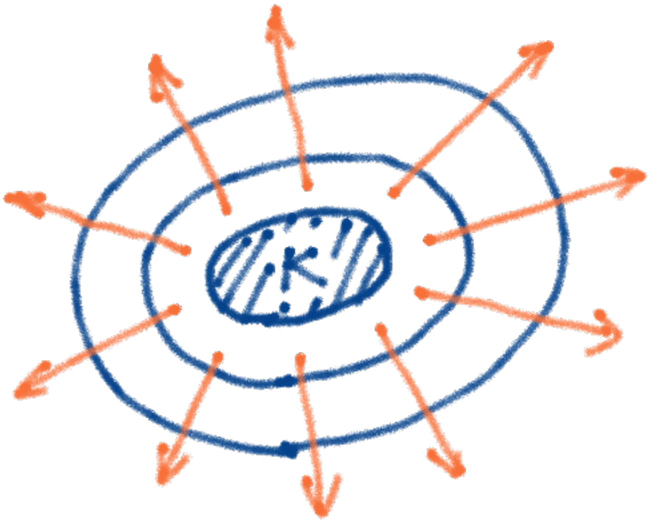
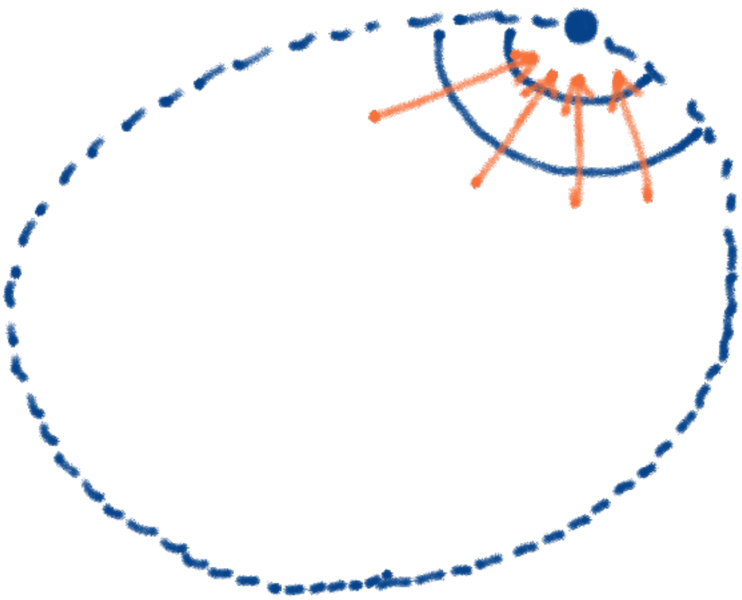
Geometric Analysis on Asymptotically Hyperbolic Spaces

AH spaces

≈ 

Locally more and more like $\mathbb{H}$

≈ 

More and more like nbhd of $p \in \partial_\infty \mathbb{H}$

"Conformal infinity" $\partial_\infty X$ carries a conf class

$\boxed{\text{Analytic properties of } X}$ ← Topology of X

$\boxed{\text{Analytic properties of } X}$ $\xleftarrow{\text{effect}}$ Conf geom of $\partial_\infty X$

History.

Fefferman ('76)

Lax-Phillips ('82)
Sullivan ('87)

Fefferman-Graham ('85)

Mazzeo-Melrose ('87)
Mazzeo ('88)

Graham-Lee ('91)

$\left.\begin{array}{l}\text{L. Andersson ('93)}\\ \text{Biquard ('00)}\\ \text{Lee ('06)}\end{array}\right\}$ Foundations

Maldacena ('98) AdS/CFT

M. Anderson
S.-Y. A. Chang
M. Gursky
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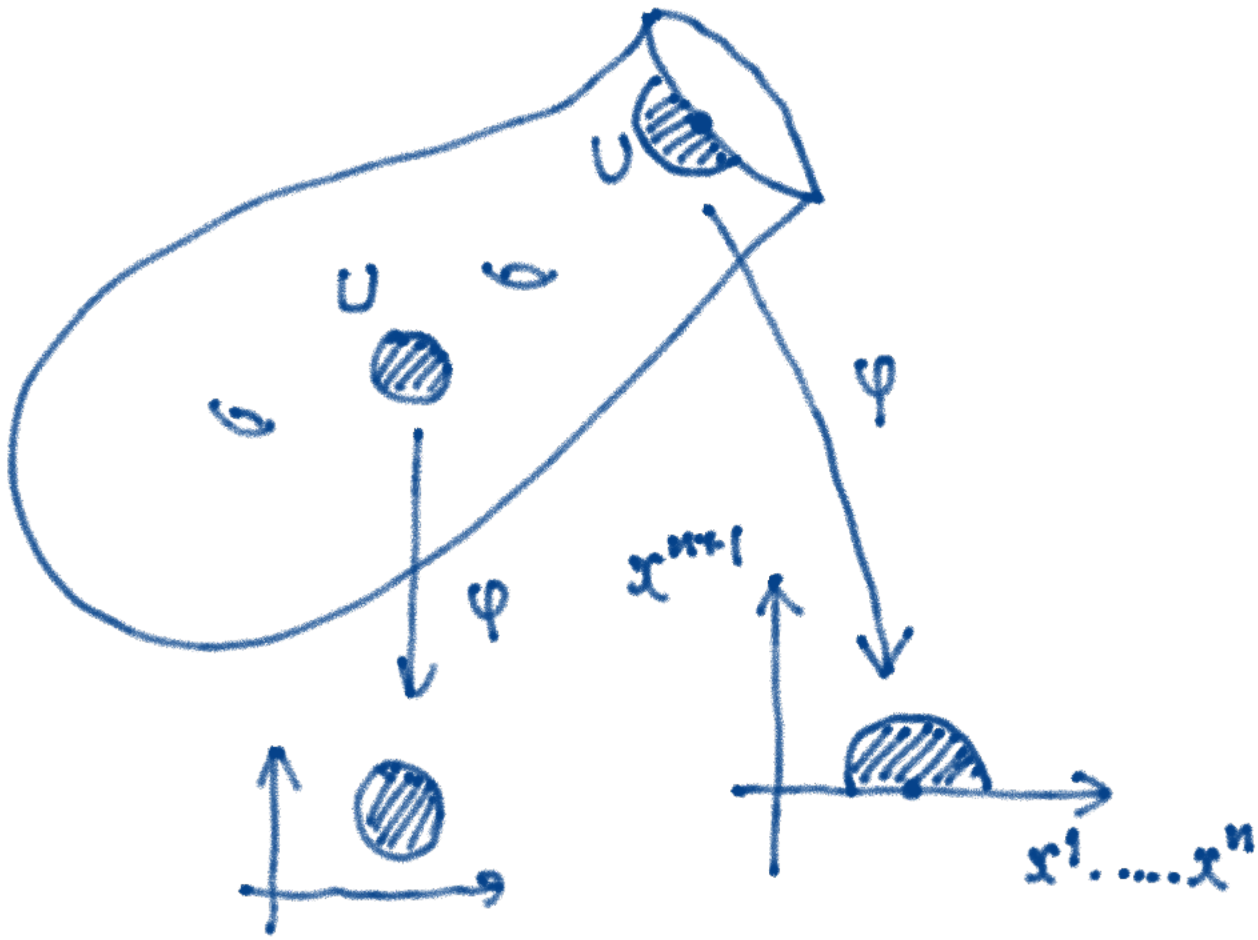
1. Definitions & Examples

1.1 scc AH spaces

(scc = smooth conformally compact)

$\overline{X}$ a smooth compact manifold-with-bdry, $\dim = n+1$



$X := \mathrm{int}(\overline{X})$

$\partial X := \overline{X} \setminus X$

$\overline{X} = X \sqcup \partial X$

Def 1.1 $\rho \in C^\infty(\bar{X})$ **boundary def func**

$\overset{\text{def}}{\Longleftrightarrow}$

& $d\rho$ nowhere vanishing on ∂X

Lem 1.2 ρ bdf. Then,

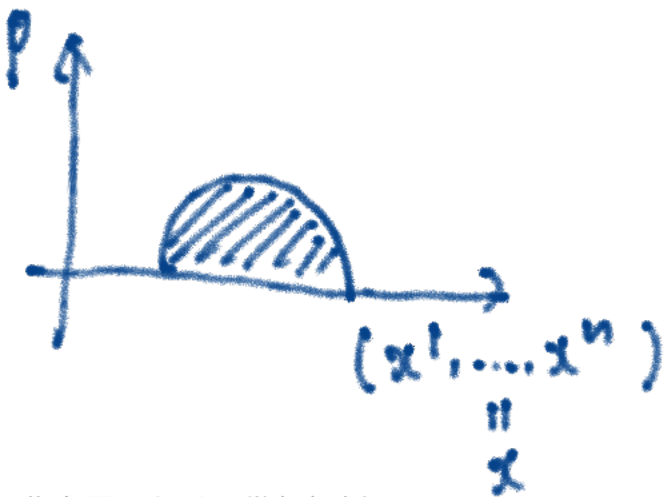
$\hat{\rho} \in C^\infty(\bar{X})$ bdf

$\Longleftrightarrow \quad \hat{\rho} = F\rho, \quad F \in C^\infty(\bar{X}),$

$F > 0$ everywhere.

[Proof] $\Leftarrow$ clear.

$\Rightarrow$ Argue locally.



$$\hat{\rho}(x,\rho) = \int_0^\rho \frac{\partial \hat{\rho}}{\partial \rho}(x,t)\,dt = \rho \underbrace{\int_0^1 \frac{\partial \hat{\rho}}{\partial \rho}(x,\rho s)\,ds}_{F} \qquad \square$$

Def 1.3 (X, g) smooth conf compact

$\overset{\text{def}}{\Longleftrightarrow}$ ρ bdf, $\bar{g} := \rho^2 g$ extends smoothly (C^∞) to $\bar{X}$ as Riem met

Ex. Poincaré ball model of $\mathbb{H}^{n+1}$

$$B^{n+1} \subset \mathbb{R}^{n+1} \qquad g = 4 \cdot \frac{g_{Euc}}{(1-|x|^2)^2}$$

$\rho = 1 - |x|^2$ bdf $\quad \therefore\ g$ scc.

Rem choice of ρ is irrelevant (Lem 1.2).

Def 1.3' (X, g) C^m cc

$\overset{\text{def}}{\Longleftrightarrow}$ g C^m Riem met

$\bar{g} = \rho^2 g$ C^m Riem met on $\bar{X}$

Prop 1.4

(1) C^0 cc $\Longrightarrow$ Metric balls are rel compact

(2) C^2 cc $\Longrightarrow$ If $\bar{g} = \rho^2 g$, then for $p \in \partial X$, as $x \to p$,

$$R_{ijkl}(x) \to -|d\rho|^2_{\bar{g}}(p) \cdot (g_{ik}g_{jl} - g_{il}g_{jk})$$

$x \in X$

in the sense that

$$|(\text{LHS}) - (\text{RHS})|_g \longrightarrow 0.$$

Def 1.5 (X, g) <u>sccAH space</u>

$\overset{\text{def}}{\Longleftrightarrow}$ (X, g) scc & $|d\rho|_{\bar{g}} = 1$ on ∂X

$h = \bar{g}|_{T\partial X}$ $[h]$ <u>conformal infinity</u>

$h \sim \hat{h} \overset{\text{def}}{\Longleftrightarrow} \exists f \in C^\infty(\partial X)$ s.t. $\hat{h} = e^{2f} h$

Toward proof Prop 1.4 (2)

Levi-Civita conn of g

$$\nabla : \Gamma(TX) \times \Gamma(TX) \longrightarrow \Gamma(TX)$$
$$\xi \qquad \eta \qquad \nabla_\xi \eta$$

induces

$$\nabla : \Gamma(TX) \times \Gamma(E) \longrightarrow \Gamma(E)$$
$$\xi \qquad u \qquad \nabla_\xi u$$

for "geometric tensor bundle" E, e.g.,

$$E = (TX)^{\otimes k} \otimes (T^*X)^{\otimes l}$$
$$\Lambda^k T^*X, \quad \mathrm{Sym}^k T^*X,$$
$$\mathrm{Sym}_0^2 T^*X = \coprod_{x \in X} \{ b \in \mathrm{Sym}^2 T_x^* X \mid \mathrm{tr}_{g_x} b = 0 \}$$

Curvature form

$$R \in \Gamma(\Lambda^2 T^*X \otimes \mathrm{End}(TX))$$

$$R(\xi, \eta)\zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta$$

$$\Lambda^2 T^*X \otimes \underbrace{\mathrm{End}(TX)}_{TX \otimes T^*X} = \Lambda^2 T^*X \otimes TX \otimes T^*X$$

Hence also curvature tensor.

g induces $TX \cong T^*X$. Hence R can be regarded as

$$R \in \Gamma(\Lambda^2 T^*X \otimes T^*X \otimes T^*X).$$

★ Local expressions

Take local frame of TX $(X_1, \ldots, X_{n+1})$

dual local frame of T^*X $(\theta^1, \ldots, \theta^{n+1})$

$\theta^i(X_j) = \delta^i{}_j$ Kronecker's delta

Then tensors are sets of coeff functions.

Ex. (1) Vector fields $\xi = \boxed{\sum_i} \xi^i X_i$

Omitted in the sequel

" $\xi = \{\xi^i\}$ " or " vector field ξ^i "

(2) 1-form $\alpha = \alpha_i \theta^i$ " 1-form α_i "

$$\alpha(\xi) = \alpha_i \xi^j \theta^i(X_j) = \alpha_i \xi^j \delta^i{}_j$$

$$= \alpha_i \xi^i$$

Evaluation is contraction

(3) Curvature Tensor

$R_{ij}{}^k{}_l$ if seen as $R \in \Gamma(\Lambda^2 T^*X \otimes TX \otimes T^*X)$

$$(R = R_{ij}{}^k{}_l \, \theta^i \otimes \theta^j \otimes X_k \otimes \theta^l)$$

R_{ijkl} if seen as $R \in \Gamma(\Lambda^2 T^*X \otimes T^*X \otimes T^*X)$

$R_{ijkl} = g_{km} R_{ij}{}^m{}_l$ "lowering an index"

$R_{ij}{}^k{}_l = g^{km} R_{ijml}$ "raising —"

g^{km} components of g^{-1}

(4) Ricci Tensor, scalar curvature

$$R_{ij} := R_{ki}{}^{k}{}_{j} = g^{kl} R_{kilj}$$

Ricci symmetric

$$R := R_i{}^i \quad \left(\text{where } R_i{}^j = g^{jk} R_{ik} \right)$$

$$= R_{j;}{}^{ji}.$$

Einstein $\overset{\text{def}}{\Longleftrightarrow}$ $\exists \lambda \in \mathbb{R}$ s.t. $R_{ij} = \lambda g_{ij}$

(5) Covar differentiation.

$$u \in \Gamma(E) \rightsquigarrow \nabla u \in \Gamma(T^*X \otimes E)$$

$$u_{i_1 \cdots}{}^{j_1 \cdots} \qquad (\nabla u)_{k i_1 \cdots}{}^{j_1 \cdots}$$

$$\| $$

$$\nabla_k u_{i_1 \cdots}{}^{j_1 \cdots}$$

Fact 1.6 (2nd Bianchi identity)

$$\nabla_i R_{jk}{}^l{}_m + \nabla_j R_{ki}{}^l{}_m + \nabla_k R_{ij}{}^l{}_m = 0.$$

Cor 1.7 (Contracted ———)

$$\nabla^j R_{ij} = \frac{1}{2} \nabla_i R.$$

Come back to Prop 1.4 (2).

Lem 1.8 g, $\hat{g} = e^{2f} g$

$$e^{-2f} \boxed{\hat{R}_{ijkl}} = \boxed{R_{ijkl}} - \cdots$$

3rd index lowered by $\hat{g}$

3rd index low – by g

(See Prob 1.2)

$f_{ij} = \nabla_j \nabla_i f$

$f_i = \nabla_i f$

$f^m = g^{ml} f_l$

[Proof of Prop 1.4 (2)]

Apply Lem 1.8 to

$(g, \hat{g}) = (\bar{g}, g)$.

$g = \rho^{-2} \bar{g} \longrightarrow f = -\log \rho$

Note: $|\bar{R}_{ijkl}|_g = O(\rho^4)$. □

1.2 AdS Schwarzschild metric

Example of (scc) AH Einstein metrics:

1. $\mathbb{H}^{n+1}$ & its convex-cocompact quot's
2. AdS Schwarzschild
   (Hawking - Page)
3. Metrics on B^{2n} with
   Berger metrics as conf inf
   (Pedersen, Page-Pope, 松本)
4. Surgery examples of
   M. Anderson

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Schwarzschild metric (1916)

$$g = -V(r)\,dt^2 + V(r)^{-1} dr^2 + r^2 g_{S^{n-1}}$$

$$V(r) = 1 - \frac{2m}{r^{n-2}} \qquad m > 0 \quad \underline{\text{mass}}$$

Ricci-flat Lorentzian metric.

$$r \to \infty \qquad g \to -dt^2 + \underbrace{dr^2 + r^2 g_{S^{n-1}}}_{g_{Euc} \text{ on } \mathbb{R}^n}$$

asymp to Minkowski.

$t \rightsquigarrow \sqrt{-1}\, t$ Wick rotation

$$g = V(r)\,dt^2 + V(r)^{-1} dr^2 + r^2 g_{S^{n-1}}$$

Ricci-flat Riem.

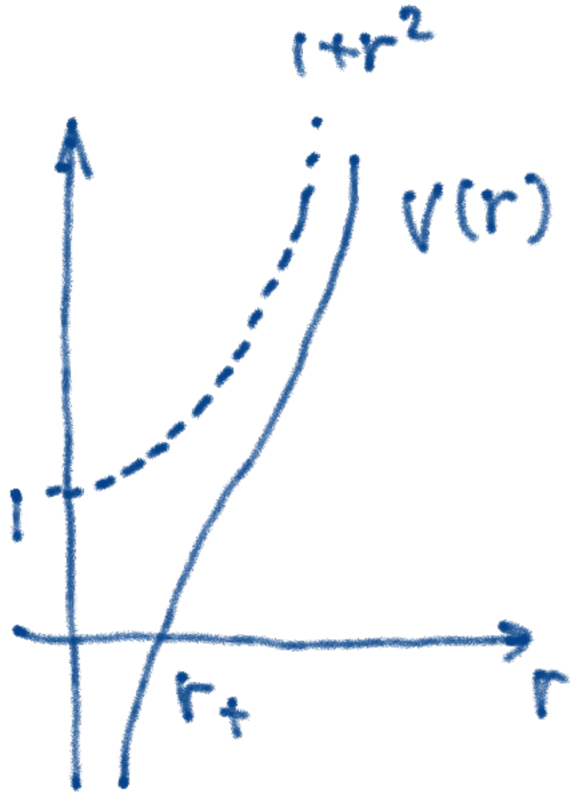
AdS Schwarzschild (wick rotated)

$$g = V(r)\,dt^2 + V(r)^{-1} dr^2 + r^2 g_{S^{n-1}}$$

$$V(r) = 1 + r^2 - \frac{2m}{r^{n-2}} \qquad m > 0.$$

Einstein, $\quad R_{ij} = -n g_{ij}$

Well-defined on $\mathbb{R} \times (r_+, \infty) \times S^{n-1}$



Reinterpretation. g is a met on

$$\mathbb{R}/_{2\pi\beta\mathbb{Z}} \times (r_+, \infty)_r \times S^{n-1}$$

$$\begin{matrix} \wr\wr & & \wr\wr \\ S^1 & & (0,\infty)_s \quad r = r_+ + s^2 \end{matrix}$$

$$\approx (\mathbb{R}^2 \setminus \{0\}) \times S^{n-1}$$

<u>Lem 1.9</u> g extends to a smooth met on $X = \mathbb{R}^2 \times S^{n-1}$ if and only if

$$\beta = \frac{2r_+}{n r_+^2 + n - 2}.$$

[Proof] $\tau = \beta\theta$, $r = r_+ + s^2$.

$$g = \beta^2 V(r_+ + s^2)\, d\theta^2 + V(r_+ + s^2)^{-1} \cdot 4s^2 ds^2 + (r_+ + s^2)^2 g_{S^{n-1}}$$

Suffices to determine when

$$4s^2 V(r_+ + s^2)^{-1} \cdot \left(ds^2 + \boxed{\frac{\beta^2 V(r_+ + s^2)^2}{4s^2}} d\theta^2 \right)$$

is smooth in $\mathbb{R}^2$.

Leading term has to be s^2 $\square$

★ Conformal compactifiability

Put $S^1 \times S^{n-1}$ at infinity.

Introduce C^∞ str by claiming $\rho = r^{-1}$ is a bdf.

$$g = \beta^2 V(\rho^{-1}) d\theta^2 + V(\rho^{-1})^{-1} \cdot \rho^{-4} d\rho^2 + \rho^{-2} g_{S^{n-1}}$$

Recall $V(r) = 1 + r^2 - \dfrac{2m}{r^{n-2}}$.

$$g = \beta^2 \left(1 + \rho^{-2} - 2m\rho^{n-2}\right) d\theta^2 + \frac{1}{1+\rho^{-2} - 2m\rho^{n-2}} \cdot \rho^{-4} d\rho^2 + \rho^{-2} g_{S^{n-1}}$$

$$= \frac{1}{\rho^2}\left(\frac{1}{1+\rho^2 - 2m\rho^n} d\rho^2 + \beta^2 \left(1+\rho^2 - 2m\rho^n\right) d\theta^2 + g_{S^{n-1}} \right).$$

$\therefore$ (X, g) is sccAH with compactification $\overline{X} = (\mathbb{R}^2 \times S^{n-1}) \sqcup (S^1 \times S^{n-1})$

$\approx \overline{B^2} \times S^{n-1}$.

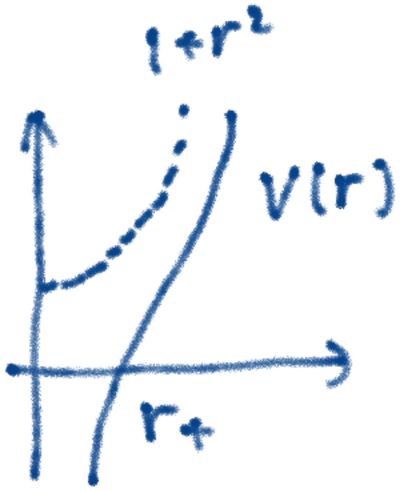
Conf. inf $= [\beta^2 d\theta^2 + g_{S^{n-1}}]$.

Dependence of conf infinity on m
(β)

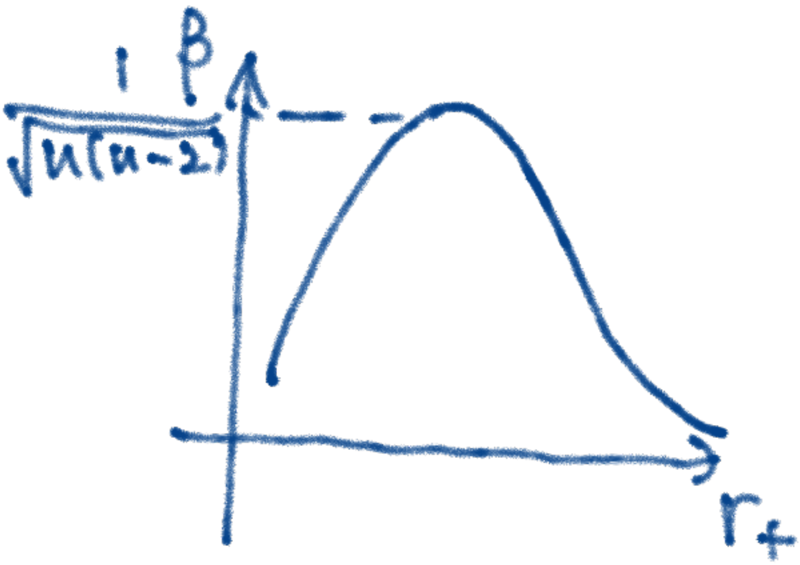
$$m \rightsquigarrow r_+ \rightsquigarrow \beta$$

$$V(r) = 1 + r^2 - \frac{2m}{r^{n-2}}$$

$$V(r_+) = 0$$



r_+ str increasing wrt m



For each $\beta < \frac{1}{\sqrt{n(n-2)}}$, $\exists$ two corresp. r_+'s ; hence two m's.

<u>Prop 1.10</u> $m_1 \neq m_2$

$\Rightarrow$ g_{m_1}, g_{m_2} not isom.

AdS Schwarzschild provides counterex to uniqueness of "AH Einstein filling."