3. Fredholm theorem for geom LDOs on AH spaces

3.1 The result

(X, g) Riem

E, F geom tensor bundles

e.g. $(TX)^{\otimes k} \otimes (T^*X)^{\otimes l}$

$\Lambda^k T^*X \qquad \mathrm{Sym}^k T^*X$

$\mathrm{Sym}_0^2 T^*X$

Def 3.1 $P : \Gamma(E) \to \Gamma(F)$ LDO is geometrically defined

$\overset{\text{def}}{\Longleftrightarrow}$ P is given in terms of

∇_i, R_{ijkl}, g_{ij}, g^{ij} and

contractions.

Ex. $\Delta^{(p)} := -(dd^* + d^*d)$ on p-forms.

α p-form

$$(d\alpha)_{i_1 \cdots i_{p+1}} = \sum_{k=1}^{p+1} (-1)^{k-1} \nabla_{i_k} \alpha_{i_1 \cdots \check{i_k} \cdots i_{p+1}}$$

$\uparrow$ remove

$$(d^*\alpha)_{i_1 \cdots i_{p-1}} = -\nabla^j \alpha_{j i_1 \cdots i_{p-1}}$$

Hence $\Delta^{(p)}$ is geom defined.

Thm 3.2 (Fredholm theorem)

(X, g) AH space E geom tensor bundle

$P: \Gamma(E) \to \Gamma(E)$

geom defined, formally self-adj,

ell LDO of order l.

Suppose: (i) A coercivity estimate holds for P on $\mathbb{H}^{n+1}$.

(ii) $R := \min_{\zeta \in \Sigma_P} \left| \mathrm{Re}\, \zeta - \frac{n}{2} \right| > 0$

where $\Sigma_P = \{ \text{char roots of } P \}$.

Then, for $-R < {}^\forall \delta < R$,

(1) $P: H^{m+l}_\delta (X, E) \to H^m_\delta (X, E)$, $m \geq 0$

(2) $P: C^{m+l, \alpha}_{\frac{n}{2}+\delta} (X, E) \to C^{m, \alpha}_{\frac{n}{2}+\delta} (X, E)$, $m \geq 0$, $0 < \alpha < 1$

are Fredholm of index 0.

Moreover, $\ker P$ remains the same no matter which func sp is chosen from (1), (2).

Def 3.3 $\mathcal{B}_1, \mathcal{B}_2$ Banach sp.

A bdd op $P : \mathcal{B}_1 \to \mathcal{B}_2$ is Fredholm

$\overset{\text{def}}{\Longleftrightarrow}$ dim ker $P < \infty$

dim coker $P < \infty$ $\left(\text{coker} = \dfrac{\mathcal{B}_2}{\text{ran } P} \right)$

$\uparrow$

difference $=:$ index P

Fredholm $=$ "close to invertible"

Fact 3.4 $P : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ Fredholm

$\Longleftrightarrow$ $\exists\, Q_1, Q_2 : \mathcal{B}_2 \longrightarrow \mathcal{B}_1$ bdd s.t.

$$Q_1 P = I_{\mathcal{B}_1} + K_1, \quad K_1 \in \mathcal{K}(\mathcal{B}_1),$$

$$P Q_2 = I_{\mathcal{B}_2} + K_2, \quad K_2 \in \mathcal{K}(\mathcal{B}_2)$$

where $\mathcal{K}$ is the set of compact operators.

$K : \mathcal{B}_1 \to \mathcal{B}_2$ bdd is compact $\overset{\text{def}}{\Longleftrightarrow}$ $K(\mathcal{B}_1)$ rel compact

Fact $\left.\begin{array}{l} (\text{bdd}) \circ (\text{compact}) \\ (\text{compact}) \circ (\text{bdd}) \end{array}\right\}$ are compact.

3.2 Function spaces

Recall : (X, g) sccAH space if
$\bar{g} = \rho^2 g$ extends to Riem met on $\bar{X}$,
(ρ bdf) $\quad |d\rho|_{\bar{g}} = 1$ on ∂X.

★ Sobolev sp

$$H^m(X,E) = \{ u \in L^2(X,E) \mid \nabla^j u \in L^2 \ (1 \le j \le m) \}$$

$$H^m_\delta(X,E) = \rho^\delta H^m(X,E)$$

$(\delta \in \mathbb{R})$

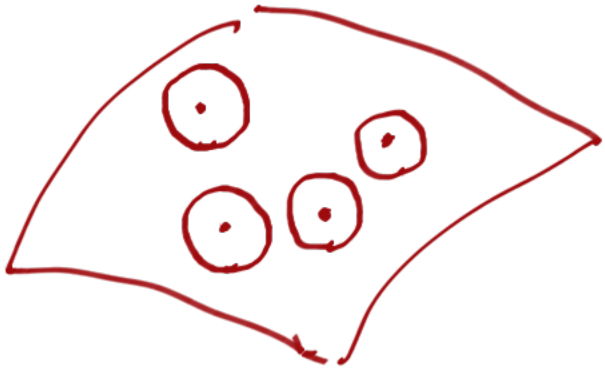
★ Hölder sp

$$C^{m,0}(X,E) = \{ u \in C^m(X,E) \mid \nabla^j u \text{ bdd } (0 \le j \le m) \}$$

<u>Lem 3.5</u> $\quad$ sccAH space (X, g) has positive injectivity radius r_{inj}.

$$r_{inj}(x) = \sup \left\{ r > 0 \;\middle|\; \begin{array}{l} T_xX \supset B_r \xrightarrow{\exp} X \text{ is} \\ \text{diffeo onto image} \end{array} \right\} > 0$$

$$r_{inj} = \inf_{x \in X} r_{inj}(x)$$



In particular,

$d(x,y) < r_{inj}$

$\Longrightarrow$ $\exists\, \gamma_{xy}$ unique minimizing geodesic

$P_{x \to y}$ parallel transl

$$C^{m,\alpha}(X,E) = \left\{ u \in C^{m,0} \;\middle|\; \nabla^m u \;\; \alpha\text{-Hölder cont} \right\}$$

$(0 < \alpha < 1)$

def $\Updownarrow$

$$[\nabla^m u]_\alpha := \sup_{\substack{x,y \in X \\ x \neq y \\ d(x,y) < r_{inj}}} \frac{|P_{x\to y} \nabla^m u(x) - \nabla^m u(y)|}{d(x,y)^\alpha} < \infty$$

$$\|u\|_{C^{m,\alpha}} = \|u\|_{C^{m,0}} + [\nabla^m u]_\alpha$$

$$C^{m,\alpha}_\delta(X,E) = \rho^\delta C^{m,\alpha}(X,E).$$

Def 3.6

(X, g) is an AH space of class $C^{m,\alpha}$

$\overset{\text{def}}{\Longleftrightarrow}$ $g = g_0 + u$,

where (X, g_0) sccAH,

$u \in C^{m,\alpha}_{\delta}(X, \mathrm{Sym}^2 T^*X)$ for some $\delta > 0$

(X, g) is simply an AH space if above holds for $\forall m$, $\forall \alpha$ (δ should be uniform).

Prop 3.7 (Lee's Möbius atlas)

sccAH (X, g) has an atlas $\{(U_\lambda, \varphi_\lambda)\}$ satisfying:

(i) $\varphi_\lambda : U_\lambda \xrightarrow{\text{diffeo}} B_2 \subset \mathbb{R}^{n+1}$

$\{\varphi_\lambda^{-1}(B_1)\}$ covers X

$x_\lambda := \varphi_\lambda^{-1}(0)$ center of U_λ

(ii)
$$\begin{tikzcd} U_\lambda \arrow[dr, "\varphi_\lambda"'] & & B_2(0) \subset \mathbb{H}^{n+1} \\ & B_2 \arrow[ur, "\exp"'] & \end{tikzcd}$$

$\exists C_m > 0$, $\exists C' > 0$ s.t. $\forall \lambda$

if $\tilde{g}_\lambda = (\varphi_\lambda^{-1})^*(g|_{U_\lambda})$,

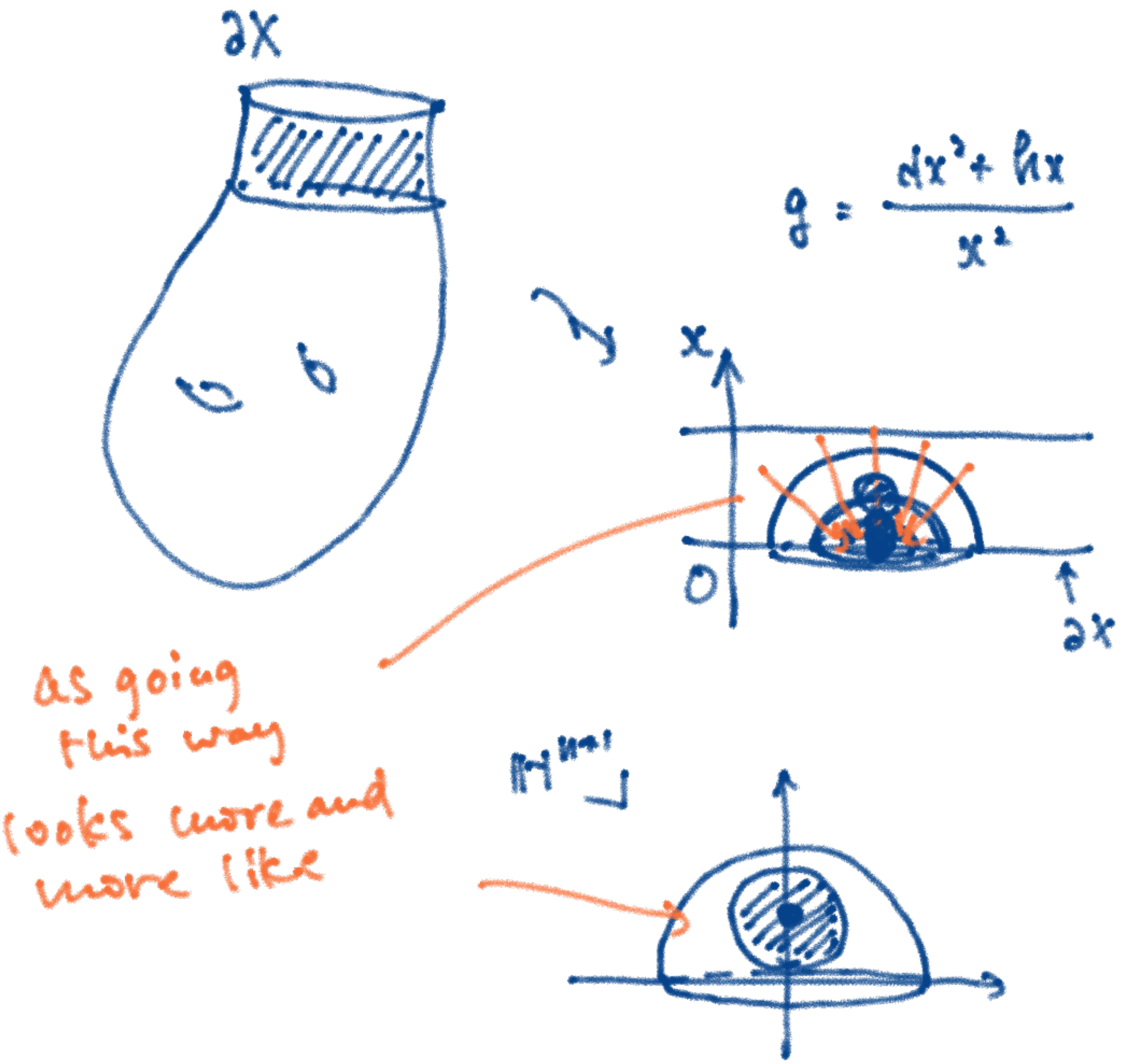
$\tilde{g}_{\mathbb{H}^{n+1}} = \exp^*(g_{\mathbb{H}^{n+1}}|_{B_2(0)})$,

Then $\|\tilde{g}_\lambda - \tilde{g}_{\mathbb{H}^{n+1}}\|_{C^m(B_2)} < C_m$,

$\sup_{B_2} |\tilde{g}_\lambda^{-1} - \tilde{g}_{\mathbb{H}^{n+1}}^{-1}| < C'$.

(iii) $\{U_\lambda\}$ unif loc finite

(iv) Any compact $K \subset X$ intersects finite # of U_λ's.



∂X
$g = \frac{dx^2 + h_x}{x^2}$
x
0
∂X
as going this way looks more and more like
$\mathbb{H}^{n+1}$

Prop 3.8 $\{(U_\lambda, \varphi_\lambda)\}$ Möbius atlas

$$\|u\|_{H^m} \sim \sum_\lambda \|(\varphi_\lambda^{-1})^* u\|_{H^m(B_2)}$$

$$\|u\|_{H^m_\delta} \sim \sum_\lambda \rho(x_\lambda)^{-\delta} \|(\varphi_\lambda^{-1})^* u\|_{H^m(B_2)}$$

Same for Hölder norms.

B_2 can be replaced with B_1.

Cor 3.9 (weighted global ell estimates)

$P : \Gamma(E) \to \Gamma(E)$ geom ell LDO order l

(1) $u \in L^2_\delta$, $Pu \in H^m_\delta \Rightarrow u \in H^{m+l}_\delta$

$$\|u\|_{H^{m+l}_\delta} \leq C\left(\|Pu\|_{H^m_\delta} + \|u\|_{L^2_\delta}\right)$$

(2) $u \in C^{0,0}_\delta$, $Pu \in C^{m,\alpha}_\delta \Longrightarrow u \in C^{m+l,\alpha}_\delta$

($0<\alpha<1$ fixed)

Similar estimate.

Cor 3.10 (Sobolev embedding)

(1) $m > m' + \frac{n+1}{2}$

$\Longrightarrow \quad H^m_\delta(E) \hookrightarrow C^{m',0}_\delta(E)$ continuous

(2) $m \geq m' + \alpha + \frac{n+1}{2}$, $0 < \alpha < 1$

$\Longrightarrow \quad H^m_\delta(E) \hookrightarrow C^{m',\alpha}_\delta(E)$ continuous

Cor 3.11 (Rellich compactness)

(1) $m_1 > m_2$, $\delta_1 > \delta_2$

$\Longrightarrow \quad H^{m_1}_{\delta_1}(E) \hookrightarrow H^{m_2}_{\delta_2}(E)$ compact

(2) $m_1 + \alpha_1 > m_2 + \alpha_2$, $\delta_1 > \delta_2$

$\Longrightarrow \quad C^{m_1,\alpha_1}_{\delta_1}(E) \hookrightarrow C^{m_2,\alpha_2}_{\delta_2}$ compact

3.3 Coercivity estimate

Def 3.12 $P : \Gamma(E) \to \Gamma(E)$ geom LDO

We say P satisfies coercive est on $\mathbb{H}^{n+1}$ if

$$\|u\|_{L^2} \le C \|Pu\|_{L^2} \quad u \in C_0^\infty(\mathbb{H}^{n+1}, E)$$

Estimate for Hodge $\Delta^{(p)} = -(dd^* + d^*d)$

Prop 3.13 (Dodziuk, Donnelly, Donnelly - Xavier)

For p-forms on $\mathbb{H}^{n+1}$,

$$\left(\frac{n}{2} - p\right)^2 \|\alpha\|^2 \le (\alpha, -\Delta^{(p)}\alpha) \quad \text{if } p < \frac{n}{2}$$

$$\left(\frac{n}{2} - (n+1-p)\right)\|\alpha\|^2 \le (\alpha, -\Delta^{(p)}\alpha) \quad \text{if } p > \frac{n+2}{2}$$

No such estimate for $p = \frac{n}{2}, \frac{n+1}{2}, \frac{n+2}{2}$.

3.4 Characteristic poly / roots

$P : \Gamma(E) \to \Gamma(E)$ geom LDO

Write P on $\mathbb{H}^{n+1}$ using upper-half sp model. Use identification

$$E \cong \underline{\mathbb{R}^r} \quad \text{(trivial bundle)}$$

induced by the frame $x\partial_x, x\partial_{y^1}, \dots, x\partial_{y^n}$ to identify P as a system of r LDOs for r unknown functions:

$$P = \begin{pmatrix} r \times r \text{ matrix} \\ \text{components: LDO} \\ \text{acting on functions} \end{pmatrix}$$

Def 3.14 The char poly $I_P[s] \in M_{r\times r}[s]$ is such that

$$P \longrightarrow I_P(x\partial_x) \quad \text{as } x \to 0$$

(as matrix of polynomials of $x\partial_x, \partial_{y^1}, \dots, \partial_{y^n}$)

$\zeta \in \mathbb{C}$ <u>char root</u> $\overset{\text{def}}{\Longleftrightarrow}$ $I_P(\zeta)$ not invertible

$$\Sigma_P = \{ \text{char roots of } P \}.$$

<u>Ex.</u> Hodge Laplacian on 1-forms

$$(\Delta^{(1)}\alpha)_i = \nabla^j \nabla_j \alpha_i - R_i{}^j \alpha_j$$

$$= \underbrace{\nabla^j \nabla_j \alpha_i}_{} + n\alpha_i$$

$$(\Delta^{(1)}\alpha)_0 = \left((x\partial_x)^2 - n\, x\partial_x \right) \alpha_0$$

$$+ x^2 \Delta_{\mathbb{R}^n} \alpha_0 + 2x \sum_{b=1}^{n} \partial_{y^b} \alpha_b$$

corresp $x\partial_x \leftrightarrow \frac{1}{x} dx$

$$(\Delta^{(1)}\alpha)_a = \left((x\partial_x)^2 - n\, x\partial_x + (n-1) \right) \alpha_a$$

$a \in \{1, \dots, n\}$

$$+ x^2 \Delta_{\mathbb{R}^n} \alpha_a - 2x\, \partial_{y^a} \alpha_0 .$$

$$\begin{pmatrix} (x\partial_x)^2 - n x\partial_x + x^2\Delta_{\mathbb{R}^n} & 2x\partial_{y^b} \\ -2x\partial_{y^a} & (x\partial_x)^2 - n x\partial_x + (n-1) + x^2\Delta_{\mathbb{R}^n} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} (x\partial_x)^2 - n x\partial_x & \\ & (x\partial_x)^2 - n x\partial_x + n-1 \end{pmatrix}$$

$$\therefore\ I_{\Delta^{(1)}}(s) = \begin{pmatrix} s^2 - ns & \\ & s^2 - ns + (n-1) \end{pmatrix}$$

$$\Sigma_{\Delta^{(1)}} = \{0, n, 1, n-1\}$$