2. Asymp Dirichlet prob for $\Delta_{\mathbb{H}^{n+1}}$

2.1 Preliminaries

The Laplacian of Riem mfd (X, g)

$$\Delta u := g^{ij} \nabla_i \nabla_j u$$

loc frame $X_1, \dots, X_{n+1}$

$$= g^{ij} (X_i X_j u - \Gamma^k{}_{ij} X_k u)$$

Fact 2.1 Δ is characterized by

$$\int_X (\Delta u) \varphi \, dV = - \int_X \langle du, d\varphi \rangle \, dV, \quad \varphi \in C_0^\infty.$$

Equivalently: $\Delta = - d^* d$.

∃ metric

Def 2.2 $P : \Gamma(E) \to \Gamma(F)$ LDO

$\rightsquigarrow$ formal adj $P^* : \Gamma(F) \to \Gamma(E)$

$$\int_X \langle P^* u, \varphi \rangle \, dV = \int_X \langle u, P\varphi \rangle \, dV$$

$$\varphi \in C_0^\infty(X, E)$$

★ Ellipticity

$P : \Gamma(E) \to \Gamma(F)$ LDO of order l

$$Pu = a^{i_1 \cdots i_l} X_{i_1} \cdots X_{i_l} u + \left(\text{lower order terms}\right)$$

$$a^{i_1 \cdots i_l}(x) \in \mathrm{Hom}(E_x, F_x)$$

Def 2.3 (1) The principal symbol

$$\sigma_P : T^*X \longrightarrow \mathrm{Hom}(E, F)$$

$$\sigma_P(x, \eta) := a^{i_1 \cdots i_l}(x)\, \eta_{i_1} \cdots \eta_{i_l}$$

$x \in X \quad \eta \in T_x^* X \qquad (\eta = \eta_i \theta^i)$

(2) P elliptic $\overset{\text{def}}{\iff}$ σ_P invertible on $T^*X \setminus (\text{zero sect})$

Ex. $\sigma_\Delta(x, \eta) = g^{ij}(x)\, \eta_i \eta_j = |\eta|^2_{g_x}$

$\therefore$ Δ elliptic.

$U \subset \mathbb{R}^N$ $\quad H^m(U)$ $\quad L^2$- Sobolev sp

$(m = 0, 1, 2, 3, \dots)$

U convex $\rightsquigarrow$ $C^{m,\alpha}(U)$ $\quad$ Hölder sp

$(m = 0, 1, 2, \dots; \ 0 \leq \alpha < 1)$

<u>Fact 2.4</u> (local elliptic estimates)

P ell LDO defined on a nbhd of $\bar{B}_1 \subset \mathbb{R}^N$ of order l

(1) $u \in L^2(B_1)$, $\quad Pu \in H^m(B_1)$

$\Longrightarrow u \in H^{m+l}(B_{1/2})$.

$$\|u\|_{H^{m+l}(B_{1/2})} \leq C\left(\|Pu\|_{H^m(B_1)} + \|u\|_{L^2(B_1)}\right)$$

(2) $u \in C^{0,0}(B_1)$, $\quad Pu \in C^{m,\alpha}(B_1)$

$(0 < \alpha < 1$ fixed$)$

$\Longrightarrow u \in C^{m+l,\alpha}(B_{1/2})$,

similar estimate.

Moreover, if

$|$coeffs of $P|$ $\quad$ unif bdd above,

$\sigma_P(x,\eta)/|\eta|^l$ $\quad$ unif bdd below,

then $\exists\, C$ uniformly.

2.2 The result for $\Delta_{\mathbb{H}^{n+1}}$

$\Delta = \Delta_{\mathbb{H}^{n+1}}$ below.

Prop 2.5

(1) Upper-half sp model.

$$\mathbb{H}^{n+1} = \underset{\substack{\cup \\ x}}{(0,\infty)} \times \underset{\substack{\cup \\ (y^1,\dots,y^n)}}{\mathbb{R}^n} \qquad g = \frac{g_{Euc}}{x^2}$$

$$(*) \quad \Delta = (x\partial_x)^2 - n\, x\partial_x + x^2 \Delta_{\mathbb{R}^n}$$

$$\partial_x = \frac{\partial}{\partial x} \qquad \left(\Delta_{\mathbb{R}^n} = \sum_{a=1}^{n} \partial_{y^a}^2 \right)$$

(2) Poincaré ball model

$$\rho = \frac{1-|x|}{1+|x|} = e^{-d(0,x)}$$

$$\mathbb{H}^{n+1} \setminus \{0\} \approx \underset{\substack{\cup \\ \rho}}{(0,1)} \times S^n$$

$$(**) \quad \Delta = (\rho\partial_\rho)^2 - n \cdot \frac{1+\rho^2}{1-\rho^2}\rho\partial_\rho + \frac{4\rho^2}{(1-\rho^2)^2}\Delta_{S^n}$$

Observe:

(*) $\Delta \xrightarrow{\text{"}x\to 0\text{"}} (x\partial_x)^2 - n\, x\partial_x$

(as noncomm poly of $x\partial_x, \partial_{y^1}, \dots, \partial_{y^n}$)

(**) $\Delta \xrightarrow{\text{"}\rho\to 0\text{"}} (\rho\partial_\rho)^2 - n\, \rho\partial_\rho$

$I(s) = s^2 - ns$ characteristic poly of Δ

$I(s) = 0 \iff s = 0, n$ characteristic roots

Thm 2.6 Let $\varphi \in C^\infty(\partial B^{n+1})$.

(1) $\exists!\, u \in C^\infty(B^{n+1}) \cap C^0(\overline{B^{n+1}})$ s.t.

$\Delta u = 0$ in B^{n+1}, $u|_{\partial B^{n+1}} = \varphi$.

(2) $u \in C^{n-1,\alpha}(\overline{B^{n+1}})$ for $0 < {}^\forall\alpha < 1$.

If $n = 2, 4, 6, \dots$ then $u \notin C^n(\overline{B^{n+1}})$ generically.

Rem Uniqueness follows from the maximum principle for Riem mfd.

(cf. Petersen GTM)

Methods. Reduce to ODE theory.

1. Use spherical harmonics decomp of φ.

2. Analytize Green's function and get mapping properties of Δ^{-1}. (We use this!)

2.3 Construction of approx solution

Why mapping properties of Δ^{-1}?

Why "$\Delta v = f$ is good $\Rightarrow$ v is good" helps?

Prop 2.7 $\forall \varphi \in C^\infty(\partial B^{n+1})$ $\exists u \in C^\infty(\overline{B^{n+1}})$

s.t.

$$\Delta u = O(\rho^n), \qquad u|_{\partial B^{n+1}} = \varphi.$$

(Note: Δu is automatically $\in C^\infty(\overline{B^{n+1}})$)

[Proof] Inductive arg. Construct $u_{(k)}$

with $\Delta u_{(k)} = O(\rho^{k+1})$, $u_{(k)}|_{\partial B^{n+1}} = \varphi$.

Step $k=0$ Any $u_{(0)}$ works.

Step $k \geq 1$ We have $u_{(k-1)}$. Try

$$u_{(k)} = u_{(k-1)} + v, \qquad v = O(\rho^k).$$

Then

$$\Delta u_{(k)} = \Delta u_{(k-1)} + (k^2 - nk)v + O(\rho^{k+1})$$

and if $k^2 - nk \neq 0$, we can take v s.t.

$$\Delta u_{(k)} = O(\rho^{k+1}).$$ $\square$

Step $k=n$ can't be carried out.

2.4 Mapping properties via Green's fn

2.4.1 Function spaces on $\mathbb{H}^{n+1}$

★ Sobolev sp H^m, H^m_δ ($\delta \in \mathbb{R}$)

defined as distribution

$$H^0 := L^2$$

$$H^m := \{ u \in L^2 \mid \nabla^j u \in L^2 \ (1 \leq j \leq m) \}$$

$$\|u\|^2_{H^m} := \sum_{j=0}^{m} \|\nabla^j u\|^2_{L^2}$$

$$H^m_\delta := \rho^\delta H^m = \{ u \mid \rho^{-\delta} u \in H^m \}$$

ρ any fixed bdf e.g. $1 - |x|^2$

$$\|u\|_{H^m_\delta} := \|\rho^{-\delta} u\|_{H^m}$$

★ Hölder sp $C^{m,\alpha}$, $C^{m,\alpha}_\delta$ $\delta \in \mathbb{R}$

$m = 0, 1, 2, \ldots$ $0 \leq \alpha < 1$

$$C^{m,0} := \{ u \in C^m \mid |\nabla^j u| \text{ bdd } (0 \leq j \leq m) \}$$

m times continuously differentiable

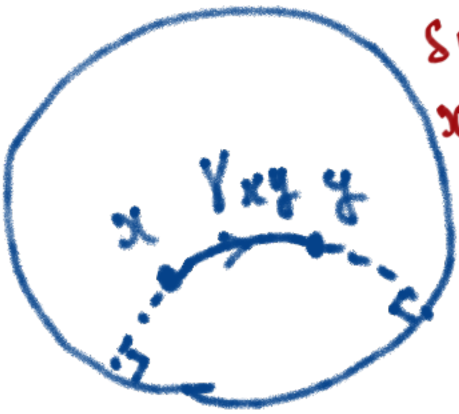
$$\|u\|_{C^{m,0}} := \sum_{j=0}^{m} \sup |\nabla^j u| .$$

$$C^{m,\alpha} := \{ u \in C^{m,0} \mid \nabla^m u \ \ \alpha\text{-Hölder cont} \}$$

$0 < \alpha < 1$

$$\sup_{\substack{x,y \in \mathbb{H}^{n+1} \\ x \neq y}} \frac{|P_{x\to y}\nabla^m u(x) - \nabla^m u(y)|}{d(x,y)^\alpha} < \infty$$

$$\sup_{\substack{x,y \in \mathbb{H}^{n+1} \\ x \neq y}} \frac{|P_{x\to y}\nabla^m u(x) - \nabla^m u(y)|}{d(x,y)^\alpha} = [\nabla^m u]_\alpha$$



$P_{x\to y}$ parallel transl along γ_{xy}

$$\|u\|_{C^{m,\alpha}} := \|u\|_{C^{m,0}} + [\nabla^m u]_\alpha$$

$$C^{m,\alpha}_\delta := \rho^\delta C^{m,\alpha} \qquad \|u\|_{C^{m,\alpha}_\delta} := \|\rho^{-\delta} u\|_{C^{m,\alpha}}$$

Prop 2.8 (Global ell estimates for Δ)

(1) Fact 2.4 (1) globalizes :

$u \in L^2$, $\Delta u \in H^m \Longrightarrow u \in H^{m+2}$.

$$\|u\|_{H^{m+2}} \leq C\left(\|\Delta u\|_{H^m} + \|u\|_{L^2} \right).$$

(2) Fact 2.4 (2) globalizes similarly.

Prop. 2.8# (Weighted ————)

Prop 2.8 holds true even if

$H^m \rightsquigarrow H^m_\delta$, $C^{m,\alpha} \rightsquigarrow C^{m,\alpha}_\delta$.

[Proof of Prop 2.8 (1)]

Take family $\{B_1(x_j)\}$ s.t.

(i) $\{B_1(x_j)\}$ covers $\mathbb{H}^{n+1}$.

(ii) $\{B_2(x_j)\}$ is unif loc finite.

$\exists N$ $\forall x$ belongs to at most N $B_2(x_j)$'s

Then

$$\|u\|_{H^{m+2}} \leq C \sum_j \|u\|_{H^{m+2}(B_1(x_j))}$$

$$\leq C' \sum_j \left(\|\Delta u\|_{H^m(B_2(x_j))} + \|u\|_{L^2(B_2(x_j))} \right)$$

$$\leq N C'' \left(\|\Delta u\|_{H^m} + \|u\|_{L^2} \right).$$

To take $\{B_1(x_j)\}$: Zorn's lemma.

$\{x_j\}$ maximal collection of pts for which $B_{\frac{1}{2}}(x_j)$'s don't intersect.

$$N = \frac{\mathrm{Vol}(B_{7/2})}{\mathrm{Vol}(B_{1/2})}$$

$\square$

2.4.2 Green's function

Prop 2.9 (Coerciveness estimate)

$$\frac{n^2}{4}\|u\|_{L^2}^2 \leq (u, -\Delta u)_{L^2}, \quad \forall u \in H^2$$

[Proof] Suffices to show for $u \in C_0^\infty$ because C_0^∞ is dense in H^2. $\square$

Cor 2.10 $\Delta : H^2 \to L^2$ is bijective.

Hence $\Delta^{-1} : L^2 \to H^2$ is bdd (by open mapping principle).

Recall: For bdd P between Hilbert spaces,

$$\operatorname{ran} P \text{ closed} \iff \|u\| \leq C\|Pu\| \quad u \in (\ker P)^\perp$$

[Proof of Cor 2.10]

Prop 2.9 $\rightsquigarrow$ $\|u\|_{L^2} \leq C \|\Delta u\|_{L^2}$

$\rightsquigarrow$ $\ker \Delta = 0$

$\operatorname{ran} \Delta$ closed.

Suffices to show $(\operatorname{ran} \Delta)^{\perp} = 0$.

$\square$

$\Delta^{-1} : L^2 \to H^2$ in particular defines

$G = \Delta^{-1} : C_0^\infty \to \mathcal{D}'$ $\underbrace{\text{continuous}}_{\text{sequentially}}$

<u>Lem 2.11</u> (Schwartz kernel thm)

X paracompact $P : C_0^\infty \to \mathcal{D}'$ seq cont

$\Longrightarrow$ $\exists K \in \mathcal{D}'(X \times X)$ s.t.

$\langle Pu, v \rangle = \langle K, v \otimes u \rangle \quad u, v \in C_0^\infty$

$(v \otimes u)(x, y) = v(x) u(y)$

Def 2.12 K of G is the **Green's function.**

Formally:

$$\int Pu(x)v(x)\,dV_x = \iint K(x,y)\,v(x)\,u(y)\,dV_x\,dV_y$$

$$\therefore \quad Pu(x) = \int K(x,y)\,u(y)\,dV_y$$

Want to prove that this makes sense for func's of some class.

Lem 2.13

(1) "$K(x,y) = K(y,x)$" in the sense that $\langle K, u \otimes v \rangle = \langle K, v \otimes u \rangle$.

(2) $\Delta_x K(x,y) = \delta_{diag}$ where $\langle \delta_{diag}, u \otimes v \rangle = (u, v)_{L^2}$.

(3) K is C^∞ away from diag.

(4) $0 \in \mathbb{H}^{n+1}$, $U \subset \mathbb{H}^{n+1}$ rel cpt nbhd of 0

$$\Rightarrow \quad K(0,x)\Big|_{\mathbb{H}^{n+1} \setminus U} \in L^2.$$

Our goal:

<u>Thm 2.14</u> For $-\frac{n}{2} < \delta < \frac{n}{2}$.

(1) $\Delta : H^{m+2}_{\delta} \longrightarrow H^{m}_{\delta}$ invertible $\forall m \geq 0$

(2) $\Delta : C^{m+2,\alpha}_{\frac{n}{2}+\delta} \rightarrow C^{m,\alpha}_{\frac{n}{2}+\delta}$ invertible $\forall m \geq 0$ $0 < \forall \alpha < 1$

Inverse given by $f \longmapsto \overset{u(x)}{=} \int K(x,y) f(y) dV_y$

To this goal —

$K(0,x)$ is rotationally inv

$\|$

$k(\rho)$ $\rho = \frac{1-|x|}{1+|x|} = e^{-d(0,x)}$ $\in (0,1)$

<u>Lem 2.15</u> $|k(\rho)| \leq C\rho^{n}$ on $(0, \rho_0)$

[Proof] $\Delta_x K(0,x) = 0$ away from $x=0$

$$\therefore \left((\rho\partial_\rho)^2 - n\cdot \frac{1+\rho^2}{1-\rho^2}\rho\partial_\rho \right) k(\rho) = 0.$$

Using Lem 2.13 (4),

$$k(\rho) = C\int_{\frac{1-\rho}{1+\rho}}^{1} \frac{(1-t^2)^{n-1}}{t^n}\,dt.$$

$\square$

Alternatively, a weaker ineq

$$|k(\rho)| \le C_\varepsilon \rho^{n-\varepsilon} \quad (\forall \varepsilon > 0)$$

follows from gen theory of ODEs.

Exponent n comes from characteristic roots.

Lem 2.16 If $\alpha+\beta > n$, $\alpha > \beta$,

$$\int_{\mathbb{H}^{n+1}} \underbrace{e^{-\alpha d(x,y)}}_{\text{better decay}} \underbrace{e^{-\beta d(y,z)}}_{\text{worse decay}} dV_y \le C e^{-\beta d(x,z)}.$$

worse decay remains true

Cor 2.17

$f \mapsto u$, $u(x) = \int K(x,y) f(y) dV_y$

defines bdd operators

$$L^2_\delta \to L^2_\delta, \quad C^{0,0}_{\frac{n}{2}+\delta} \to C^{0,0}_{\frac{n}{2}+\delta},$$

for $-\frac{n}{2} < \delta < \frac{n}{2}$.

Thm 2.14 = Cor 2.17 + Prop 2.8 #

Cor 2.18 (of Thm 2.14) $0<\alpha<1$, $-\frac{n}{2}<\delta<\frac{n}{2}$, $m+2+\alpha<\frac{n}{2}+\delta$

$$f\in\rho^{\frac{n}{2}+\delta}C^{m,\alpha}(\overline{B^{n+1}})$$

$$\Longrightarrow \exists!\, v\in C^{m+2,\alpha}(\overline{B^{n+1}})$$

$$\text{s.t. } \Delta v=f,\quad v|_{\partial B^{n+1}}=0.$$

[Proof]

$$f\in\rho^{\frac{n}{2}+\delta}C^{m,\alpha}(\overline{B^{n+1}})\subset C^{m,\alpha}_{\frac{n}{2}+\delta}$$

$$\rightsquigarrow \exists\, v\in C^{m+2,\alpha}_{\frac{n}{2}+\delta} \text{ s.t. } \Delta v=f.$$

General fact:

$$C^{m+2,\alpha}_{\frac{n}{2}+\delta}\subset C^{m+2,\alpha}(\overline{B^{n+1}})$$

$$\text{if } m+2+\alpha<\frac{n}{2}+\delta.$$

$\square$

Proof of Thm 2.6 is completed by Cor 2.18.

Characteristic root

Coerciveness estimate

Existence range of Approx sol'n

Green's fn decay estimate

Mapping properties of Δ^{-1}