^n$, $u = O(x)$, $v = O(x^k)$, $k \geq 1$

$$\Phi_{g_{\mathbb{H}^{n+1}}}(u+v)_{00} = \Phi_{g_{\mathbb{H}^{n+1}}}(u)_{00} + (-k^2+nk+2n)v_{00} + O(x^{k+1})$$

This remains true for any scc AH space (X, g), with $x \to \rho$ bdf.

Cor 5.8

$\forall\, (X, g_0)$ scc AH

$\exists\, u$ smooth on $\overline{X}$, $u = O(\rho)$,

$$\Phi_{g_0}(u) = O(\rho^n).$$

5.5 Einstein deformation

Thm 5.9 (Graham-Lee 1991, Biquard 2000, Lee 2006)

(X, g_0) AH Einstein, $[h_0]$ conf inf

Suppose $\ker_{L^2} P = 0$ on (X, g_0).

$\Rightarrow$ $\forall m \geq 2$, $0 < \forall \alpha < 1$

$\forall h$ (smooth) metric on ∂X sufficiently close to h_0 in $C^{m,\alpha}$ topology

$\exists g$ AH Einstein with conf infinity $[h]$

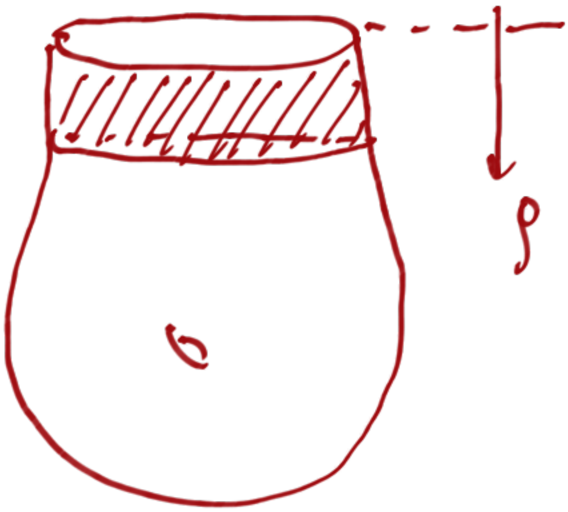
[Proof] Let $g_0 \in g_0^{sce} + C_\delta^{m',\alpha'}$ $(\forall m', \forall \alpha')$
$\delta > 0$

Consider

$$h_0 \in U_1 \subset \{ C^{m,\alpha} \text{ metrics on } \partial X \}$$

$$0 \in U_2 \subset C^{m,\alpha}_{\delta_1}(X, \mathrm{Sym}^2 T^*X)$$

$$\delta_1 = \min(\delta, 1)$$



Take a bdf ρ s.t. $(\rho^2 g)|_{T\partial X} = h_0$.

Take a collar nbhd of ∂X s.t.

$$g_0^{scc} = \frac{d\rho^2 + h_0}{\rho^2} + O(\rho)$$

Let $g_h = g_0 + \dfrac{h - h_0}{\rho^2}$ for $h \in U_1$.

(extend to X)

Then $\Phi_{g_h}(0) \in C^{m-2,\alpha}_{\delta_1}$. $(\rho^2 g_h)|_{T\partial X} = h$.

$$\delta_1 = \min(\delta, 1)$$

$$Q : U_1 \times U_2 \longrightarrow C^{m-2,\alpha}_{\delta_1}$$

$$Q(h, u) := \Phi_{g_h}(u).$$

Then

$$Q(h_0, 0) = \Phi_{\underbrace{g_{h_0}}_{= g_0}}(0) = 0$$

$$\underbrace{(dQ)_{(h_0,0)}(0, \cdot)}_{= (d\Phi_{g_0})_0 = P_{g_0}} : C^{m,\alpha}_{\delta_1} \to C^{m-2,\alpha}_{\delta_1}$$

By Thm 3.2, Prop 5.6, Prop 5.7, and $\ker_{L^2} P_{g_0} = 0$, this is an isomorphism.

By Lem 5.2, after shrinking U_1, U_2 if necessary,

$$\exists f : U_1 \longrightarrow U_2 \quad \text{s.t.}$$

$$Q(h, f(h)) = \Phi_{g_h}(f(h)) = 0.$$

By Prop. 5.3, $g_h + f(h)$ is Einstein (construction of g_h should be carefully done so that $\mathrm{Ric}(g_h) < 0$ and hence $\mathrm{Ric}(g_h + f(h)) \cancel{=} < 0$).

When h is smooth, the added term $f(h)$ is $\in C^{m',\alpha'}_{\delta,1}$ $(\forall m', \forall \alpha')$ (cf. proof of Prop 5.3). $\square$