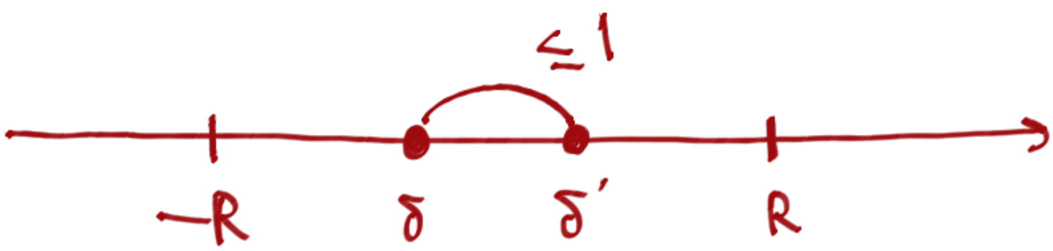
3.5 On the proof of Thm 3.2

Thm 3.2 follows from below + Cor 3.11 (Rellich)

Thm 3.15 P as in Thm 3.2.

If



then

$$\exists\, Q_1, Q_2 : H^m_\delta \to H^{m+l}_\delta \quad \text{bdd},$$

$$\exists\, S_1 : H^{m+l}_\delta \to H^{m+l+1}_{\delta'} \quad \text{bdd},$$

$$\exists\, S_2 : H^m_\delta \to H^{m+1}_{\delta'} \quad \text{bdd}$$

s.t.

$$Q_1 P u = u + S_1 u, \quad u \in H^{m+l}_\delta,$$

$$P Q_2 u = u + S_2 u, \quad u \in H^m_\delta.$$

$$Q_1 P = I_{H^{m+l}_\delta} + \iota S_1$$

Same for Hölder spaces

$$\iota : H^{m+l+1}_{\delta'} \hookrightarrow H^{m+l}_\delta \ \text{cpt}$$

Q_1, Q_2 are constructed by patching $P^{-1}_{\mathbb{H}^{n+1}}$ by partition of unity.

<u>Cor 3.16</u>

P as in Thm 3.2. $-R<\delta<\delta'<R$.

(1) $u \in L^2_\delta$, $Pu \in H^m_{\delta'} \Rightarrow u \in H^{m+l}_{\delta'}$.

improved

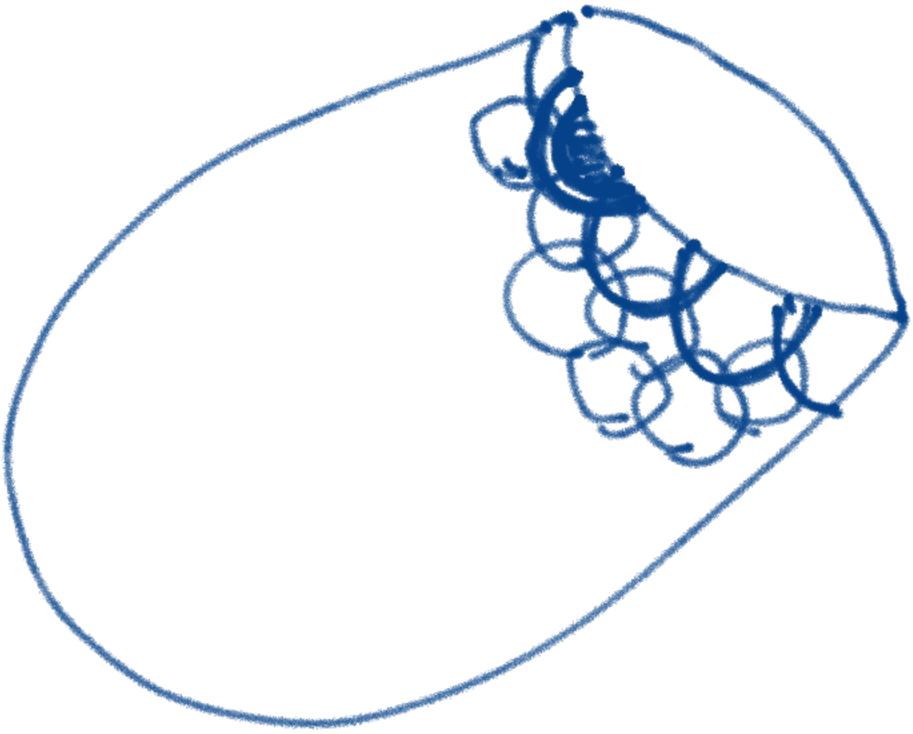
(2) $u \in C^{0,0}_{\frac{n}{2}+\delta}$, $Pu \in C^{m,\alpha}_{\frac{n}{2}+\delta'} \Rightarrow u \in C^{m+l,\alpha}_{\frac{n}{2}+\delta'}$

$(0<\alpha<1)$

[Proof] If $\delta'-\delta \leq 1$,

$$u = \underbrace{Q_1 \underbrace{Pu}_{\in H^m_{\delta'}}}_{\in H^{m+l}_{\delta'}} - \underbrace{S_1 \underbrace{u}_{\in H^{m+l}_{\delta}}}_{\in H^{m+l+1}_{\delta'}} \in H^{m+l}_{\delta'}.$$

In general, iterate. □



4. L^2 harmonic forms on AH spaces

4.1 Result

(X, g) Riem

A (smooth) p-form is <u>L^2 harmonic</u>

$\overset{\text{def}}{\Longleftrightarrow}$ $\Delta\alpha = 0$, $\alpha \in L^2$

$\left(\int |\alpha|^2 dV < \infty \right)$

$$\mathcal{H}^p_{(2)} := \{ L^2 \text{ harmonic } p\text{-forms} \}$$

On AH spaces,

$$\mathcal{H}^p_{(2)} = \ker\left(\Delta : H^2(X, \Lambda^p) \to L^2(X, \Lambda^p) \right)$$

On closed manifolds, $\mathcal{H}^p \cong H^p$

because $\Omega^p = \operatorname{im} \Delta \oplus \mathcal{H}^p$

$= \operatorname{im} d \oplus \operatorname{im} d^* \oplus \mathcal{H}^p$

In general, $\Omega^p_{(2)} = \overline{\operatorname{im} \Delta} \oplus \mathcal{H}^p_{(2)}$

L^2 p-forms $= \overline{\operatorname{im} d} \oplus \overline{\operatorname{im} d^*} \oplus \mathcal{H}^p_{(2)}$.

<u>Thm 4.1</u> (Mazzeo 1988)

(X, g) AH space dim $n+1$

$$\mathcal{H}^p_{(2)} \cong \begin{cases} H^p_c(X) & p < \frac{n}{2} \\ H^p(X) & p > \frac{n+2}{2} \end{cases}$$

$$H^p_c(X) := \frac{\ker(d : \Omega^p_c \to \Omega^{p+1}_c)}{\operatorname{im}(d : \Omega^{p-1}_c \to \Omega^p_c)}.$$

<u>Rem</u> $p = \frac{n}{2}, \frac{n+2}{2}$ $\quad$ $\dim \mathcal{H}^p_{(2)} < \infty$

$p = \frac{n+1}{2}$ $\quad$ $\dim \mathcal{H}^p_{(2)} = \infty$

4.2 Digression: Relative cohomology

$$H^p(\bar{X}, \partial X) := \frac{\ker(d : \Omega^p_{rel} \to \Omega^{p+1}_{rel})}{\operatorname{im}(d : \Omega^{p-1}_{rel} \to \Omega^p_{rel})}$$

$$\Omega^p_{rel} := \{ \alpha \in \Omega^p(\bar{X}) \mid \iota^*\alpha = 0 \}$$

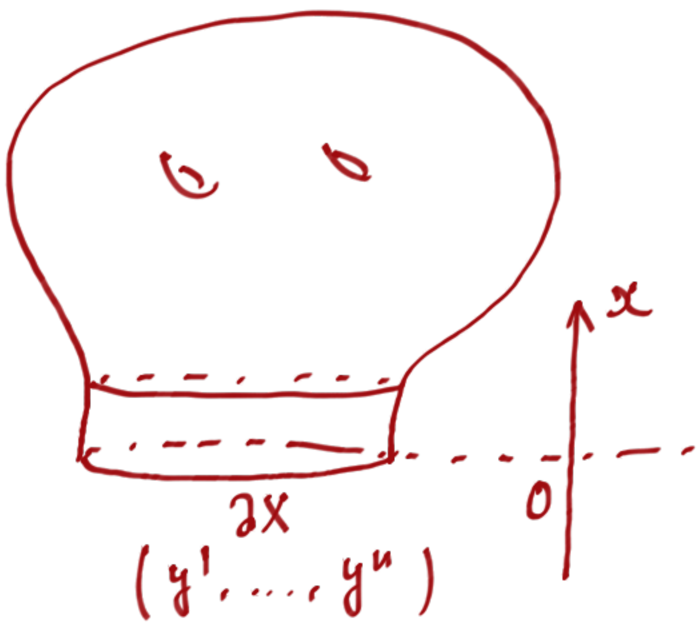
$$\iota : \partial X \hookrightarrow \bar{X}$$

Prop 4.2 $H^p_c(X) \xrightarrow{\cong} H^p(\bar{X}, \partial X)$.

Methods: ① Long exact seq + five lemma.

② de Rham theory

[Proof] Chain homotopy op near ∂X.



$\alpha \in \Omega^p(\bar{X})$

$\iota : \partial X \hookrightarrow \bar{X}$

$\pi : (x, y) \mapsto (0, y)$

$$\alpha = \alpha^{(t)}_{a_1\cdots a_p} dy^{a_1}\wedge\cdots\wedge dy^{a_p} + \alpha^{(n)}_{a_1\cdots a_{p-1}} dx\wedge dy^{a_1}\wedge\cdots\wedge dy^{a_{p-1}}$$

$$(R\alpha)_{a_1\cdots a_{p-1}}(x,y) = \int_0^x \alpha^{(n)}_{a_1\cdots a_{p-1}}(t,y)\,dt$$

$$R\alpha = (R\alpha)_{a_1\cdots a_{p-1}} dy^{a_1}\wedge\cdots\wedge dy^{a_p}$$

Then $dR\alpha + Rd\alpha = \alpha - \pi^*\iota^*\alpha$ (near ∂X).

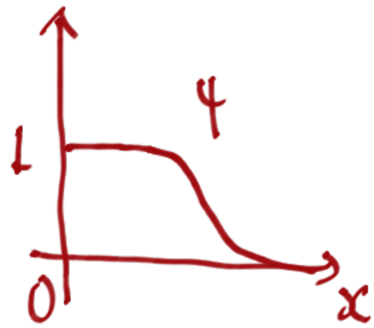
Hence if $\iota^*\alpha = 0$, then $dR\alpha + Rd\alpha = \alpha$ (near ∂X).

Extend $R\alpha$

$$\tilde{R}\alpha := \psi R\alpha \in \Omega^{p-1}(\bar{X})$$

$$\tilde{R} : \Omega^p(\bar{X}) \longrightarrow \Omega^{p-1}(\bar{X})$$

$$\Omega^p_{rel} \longrightarrow \Omega^{p-1}_{rel}$$



Injectivity

$\alpha \in \Omega^p_c$, $d\alpha = 0$

Suppose $\alpha = d\beta$, $\beta \in \Omega^{p-1}_{rel}$.

$$d\tilde{R}\beta + \tilde{R}\underbrace{d\beta}_{\substack{\| \\ \alpha \,=\, 0 \text{ near } \partial X}} = \beta \quad (\text{near } \partial X)$$

$\therefore \; d\tilde{R}\beta = \beta$ near ∂X

$\therefore \; d(\beta - d\tilde{R}\beta) = d\beta = \alpha$, $\beta - d\tilde{R}\beta \in \Omega^{p-1}_c$.

Surjectivity

Take $[\alpha] \in H^p(\bar{X}, \partial X)$. $\alpha \in \Omega^p_{rel}$, $d\alpha = 0$.

$$d\tilde{R}\alpha + \tilde{R}\underbrace{d\alpha}_{\substack{\| \\ 0}} = \alpha \quad (\text{near } \partial X)$$

$\therefore \; d\tilde{R}\alpha = \alpha$ near ∂X.

$\alpha - d\tilde{R}\alpha \in \Omega^p_c$, $d(\alpha - d\tilde{R}\alpha) = 0$.

$$[\alpha - d\underbrace{\tilde{R}\alpha}_{\substack{\cap \\ \Omega^{p-1}_{rel}}}] = [\alpha] \text{ in } H^p(\bar{X}, \partial X)$$

$\square$

4.3 Fredholm theorem for $\Delta^{(p)}$ and decay of L^2 harm forms

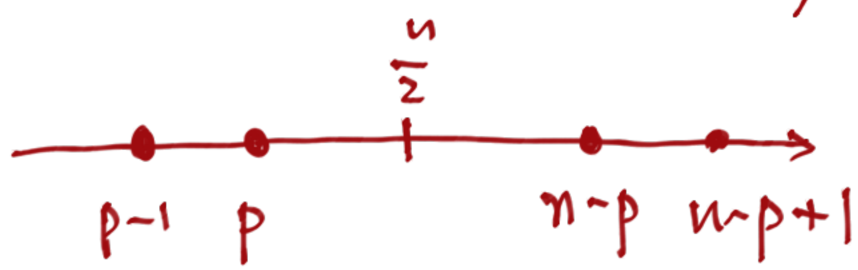
We apply Thm 3.2 to $\Delta = \Delta^{(p)}$.

- Coercivity estimate
  Prop 3.13. $p \neq \frac{n}{2}, \frac{n+1}{2}, \frac{n+2}{2}$

- char roots

$$I_{\Delta^{(p)}}(s) = \begin{pmatrix} s^2 - ns + (p-1)(n-p+1) & 0 \\ 0 & s^2 - ns + p(n-p) \end{pmatrix}$$

$\Sigma_{\Delta^{(p)}} =$



<u>Lem</u> On upper-half sp model of $\mathbb{H}^{n+1}$,

$\Gamma^k{}_{ij}$ wrt $\{x\partial x, x\partial y^1, \dots, x\partial y^n\}$ is

index $0, 1, \dots, n$

$$\Gamma^0{}_{00} = \Gamma^0{}_{0b} = \Gamma^c{}_{00} = \Gamma^c{}_{0b} = 0,$$

$$\Gamma^0{}_{a0} = 0, \qquad \Gamma^0{}_{ab} = \delta_{ab},$$

$$\Gamma^c{}_{a0} = -\delta_a{}^c, \quad \Gamma^c{}_{ab} = 0.$$

where a, b, c run through

$\{1, \dots, n\}$.

Therefore:

<u>Cor 4.3</u> (X, g) AH

$$R = \begin{cases} \frac{n}{2} - p & p < \frac{n}{2} \\ p - 1 - \frac{n}{2} & p > \frac{n+2}{2} \end{cases}$$

Then, for $-R < {}^\forall \delta < R$,

(1) $\Delta : H_\delta^{m+2}(X, \Lambda^p) \to H_\delta^m(X, \Lambda^p)$

(2) $\Delta : C_{\frac{n}{2}+\delta}^{m+2, \alpha}(X, \Lambda^p) \to C_{\frac{n}{2}+\delta}^{m, \alpha}(X, \Lambda^p)$

are Fredholm of index 0.

$$\ker \Delta = \ker(\Delta : H^2 \to L^2)$$
$$= \mathcal{H}^p_{(2)}.$$

Cor 4.4

If $\alpha \in \mathcal{H}^p_{(2)}$, $p < \frac{n}{2}$, then

$$\alpha \in C^{0,0}_{n-p-\varepsilon}(X, \Lambda^p), \quad \forall \varepsilon > 0.$$

i.e., $|x^{-n+p+\varepsilon} \alpha|_g = O(1)$

$$|\alpha|_g = O(x^{n-p-\varepsilon})$$

$$\therefore \quad |\alpha|_{\bar{g}} = O(x^{n-2p-\varepsilon})$$

Rem dim coker $\Delta < \infty$ implies im Δ closed. Hence

$$\begin{aligned} \Omega^p_{(2)} &= \overline{\operatorname{im} \Delta} \oplus \mathcal{H}^p_{(2)} \\ &= \operatorname{im} \Delta \oplus \mathcal{H}^p_{(2)} \\ &= \operatorname{im} d \oplus \operatorname{im} d^* \oplus \mathcal{H}^p_{(2)} \end{aligned}$$

$$\mathcal{H}^p_{(2)} \cong H^p_{(2)} := \frac{\ker(d \text{ on } L^2)}{\operatorname{im}(d \text{ on } L^2)}.$$

4.4 Proof of Thm 4.1 (orientable case)

Lem 4.5 (X, g) AH. $\dim = n+1$

(1) $\dim H^p(X) < \infty$, $\dim H^p_c(X) < \infty$,
Moreover, if X orientable,

$$(H^p_c(X))^* \cong H^{n+1-p}(X).$$

(2) If $p < \frac{n}{2}$, $\dim \mathcal{H}^p_{(2)} < \infty$,
$\dim \mathcal{H}^{n+1-p}_{(2)} < \infty$.

and

$$(\mathcal{H}^p_{(2)})^* \cong \mathcal{H}^{n+1-p}_{(2)}.$$

Because of this, Thm 4.1 is reduced to:

Prop 4.6 Let $p < \frac{n}{2}$.

(1) $H^p_c(X) \to H^p_{(2)}(X)$ is surj.

(2) $\mathcal{H}^{n+1-p}_{(2)} \to H^{n+1-p}(X)$ is surj.

[Proof]

(1) Take a class $\in H^p_{(2)}(X)$.

Represented by $\alpha \in \mathcal{H}^p_{(2)}$.

$$R\alpha = \left(\int_0^x \alpha^{(n)}_{a_1 \cdots a_{p-1}}(t, y)\, dt \right) dy^{a_1} \wedge \cdots \wedge dy^{a_{p-1}}$$

$$\tilde{R}\alpha = \psi R\alpha$$

If $\tilde{R}\alpha$ is L^2 and $d\tilde{R}\alpha = \alpha$ near ∂X,

then $\alpha - d\tilde{R}\alpha \in \Omega^p_{(2),c}$ and

$[\alpha - d\tilde{R}\alpha] = [\alpha] \in H^p_{(2)}(X)$.

Cor 4.4 $\rightarrow$ $|\alpha|_{\tilde{g}} = O(x^{n-2p-\varepsilon})$

$\therefore$ $|\tilde{R}\alpha|_{\tilde{g}} = O(x^{n-2p-\varepsilon+1})$

$|\tilde{R}\alpha|_g = O(x^{n-p-\varepsilon})$

$$\|\tilde{R}\alpha\|^2 = \int O(x^{2n-2p-2\varepsilon}) \cdot \frac{dx\,dy^1 \dots dy^n}{x^{n+1}}$$

$$= \int \underbrace{O(x^{n-2p-2\varepsilon-1})}_{\text{better than } O(x^{-1})} dx\,dy^1 \dots dy^n$$

$< \infty$.

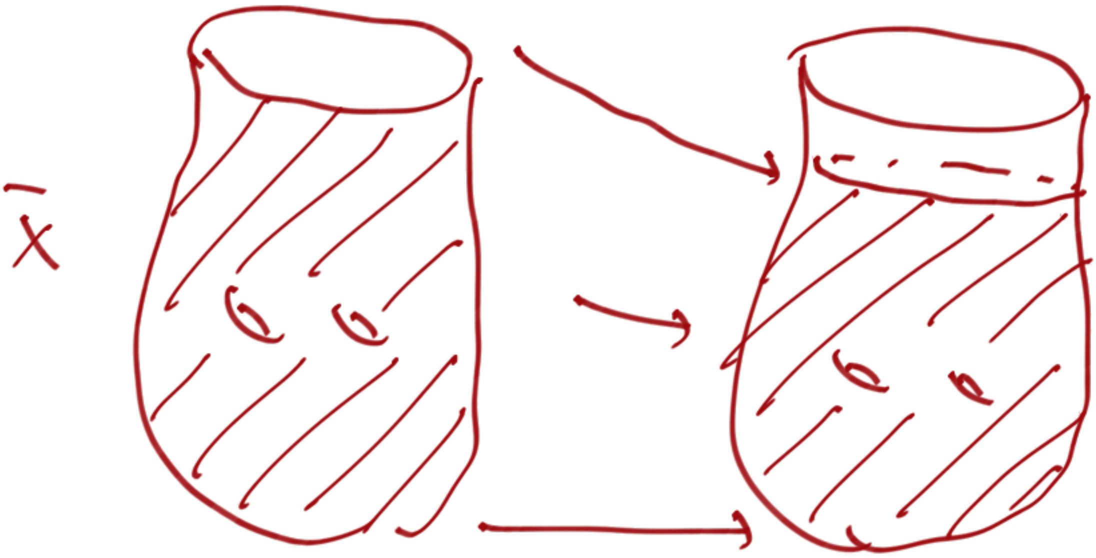
Moreover, if $\mu_\tau : (x, y) \to (\tau x, y)$

$$\underbrace{d \int_{\tau x}^{x} \alpha^{(n)}(t, y)\,dt}_{\substack{L^2 \downarrow \tau \to 0 \\ \tilde{R}\alpha}} = \alpha - \underbrace{\mu_\tau^* \alpha}_{\substack{\downarrow L^2 \\ 0}}$$

(2) $\mathcal{H}^{n+1-p}_{(2)} \to H^{n+1-p}(X)$ surj

Take $[\alpha] \in H^{n+1-p}(X)$.

May assume α is smooth on $\bar{X}$



$$\|\alpha\|^2_{L^2} = \int |\alpha|^2_g \, dV_g = \int \rho^{2(n+1-p)} |\alpha|^2_{\bar{g}} \cdot \rho^{-n-1} dV_{\bar{g}}$$

$$= \int \underbrace{\rho^{n+1-2p}}_{\text{bdd}} |\alpha|^2_{\bar{g}} \, dV_{\bar{g}} < \infty$$

$\alpha = \Delta\beta + \omega, \quad \omega \in \mathcal{H}^{n+1-p}_{(2)}$

$(\because \ \mathrm{im}\,\Delta$ closed$)$

$$\alpha = dd^*\beta + d^*d\beta + \omega$$

Since $d\alpha = 0$, $dd^*d\beta = 0$ and hence

$d^*d\beta = 0$.

$\left(\because\ 0 = (dd^*d\beta, d\beta) = (d^*d\beta, d^*d\beta) \right)$

$\therefore\ \alpha = dd^*\beta + \omega$.

$\therefore\ [\alpha] = [\omega]$. $\square$

Rem (Yeganefar 2004)

$$H^p_{(2)}(X) \cong \begin{cases} H^p_c(X) & \text{if } p < \frac{n+1}{2} \\ H^p(X) & \text{if } p > \frac{n+1}{2} \end{cases}.$$