ON THE CANONICAL CONTACT STRUCTURE OF LINKS OF
COMPLEX SURFACE SINGULARITIES

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1. Introduction

In this note we discuss two main topics related with the canonical contact structure of the link of a complex normal surface singularity. The first is their Stein fillability. In section 2 we present the case of cyclic quotient singularities when the problem is totally understood. Section 3 contains some partial results about the more general case of sandwiched singularities. We present these case with the hope that they might serve as models for the general case which is still open.

The second topic captures some invariants like support genus, binding number or norm associated with contact structures. Since all the Milnor open book decompositions (i.e. those which are the Milnor fibrations cut out by analytic germs) support the canonical contact structure, these invariants associated with the canonical contact structure can be studied via the Milnor open book decompositions. In section 4 we transport the problem in the language of the lattices associated with the resolution graphs and we present some precise formulas.

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2. Preliminaries, questions and comments

2.1. The link. One of the objects which connects local complex analytic singularities with low-dimensional topology is the link of isolated singularities. Namely, if \((\mathcal{X}, o)\) is an isolated complex analytic germ of complex dimension \(n\), then for any real analytic map (so-called rug-function) \(\rho : (\mathcal{X}, o) \to [0, \infty)\) with \(\rho^{-1}(0) = \{o\}\), the diffeomorphism type of \(M_{\epsilon} := \rho^{-1}(\epsilon)\ (0 < \epsilon \ll 1)\) is independent of the choice of \(\rho\) and \(\epsilon\). This oriented real \((2n - 1)\)-manifold \(M\) is called the link of \((\mathcal{X}, o)\), it is connected whenever \(n \geq 2\). It determines the topology of \((\mathcal{X}, o)\) completely.

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One of the big challenges of singularity theory is to relate invariants, properties of the analytic structure with the topology of $M$.

If $n = 1$ then $M$ determines only the number of irreducible components of $(\mathcal{X}, o)$, i.e. it does not even differentiate smooth and non-smooth germs. But for $n = 2$ the information in $M$ is extremely rich (hidden, presumably, in its fundamental group). Notice that although the classification of 3-manifolds is an open problem, singularity links are classified: they are the outputs of negative definite plumbings.

The list of relations connecting analytic and topological properties started with the following result of Mumford [30]: $M$ is homeomorphic to $S^3$ (or, equivalently, $\pi_1(M)$ is trivial) if and only if $(\mathcal{X}, o)$ is smooth. This is just the top of the iceberg: e.g., $M$ characterizes rational and elliptic singularities providing even their Hilbert functions (cf. [2, 3, 23, 31]), or, if $M$ is rational homology sphere and the analytic structure of $(\mathcal{X}, o)$ has also some special properties (e.g. weighted homogeneous, or splice-quotient, or Q-Gorenstein) then sheaf-cohomology invariants (e.g. the geometric genus) can also be recovered from $M$ (see e.g. [52, 34, 35, 36, 43, 44, 45, 37, 38, 9, 33]).

We emphasize that most of these results (e.g. Mumford’s theorem) cannot be generalized to higher dimensions. This imposes the necessity to enrich $M$ with some additional structure induced from the analytic structure of $(\mathcal{X}, o)$. One of the candidates is the contact structure on $M$. In fact, this has a crucial role even for surface singularities creating the bridge between contact 3-manifolds and symplectic 4-manifolds/complex surface theory. In the present talk, we will restrict ourselves to the situation $n = 2$.

2.2. Contact structures. One can define a contact structure on a singularity link as follows. One takes a rug function of type $\rho := \sum_{k=1}^{N} |\phi_k|^2$, where $\phi_1, \cdots, \phi_N$ are germs from the maximal ideal of $(\mathcal{X}, o)$ defining an immersion of $\mathcal{X} \setminus \{o\}$ into $\mathbb{C}^N$. In this case $\rho$ is strict pseudosubharmonic, and $\xi := TM \cap J(TM)$, the $J$-invariant subspace of $TM$, defines a contact structure on $M$. Its isotopy class is called the canonical contact structure of $M$ induced by the analytic structure of $(\mathcal{X}, o)$. (Here $J$ is the almost complex structure of $T\mathcal{X}$.)

This definition imposes several questions/problems for any fixed $M$:

1. Classify all the possible contact structures induces by different analytic structures supported by the topological type determined by $M$.

2. How is the subclass from (1) related with the class of all contact structures of $M$?

A partial answer to (1) was given in [10, 11]: all the possible (canonical) contact structures induces by different analytic structures are contactomorphic. Notice again that this fact is not valid in higher dimensions, see Ustilovsky’s work [61].

In fact, we conjecture that all the possible (canonical) contact structures induces by different analytic structures are even isotopic. But, definitely, in order to prove this (or even to state this) one needs first to provide a canonical construction which identifies the link $M$ up to an isotopy (and independently of the
supported analytic structure); the existing plumbing construction identifies \( M \) only up to an orientation preserving diffeomorphism.

Although there is an intense activity in classification of contact structures, and a considerably impressive list of positive results finishing the classification for lens spaces, torus bundles over circles, circle bundles over surfaces, some Seifert manifolds (thanks to the work of Giroux, Etnyre, Honda, Lisca, Stipsicz and others, see e.g. [15, 16, 19, 20, 28] and the references listed therein), Part (2) is mainly open. Recall that a contact structure is either overtwisted or tight, and all overtwisted structure are characterized by the homotopy of their underlying oriented plane field (by a result of Eliashberg). Hence, the difficulty appears in the classification of tight structures. The canonical structure is one of them. Surprisingly, even the finiteness of the possible tight structures on \( M \) is not guaranteed in general (as was shown by Colin, Giroux and Honda): although the number of corresponding homotopy classes of plane fields is finite, the number of isotopy classes of tight contact structures in finite if and only if \( M \) is non-toroidal (i.e. the resolution graph is star-shaped). It is also not clear at all how one can identify in this multitude of structures the canonical one.

In the sequel the canonical contact structure will be denoted by \((M, \xi_{\text{can}})\).

2.3. Fillings of \((M, \xi_{\text{can}})\). Although in the literature there are many different versions of fillability (holomorphic, Stein, strong/weak symplectic), here we will deal only with Stein one: any Stein manifold whose contact boundary is contactomorphic to \((M, \xi_{\text{can}})\) is a Stein filling of \((M, \xi_{\text{can}})\). For a singularity link, the following questions/problems are natural:

1. Is any \((M, \xi_{\text{can}})\) Stein fillable?
2. Classify all the Stein fillings of \((M, \xi_{\text{can}})\).
3. Determine all the Stein fillings ‘coming from singularity theory’.

The answer to (1) is yes: Consider the minimal resolution of \((X, o)\). It is a holomorphic, non-Stein filling of \((M, \xi_{\text{can}})\), but by a theorem of Bogomolov and de Oliveira [8] this holomorphic structure can be deformed into a Stein one. This construction already provides an example for (3); another one is given by the Milnor fibers of the smoothings of different analytic realizations \((X, o)\) of the topological type fixed by \( M \) (if there are any). More precisely, the existence (and uniqueness) of the miniversal deformation of isolated singularities is guaranteed by results of Schlessinger [55] and Grauert [18]. In general, its base space has many irreducible components. A component is called smoothing if the generic fiber over it (the so-called Milnor fiber) is smooth. In general, different analytic structures might have different smoothings; or for a fixed \((X, o)\), different smoothing components might produce diffeomorphic Milnor fibers. It might also happen that smoothing does not exist at all (for the last two situation see e.g. the case of some simple elliptic singularities [25, 46]).

In fact, in the literature basically only these two constructions are present regarding (3); it is high desire to find some other general constructions too. We notice that rational singularities are always smoothable, moreover, the Milnor
fiber of one of the smoothing component (the Artin component) is diffeomorphic with the space of the minimal resolution.

The list regarding part (2) starts with a result of Eliashberg [13] showing that $(S^4, \xi_{can})$ has only one filling, namely the ball. For links of simple and simple elliptic singularities the classification was finished by Ohta and Ono [46, 47]. Moreover, in all these cases all possible Stein fillings are provided by the minimal resolution or Milnor fibers. This parallelism sometimes is really striking. E.g. for simple elliptic singularities with degree $k > 0$, Ohta and Ono proved that the existence of a Stein filling with vanishing first Chern class imposes $k \leq 9$. This can be compared with the fact that in the case of a Milnor fiber the Chern class is vanishing (by [56]) and the smoothability condition is the same $k \leq 9$ (cf. [51]).

Fillings of links of quotient surface singularities were classified by Bhupal and Ono [7], and of lens spaces $L(p, q)$ by Lisca [26, 27] (as a generalization a result of McDuff [29] valid for the spaces $L(p, 1)$, for all $p \geq 2$). We will return to Lisca’s result in the next section showing that Lisca’s list agrees perfectly with the list of Milnor fibers (or, with the smoothing/all deformation components).

We would like to notice that this phenomenon, namely that all the Stein fillings are obtained either by minimal resolution or Milnor fibers – at some point of the singularity complexity – might stop. This emphasizes the importance of the research in direction (3) even more.

Moreover, in general, the finiteness of the Stein fillings might fail too: Ohta and Ono produced on some singularity links infinitely many symplectic fillings [48], also Ozbagci and Stipsicz in [50], and independently Smith in [57], have shown that certain contact structures have infinitely many Stein fillings (although they are not singularity links, one expects that at some moment similar fact will be established for some singularity links as well). For similar result see also the recent article [1] too.

2.4. $(M, \xi_{can})$ and open books. By a result of Giroux [17], there is a one-to-one correspondence between open book decompositions of $M$ (up to stabilization) and contact structures on $M$ (up to isotopy).

The link $M$ of a normal surface singularity $(\mathcal{X}, o)$ admits a natural family of open book decompositions, the so-called Milnor open books. They are cut out by analytic germs $f : (\mathcal{X}, o) \to (\mathbb{C}, 0)$ which define isolated singularities. If $L_f \subset M$ denotes the (transversal) intersection of $f^{-1}(0)$ with $M$, then the Milnor fibration of $f$ defines an open book decomposition of $M$ with binding $L_f$. By [11], all the Milnor open book decompositions support the same contact structure on $M$, namely the canonical contact structure $(M, \xi_{can})$.

The last section contains a more detailed discussion about Milnor open books and their invariants.

3. Fillings. The case of cyclic quotient singularities

3.1. In this section we assume that $(\mathcal{X}, o)$ is a cyclic quotient (or Hirzebruch-Jung) singularity. More precisely, for any coprime integers $p, q$ with $p > q > 0$, let $(\mathcal{X}_{p,q}, o)$ be the germ of the quotient $\mathcal{X}_{p,q} \subset \mathbb{C}^2$ by the action $(x, y) \to (\xi x, \xi^q y)$
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of the cyclic group \( \{ \xi \in \mathbb{C} : \xi^n = 1 \} \simeq \mathbb{Z}/p\mathbb{Z} \). Its oriented link is the oriented lens space \( L(p, q) \). \( \mathcal{X}_{pq, o} \) is taut (i.e. the topological type supports only one analytic structure), and is rational (i.e. its geometric genus is zero).

For cyclic quotient singularities the classification of the Stein fillings of \((M, \xi_{can})\) and their connection with Milnor fibers and Milnor open books is completely understood. In this section we review this problem (the presentation follows [39]). For more about Milnor open books see also the last section.

There is an intimate connection of the theory of cyclic quotients with continued fractions. The next paragraph contains some needed notation about this.

3.2. Notations. If \( x = (x_1, \ldots, x_n) \) are variables, the Hirzebruch-Jung continued fraction \([x_1, \ldots, x_n]\) can be defined by induction on \( n \) through the formulæ:

\[
[x_1] = x_1 \quad \text{and} \quad [x_1, \ldots, x_n] = x_1 - 1/[x_2, \ldots, x_n] \quad \text{for} \quad n \geq 2.
\]

One shows that:

\[
[x_1, \ldots, x_n] = \frac{Z_n(x_1, \ldots, x_n)}{Z_{n-1}(x_2, \ldots, x_n)},
\]

where the polynomials \( Z_n \in \mathbb{Z}[x_1, \ldots, x_n] \) satisfy the inductive formulæ:

\[
(3.2.1) \quad Z_n(x_1, \ldots, x_n) = x_1 \cdot Z_{n-1}(x_2, \ldots, x_n) - Z_{n-2}(x_3, \ldots, x_n) \quad \text{for all} \quad n \geq 1,
\]

with \( Z_{-1} \equiv 0, Z_0 \equiv 1 \) and \( Z_1(x) = x \). In fact, \( Z_n(x) \) equals the determinant of the \( n \times n \)-matrix \( M(x) \), whose entries are \( M_{ij} = x_i, M_{i,j} = -1 \) if \(|i-j| = 1\), \( M_{i,j} = 0 \) otherwise. Hence, besides (3.2.1), they satisfy many ‘determinantal relations’ too; e.g. \( Z_n(x_1, \ldots, x_n) = Z_n(x_n, \ldots, x_1) \). Following [49], \( x \in \mathbb{N}^n \) is admissible if the matrix \( M(x) \) is positive semi-definite of rank \( \geq n - 1 \). Denote by \( \text{adm}(\mathbb{N}^n) \) the set of admissible \( n \)-tuples.

If \( x \) is admissible and \( n > 1 \), then each \( x_i > 0 \). Moreover, if \([x_1, \ldots, x_n]\) is admissible then \([x_n, \ldots, x_1]\) is admissible too. For any \( r \geq 1 \), denote:

\[
K_r := \{ k = (k_1, \ldots, k_r) \in \text{adm}(\mathbb{N}^r) \mid [k_1, \ldots, k_r] = 0 \}.
\]

For \( k = (k_1, \ldots, k_r) \in K_r \), set \( k' = (k_r, \ldots, k_1) \in K_r \).

For \( p, q \) as above and HJ-expansion \( \frac{p}{p,q} = [a_1, \ldots, a_r] \), set:

\[
K_r(\frac{p}{p,q}) = K_r(a) := \{ k \in K_r \mid k \leq a \} \subset K_r.
\]

Here, \( k \leq a \) means that \( k_i \leq a_i \) for all \( i \).

3.3. Lisca’s Conjecture. We recall briefly the classification of fillings of lens spaces established by Lisca. He provides by surgery diagrams a list of compact oriented 4-manifolds \( W_{p,q}(k) \) with boundary \( L(p, q) \). They are parametrized by the set \( K_r(\frac{p}{p,q}) \) of sequences of integers \( k \in \mathbb{N}^r \) (cf. (3.2)). He showed that each manifold \( W_{p,q}(k) \) admits a structure of Stein surface, filling \( (L(p, q), \xi_{can}) \), and that any symplectic filling of \( (L(p, q), \xi_{can}) \) is orientation-preserving diffeomorphic to a manifold obtained from one of the \( W_{p,q}(k) \) by a composition of blow-ups. In general, the oriented diffeomorphism type of the boundary and the parameter \( k \) does not determine uniquely the (orientation-preserving) diffeomorphism type of the fillings: for some pairs the corresponding types might coincide (see subsection (3.8) for this ‘ambiguity’).
Lisca also noted that, following the works of Christophersen [12] and Stevens [59], $K_r(\frac{p}{p-q})$ parametrizes also the irreducible components of the reduced miniversal base space of deformations of the cyclic quotient singularity $X_{p,q}$. Since each component of the miniversal space is in this case a smoothing component, Lisca conjectured in [27, page 768] that the Milnor fiber of the irreducible component of the reduced miniversal base space of the cyclic quotient singularity $X_{p,q}$, parametrized in [59] by $k \in K_r(\frac{p}{p-q})$ is diffeomorphic to $W_{p,q}(k)$.

On the other hand, in [21], de Jong and van Straten studied by an approach completely different from Christophersen and Stevens the deformation theory of cyclic quotient singularities (as a particular case of sandwiched singularities). They also parametrized the Milnor fibers of $X_{p,q}$ using the elements of the set $K_r(\frac{p}{p-q})$. Therefore, one can formulate the previous conjecture for their parametrization as well.

### 3.4. The answers to the (improved) conjecture.

[39] answers positively these questions. Its main results are the following:

1. One defines an additional structure associated with any (non-necessarily oriented) lens space: the ‘order’. Its meaning is the following: geometrically it is a (total) order of the two solid tori separated by the (unique) splitting torus of the lens space; in plumbing language, it is an order of the two ends of the plumbing graph (provided that this graph has at least two vertices). Then one shows that the oriented diffeomorphism type and the order of the boundary, together with the parameter $k$ determines uniquely the filling.

2. One endows in a natural way all the boundaries of the spaces involved (Lisca’s fillings $W_{p,q}(k)$, Christophersen-Stevens’ Milnor fibers $F_{p,q}(k)$, and de Jong-van Straten’s Milnor fibers $F'_{p,q}(k)$) with orders (this extra structure is denoted by $*$). Then one proves that all these spaces are connected by orientation diffeomorphisms which preserve the order of their boundaries: $W_{p,q}(k)^* \simeq F_{p,q}(k)^* \simeq F'_{p,q}(k)^*$. This is an even stronger statement than the result expected by Lisca’s conjecture since it eliminates the ambiguities present in Lisca’s classification.

3. In fact, [39] even provides a fourth description of the Milnor fibers constructed by a minimal sequence of blow ups of the projective plane which eliminates the indeterminacies of a rational function which depends on $k$. This is in the spirit of Balke’s work [5].

4. As a byproduct it follows that both Christophersen-Stevens and de Jong-van Straten parametrized the components of the miniversal base space in the same way.

5. Moreover, one obtains that the Milnor fibers corresponding to the various irreducible components of the miniversal space of deformations of $X_{p,q}$ are pairwise non-diffeomorphic by orientation-preserving diffeomorphisms whose restrictions to the boundaries preserve the order.
In the sequel we will not say more about Lisca’s construction, instead, we will describe briefly the Milnor fibers associated with the different smoothing components, with a special emphasis on the construction (3.4)(3). This will be compatible with the description of Christophersen and Stevens on the structure of the reduced miniversal base space of cyclic quotients [6, 12, 60] (cf. also with [4]). (The de Jong-van Straten construction will be reviewed in the next section.)

3.5. First we concentrate on $\mathcal{X}_{p,q}$. It can be embedded into $\mathbb{C}^{r+2}$ by some regular functions $z_0, \ldots, z_{r+1}$. Some of the equations of the embedding are

\[(3.5.1) \quad z_{i-1} z_{i+1} - z_i^{a_i} = 0 \quad \text{for all } i \in \{1, \ldots, r\}.\]

Using equations (3.5.1) and induction, one shows that the restriction of each $z_i$ to $\mathcal{X}_{p,q}$ is a rational function in $(z_0, z_1)$ of the form

$$z_i = z_1^{Z_{i-1}(a_1, \ldots, a_{i-1})} z_0^{-Z_{i-2}(a_2, \ldots, a_{i-1})} \quad \text{for } i \in \{1, \ldots, r+1\}.$$  

In particular, the restriction $pr_{01}$ of the projection $(z_0, \ldots, z_{r+1}) \mapsto (z_0, z_1)$ to $\mathcal{X}_{p,q}$ is birational, i.e. it is a ‘sandwiched representation’ of $\mathcal{X}_{p,q}$ (cf. next section for the terminology). (Here a comment is in order. Recall that $\mathcal{X}_{p,q}$ is minimal, hence, by a general construction, it admits a canonical sandwiched representation $pr : \mathcal{X}_{p,q} \to \mathbb{C}^2$, see e.g. (4.1). We wish to emphasize that the two birational maps $pr_{01}$ and $pr$ are different capturing two different geometrical aspects about $\mathcal{X}_{p,q}$.)

The equations of $\mathcal{X}_{p,q}$ are weighted homogeneous, however the weights $w_i := w(z_i)$ are not unique. With the choice $w_0 = w_1 = 1$ one has $w_i = Z_{i-1}(a_1, \ldots, a_{i-1}) - Z_{i-2}(a_2, \ldots, a_{i-1})$ for all $i \geq 1$, and $1 = w_0 = w_1 \leq w_2 \leq \cdots \leq w_{r+1} = q$.

3.6. The deformations. Next, we fix $k \in K_r(q)$, and we denote by $S_{k}^{CS}$ the corresponding deformation component (as it is described by Christophersen and Stevens). Then, we consider a special 1-parameter deformation with equations $\mathcal{E}_k^t$ of $\mathcal{X}_{p,q}$. This deformation is determined by the deformed equations of (3.5.1) (cf. [4], [59, (2.2)]). These are:

\[(3.6.1) \quad z_{i-1} z_{i+1} = z_i^{a_i} + t \cdot z_i^{k_i} \quad \text{for all } i \in \{1, \ldots, r\},\]

where $t \in \mathbb{C}$. Let $\mathcal{X}_k^t$ be the affine space determined by the equations $\mathcal{E}_k^t$ in $\mathbb{C}^{r+2}$. One proves that the deformation $t \mapsto \mathcal{X}_k^t$ has negative weight and is a smoothing belonging to the component $S_{k}^{CS}$. Hence, $\mathcal{X}_k^t$ is a smooth affine space for $t \neq 0$ (and by [62, (2.2)]) it is diffeomorphic to the Milnor fiber of $S_{k}^{CS}$.

3.7. $\mathcal{X}_k^t$ as a rational surface. Similarly as for $\mathcal{X}_{p,q}$, using (3.6.1), on $\mathcal{X}_k^t$ all the coordinates $z_i$ can be expressed as rational functions in $(z_0, z_1)$. Indeed, for each $i \in \{1, \ldots, r+1\}$, on $\mathcal{X}_k^t$ one has:

$$z_i = z_0^{-Z_{i-2}(a_2, \ldots, a_{i-1})} P_i$$

for some $P_i \in \mathbb{Z}[t, z_0, z_1]$. The polynomials $P_i$ satisfy the inductive relations:

\[(3.7.1) \quad P_{i-1} \cdot P_{i+1} = P_i^{a_i} + t P_i^{k_i} \cdot z_0^{-Z_{i-2}(a_2, \ldots, a_{i-1})}\]

with $P_1 = z_1$ and with the convention $P_0 = 1$.  

Consider the application \( \pi : \mathbb{C}^2 \setminus \{ z_0 = 0 \} \rightarrow \mathcal{X}_k^L \) given by

\[
(z_0, z_1) \mapsto (z_0, z_1, P_2, \ldots, z_0^{-a_{i-2}(a_2, \ldots, a_i)} P_i, \ldots, z_0^{-(p-q)} P_{r+1}) \in \mathbb{C}^{r+2}.
\]

(3.7.2)

We are interested in the birational map \( \mathbb{C}^2 \rightarrow \mathcal{X}_k^L \), still denoted by \( \pi \), and its extension \( \tilde{\pi} : \mathbb{P}^2 \rightarrow \mathcal{X}^L_k \), where \( \mathcal{X}^L_k \) is the closure of \( \mathcal{X}^L_k \) in \( \mathbb{P}^{r+2} \).

Let \( \rho_k : B^r \mathbb{P}^2 \rightarrow \mathcal{X}^L_k \) be the minimal sequence of blow ups such that \( \tilde{\pi} \circ \rho_k \) extends to a regular map \( B^r \mathbb{P}^2 \rightarrow \mathcal{X}^L_k \). Let \( L_\infty \subset \mathbb{P}^2 \) be the line at infinity and by \( L_0 \) the closure in \( \mathbb{P}^2 \) of \( \{ z_0 = 0 \} \). We use the same notations for their strict transforms via blow ups of \( \mathbb{P}^2 \). Since the projection \( pr : \mathbb{P}^2 \rightarrow \mathbb{C}^2 \) is regular and \( pr \circ \pi \) is the identity, one gets that \( \tilde{\pi} \circ \rho_k \) sends \( L_0 \) and the total transform of \( L_\infty \) in the curve at infinity \( \mathcal{X}^L_k \setminus \mathcal{X}^L_k \).

Therefore, let us modify \( \rho_k \) into \( \rho' \) : \( B^r \mathbb{P}^2 \rightarrow \mathbb{P}^2 \), the minimal sequence of blow ups which resolve the indeterminacies of \( \tilde{\pi} \) sitting in \( \mathbb{C}^2 \) (hence \( \rho_k \) and \( \rho_k \) over \( \mathbb{C}^2 \) coincide). Denote by \( E_\pi \) its exceptional curve and by \( C_\pi \) the union of those irreducible components of \( E_\pi \) which are sent to \( C^\infty_k \). Set \( BC^2 := B^r \mathbb{P}^2 \setminus L_\infty \).

Summing up all the above discussions, one obtains:

**Theorem 3.7.3.** The restriction of \( \tilde{\pi} \circ \rho_k \) induces an isomorphism \( BC^2 \setminus (L_0 \cup C_\pi) \rightarrow \mathcal{X}^L_k \). In particular, the Milnor fiber can be realized as the complement of the projective curve \( L_\infty \cup L_0 \cup C_\pi \) in \( B^r \mathbb{P}^2 \).

The point is that the indeterminacies of \( \tilde{\pi} \) above \( \mathbb{C}^2 \), hence the modification \( \rho_k \) too, can be described precisely. This leads to the following description of the Milnor fiber.

**Corollary 3.7.4.** Consider the lines \( L_\infty \) and \( L_0 \) on \( \mathbb{P}^2 \) as above. Blow up \( r - 1 + \sum_{i=1}^r (a_{i-2}(a_2, \ldots, a_i)) \) infinitely closed points of \( L_0 \) in order to get the dual graph in Figure 1 of the configuration of the total transform of \( L_\infty \cup L_0 \) (this procedure topologically is unique, and its existence is guaranteed by the fact that \( k \in K_\mathbb{C}(\mathbb{C}) \)). Denote the space obtained by this modification by \( B^r \mathbb{P}^2 \). Then the Milnor fiber \( \mathcal{X}_k^L \) of \( S^C_k \) is diffeomorphic to \( B^r \mathbb{P}^2 \setminus (\bigcup_{j=0}^r V_j) \).

Moreover, let \( T \) be a small open tubular neighbourhood of \( \bigcup_{j=0}^r V_j \), and set \( F_{p,q}(k) = B^r \mathbb{P}^2 \setminus T \). Then \( F_{p,q}(k) \) is a representative of the Milnor fiber of \( S^C_k \) as a manifold with boundary whose boundary is \( L(p, q) \).

Furthermore, the marking \( \{ V_j \} \), as in the Figure 1, defines on the boundary of \( F_{p,q}(k) \) an order; denote this supplemented space by \( F_{p,q}(k)^* \). Then its ordered boundary is \( L(p, q)^* \) endowed with the preferred order.
Remark 3.7.5. In fact, $\rho_k$ serves also as the minimal modification which eliminates the indeterminacy of the last component of $\pi$ from (3.7.2), namely of the rational function $z_{r+1} = P_{r+1}/z_0^{-p_q}$, in particular, we find the following alternative description of the Milnor fiber $F_{p,q}(k)$:

For each $k \in K_v(a)$, define the polynomial $P_{r+1}$ via the inductive system (3.7.1). Let $\rho_k: B\mathbb{P}^2 \to \mathbb{P}^2$ be the minimal modification of $\mathbb{P}^2$ which eliminates the indeterminacy points of $P_{r+1}/z_0^{-p_q}$ sitting in $\mathbb{C}^2$. Then the dual graph of the total transform of $L_\infty \cup L_0$ has the form indicated in Figure 1, and $F_{p,q}(k)$ is orientation-preserving diffeomorphic to $B\mathbb{P}^2 \setminus (\cup_{j=0}^r V_j)$.

3.8. An identification criterion of the Milnor fibers. The next criterion generalizes Lisca’s criterion [27, §7] to recognize the fillings of $(L(p, q), \xi_{can})$, e.g. one of the Milnor fibers considered above. Set $V$ for the closed 4-manifold obtained by gluing $F$ and $\Pi(a)$ via an orientation-preserving diffeomorphism $\phi: \partial F \to \partial(-\Pi(a))$ of their boundaries. Denote by $\{s_i\}_{1 \leq i \leq r}$ the classes of 2-spheres $\{S_i\}_{1 \leq i \leq r}$ in $H_2(\Pi(a))$ (listed in the same order as $\{a_i\}_{1 \leq i \leq r}$, and also their images via the monomorphism $H_2(\Pi(a)) \to H_2(V)$ induced by the inclusion.

Lisca’s criterion (implied also by the results of (3.7)) reads as follows:

Proposition 3.8.1. For all $i \in \{1, \ldots, r\}$ one has

$$\#\{e \in H_2(V) \mid e^2 = -1, s_i \cdot e \neq 0, s_j \cdot e = 0 \text{ for all } j \neq i\} = 2(a_i - k_i)$$

for some $k \in K_v(a)$. In this way one gets the pair $(a, k)$ and $F$ is orientation-preserving diffeomorphic to $F_{p,q}(k)$ ($\simeq F_{p,q}(k')$).

One verifies that the above criterion is independent of the choice of the diffeomorphism $\phi$. In fact, even the diffeomorphism type of the manifold $V$ is independent of the choice of $\phi$.

Notice that $\{S_i\}_{1 \leq i \leq r}$ and $\{S_{r+1}\}_{1 \leq i \leq r}$ cannot be distinguished, hence the above algorithm does not differentiate $(a, k)$ from $(a', k')$, or $F_{p,q}(k)$ from $F_{p,q}(k')$. On
the other hand, these are the only ambiguities. (In fact, if \( r = 1 \), or even of \( r > 1 \) but \( q \) and \( k \) are symmetric, then there is no ambiguity, since \((p, q, k) = (p, q', k')\).

Using the notion of order of the boundaries, one can eliminate the above ambiguity. Notice that any diffeomorphism \( F_{p,q}(k) \rightarrow F_{p,q'}(k') \) (whenever \((p, q, k) \neq (p, q', k')\)) does not preserve any fixed order of the boundary. One has:

**Theorem 3.8.2.** All the spaces \( F_{p,q}(k) \) are different, hence their boundaries \( L(p, q)^* \) and \( k \in K_r(q) \) determine uniquely all the Milnor fibers up to orientation-preserving diffeomorphisms which preserve the order of the boundary.

In order to prove this, the criterion (3.8.1) is modified as follows. Let \( F^* \) be a Stein filling of \((L(p, q), \xi_{can})\) with an order on its boundary. Consider \( \Pi(q)^* \) with its preferred order (provided a well-determined order of the \( s_i \)'s, cf. [39]). Construct \( V \) as in (3.8.1), and consider the two pairs \((q, k)\) and \((q', k')\) provided (but undecided) by (3.8.1).

**Proposition 3.8.3.** If \( \phi \) preserves (resp. reverses) the orders of the boundary then \( F^* \) is orientation and order preserving diffeomorphic to \( F_{p,q}(k)^* \) (resp. to \( F_{p,q'}(k')^* \)).

4. Fillings. The case of sandwiched singularities

4.1. Basic facts about sandwiched singularities. A normal surface singularity \((\mathcal{X}, o)\) is called sandwiched if it is analytically isomorphic to a germ of algebraic surface which admits a birational map \( \mathcal{X} \rightarrow \mathbb{C}^2 \) (see [58]). These singularities are rational, and the exceptional curve of their minimal resolution has normal crossings, and the associated weighted graph is a negative-definite tree of rational curves. They are characterized (like the rational singularities) by their dual resolution graphs: A graphs \( \Gamma \) is sandwiched if by gluing (via new edges) to some of the vertices of \( \Gamma \) new rational vertices with self-intersections \(-1\), one may obtain a ‘smooth graph’ (i.e. the dual tree of a configuration of \( \mathbb{P}^1 \)'s which blows down to a smooth point). In general, the way to add such \((-1)\)-vertices is not unique. The contraction of the smooth configuration of curves provides a ‘sandwiched representation’ \( \mathcal{X} \rightarrow \mathbb{C}^2 \) (the defining birational map).

A special family of sandwiched singularities consists of the minimal singularities, those characterized by \((E, E_i) \leq 0 \) (for all \( i \)), where \( E \) is the reduced exceptional curve of a (any) resolution, and \( E_i \) are its irreducible components. In this case there is a ‘canonical’ way to put the \((-1)\)-vertices: \(-(E, E_i)\) on each vertex except in one place where one puts one less.

In [21] de Jong and van Straten related the theory of sandwiched surface singularities with decorated plane curve singularities.

Consider a reduced germ of plane curve \((C, 0) \subset (\mathbb{C}^2, 0)\) with numbered branches (irreducible components) \( \{C_i\}_{1 \leq i \leq r} \). The multiplicity sequence associated with \( C_i \) is the sequence of multiplicities on the successive strict transforms of \( C_i \), starting from \( C_i \) itself and not counting the last strict transform. The total multiplicity \( m(i) \) of \( C_i \) with respect to \( C \) is the sum of the sequence of multiplicities of \( C_i \).
A decorated germ of plane curve is a weighted germ \((C, l)\) such that \(l = (l_i)_{1 \leq i \leq r} \in (\mathbb{N}^*)^r\) and \(l_i \geq m(i)\) for all \(i \in \{1, \ldots, r\}\).

Starting from a decorated germ, one can blow up iteratively points infinitely near 0 on the strict transforms of \(C\), such that the sum of multiplicities of the strict transform of \(C_i\) at such points is exactly \(l_i\). Such a composition of blow-ups is determined canonically by \((C, l)\). If \(l_i\) is sufficiently large (in general, larger than \(m(i)\)) then the union of the exceptional components which do not meet the strict transform of \(C\) form a connected configuration of (compact) curves \(E(C, l)\). After the contraction of \(E(C, l)\), one gets a sandwiched singularity \(X(C, l)\), determined uniquely by \((C, l)\) (for details see [21]). This follows from the fact that the collection of the irreducible components of the exceptional curve \(E\) not contained in \(E(C, l)\) are exactly the \((-1)\) curves involved in the definition of sandwiched singularities. Conversely, for any sandwiched singularity \(X\) one can find \((C, l)\) such that \(X\) can be represented as \(X(C, l)\).

4.2. Markings of the links [40]. By a result of Neumann [42], the information codified in the link of a surface singularity up to an orientation-preserving homeomorphisms and in the weighted dual graph of the minimal good resolution are equivalent. In [53, Theorems 9.1 and 9.7] this is generalized as follows. From the resolution, the abstract link inherits a plumbing structure, that is, a family of pairwise disjoint embedded tori whose complement is fibered by circles, and such that on each torus the intersection number of the fibers from each side is \(\pm 1\). (The tori correspond to the edges of the dual graph and the connected components of their complement – the “pieces of \(M\)” – correspond to the ‘un-numbered vertices’.) Then, by [53], the plumbing structure corresponding to the minimal normal crossings resolution is determined up to an isotopy by the oriented link.

Consider again \(X(C, l)\) associated with some \((C, l)\), and let \(M\) be its link. Using the notations of the previous subsection, write \(E = E(C, l) + \sum_{i=1}^r E_i\), where \((E_i)_{1 \leq i \leq r}\) is numbered such that \(E_i\) is the unique irreducible component which intersects the strict transform of \(C_i\). Denote by \(F_i\) the unique irreducible component of \(E(C, l)\) which intersects \(E_i\). To \(F_i\) corresponds a well-defined “piece” of \(M\). In this way, one gets a map from the set \(\{1, \ldots, r\}\) to the set of pieces of \(M\).

Definition 4.2.1. A map from \(\{1, \ldots, r\}\) (the index set of the numbered branches of \(C\)) to the set of “pieces” of \(M\) obtained as above is called a marking of \(M\).

Hence, each realization of \(M\) as the link of some \(X(C, l)\), where \((C, l)\) is a decorated curve with numbered branches, provides a well-defined marking of \(M\).

4.3. Deformation of sandwiched singularities after de Jong and van Straten. The point is that the above correspondence between sandwiched singularities and decorated plane curves extends to their deformation theory as well.

The total multiplicity of \(C_i\) with respect to \(C\) may be encoded also as the unique subscheme of length \(m(i)\) supported on the preimage of 0 on the normalization of \(C_i\). This allows to define the total multiplicity scheme \(m(C)\) of any reduced curve contained in a smooth complex surface, as the union of the total multiplicity schemes of all its germs. Given a smooth complex analytic surface \(\Sigma\), a pair \((C, l)\)
consisting of a reduced curve $C \hookrightarrow \Sigma$ and a subscheme $l$ of the normalization $\tilde{C}$ of $C$ is called a decorated curve if $m(C)$ is a subscheme of $l$ ([21, (4.1)]). The deformations of $(C, l)$ considered by de Jong and van Straten are:

**Definition 4.3.1.** (i) [21, p. 476] A 1-parameter deformation of a decorated curve $(C, l)$ over a germ of smooth curve $(S, 0)$ consists of:

1. a $\delta$-constant deformation $C_S \to S$ of $C$;
2. a flat deformation $l_S \subset \tilde{C}_S = \tilde{C} \times S$ of the scheme $l$, such that:
3. $m_S \subset l_S$, where the relative total multiplicity scheme $m_S$ of $\tilde{C}_S = C_S$ is defined as the closure $\bigcup_{s \in S \setminus 0} m(C_s)$.

(ii) A 1-parameter deformation $(C_S, l_S)$ is called a picture deformation if for generic $s \neq 0$ the divisor $l_s$ is reduced.

The singularities of $C_{s \neq 0}$ are only ordinary multiple points. Hence, it is easy to draw a real picture of a deformed curve, which motivates the terminology.

**Theorem 4.3.2.** [21, (4.4)] All the 1-parameter deformations of $\mathcal{X}(C, l)$ are obtained by 1-parameter deformations of the decorated germ $(C, l)$. Moreover, picture deformations provide all the smoothings of $\mathcal{X}(C, l)$.

Next, let us fix a decorated germ $(C, l)$ and one of its picture deformations $(C_S, l_S)$. Fix a closed Milnor ball $B$ for the germ $(C, 0)$. For $s \neq 0$ sufficiently small, $C_s$ will have a representative in $B$, denoted by $D$, which meets $\partial B$ transversally. It is a union of immersed discs $\{D_i\}_{1 \leq i \leq r}$ canonically oriented by their complex structures (and whose set of indices correspond canonically to those of $\{C_i\}_{1 \leq i \leq r}$). The singularities of $D$ consist of ordinary multiple points.

Denote by $\{P_j\}_{1 \leq j \leq n}$ the set of images in $B$ of the points in the support of $l_s$. It is a finite set of points which contains the singular set of $D$ (because $m_s \subset l_s$ for $s \neq 0$), but it might contain some other ‘free’ points as well. There is a priori no preferred choice of their ordering. (Hence, the matrix introduced next is well-defined only up to permutations of columns.)

The Milnor fiber of the smoothing associated with the fixed picture deformation has the following description. Let $\beta : (\tilde{B}, D) \to (B, D)$ be the simultaneous blow-up of the points $P_j$ of $D$. Here $\tilde{D} := \bigcup_{1 \leq i \leq r} \tilde{D}_i$, where $\tilde{D}_i$ is the strict transform of the disc $D_i$ by the modification $\beta$. Let $T_i$ be a sufficiently small open tubular neighbourhood of $\tilde{D}_i$ in $\tilde{B}$ (with pairwise disjoint closures).

**Proposition 4.3.3.** [21, (5.1)] Let $(C, l)$ be a standard decorated germ. Then the Milnor fiber of the smoothing of $\mathcal{X}(C, l)$ corresponding to the picture deformation $(C_S, l_S)$ is orientation-preserving diffeomorphic to the compact oriented manifold with boundary $F' := \tilde{B} \setminus \bigcup_{1 \leq i \leq r} T_i$ (whose corners are smoothed).

Moreover, by this presentation (for a fixed $(C, l)$) one can also canonically identify the boundaries of all the Milnor fibers with the link.

Finally, one reads from the above deformation a homological object too, which will be crucial in the sequel:
Definition 4.3.4. [21, page 483] The \textit{incidence matrix} of a picture deformation \((C_S, l_S)\) is the matrix \(I(C_S, l_S)\) with \(r\) rows and \(n\) columns whose entry at the \(i\)-th row and the \(j\)-th column is the multiplicity of \(P_j\) as a point of \(D_i\).

4.4. The characterization of the Milnor fibers by the incidence matrix. First, let us start with some general remarks. Obviously, the main goal would be to extend for sandwiched singularities the statements of (3.4), valid for cyclic quotients. This is obstructed seriously in both sides of the correspondence. First, at this moment there is no classification of the Stein fillings of the canonical contact structures of sandwiched singularity links. Second, there is no classification of the smoothing components either, or to the possible Milnor fibers. (Recall that in principle it might happen that several different smoothing components have the same Milnor fiber. Also, it is also still open the characterization of those ‘combinatorial’ picture deformations which can be analytically realized.) Nevertheless, we have the following characterization result, which provides a homological/combinatorial method to ‘separate’ some Milnor fibers associated with different smoothing components. In the next paragraphs we follow [40].

Theorem 4.4.1. (a) The incidence matrix \(I(C_S, l_S)\) associated to a picture deformation of a decorated germ \((C, l)\) is determined (up to a permutations of its columns) by the associated Milnor fiber and the marking of the link.

(b) Consider two topologically equivalent decorated germs of plane curves, and for each one of them a picture deformation. If their incidence matrices are different up to permutation of columns, then their associated Milnor fibers are not diffeomorphic by an orientation preserving diffeomorphism which preserves the markings of the boundaries.

The proof is based on a construction which glues a special universal “cap” to each Milnor fiber. This is explained in the next subsection.

4.5. Closing the boundary of the Milnor fiber. Let \((C_S, l_S)\) be a picture deformation of the decorated germ \((C, l)\).

As the disc-configuration \(D\) is obtained by deforming \(C\), its boundary \(\partial D := \bigcup_{1 \leq i \leq r} D_i \hookrightarrow \partial B\) is isotopic as an oriented link to \(\partial C \hookrightarrow \partial B\). Therefore, we can isotope \(D\) outside a compact ball containing all the points \(P_j\) till its boundary coincides with the boundary of \(C\). From now on, \(D\) will denote the result of this isotopy. Let \((B', C')\) be a second copy of \((B, C)\), and define:

\[(V, \Sigma) := (B, D) \cup_{id} (-B', -C').\]

Here \(V\) is the oriented 4-sphere obtained by gluing the boundaries of \(B\) and \(-B'\) by the tautological identification, and \(\Sigma := \bigcup_{i=1}^r \Sigma_i\), where \(\Sigma_i\) is obtained by gluing \(D_i\) (perturbed by the above isotopy) and \(-C_i\) along their common boundaries. Moreover, one also glues \((-B', -C')\) with \((\tilde{B}, \tilde{D})\) in such a way that the blow-up morphism \(\beta\) may be extended by the identity on \(-B'\), yielding \(\beta : (V, \Sigma) \longrightarrow (V, \Sigma)\). Here \(\Sigma := \bigcup_{i=1}^r \Sigma_i\), where \(\Sigma_i\) denotes the strict transform of \(\Sigma_i\), i.e. \(\Sigma_i = \tilde{D}_i \cup -C'_i\). Since \(C'_i\) is a topological disc, and \(D_i\) is an immersed
disc, $\Sigma_i$ is a topologically embedded 2-sphere in $\hat{V}$. Write $T := \bigcup_{1 \leq i \leq r} T_i$ and set $U := -B' \cup T$.

Since $F' = \hat{B} \setminus T$ (cf. 4.3.3), the closed oriented 4-manifold $\hat{V}$ is obtained by closing the boundary of $F'$ by the cap $U$. The point is that $U$ is independent of the chosen picture deformation and it is always glued in the same way (up to an isotopy) to the boundaries of all the involved Milnor fibers.

In particular, the homology of $\hat{V}$ serves as an invariant of the Milnor fiber $F'$. More precisely one has the next statement (which generalizes the ‘identification criterion’ (3.8.1), and implies (4.4.1) too):

**Proposition 4.5.1.** (a) Up to permutations of columns, there exists a unique basis $(e_j)_{1 \leq j \leq n}$ of $H_2(\hat{V})$ such that $e_j^2 = -1$ for all $j$, and the matrix

$$N(C_S, l_S) := ([\Sigma_i] \cdot e_j)_{1 \leq i \leq r, 1 \leq j \leq n}$$

has only non-negative entries.

(b) For any picture deformation $(C_S, l_S)$, the incidence matrix $I(C_S, l_S)$ is equal to $N(C_S, l_S)$, up to permutations of the columns.

**Corollary 4.5.2.** Let $(M, \xi_{\text{can}})$ be a link of a sandwiched singularity endowed with its canonical contact structure. Fix the topological type of a defining decorated germ. Then there are at least as many Stein fillings (up to diffeomorphisms fixed on the boundary) of $(M, \xi_{\text{can}})$ as there are incidence matrices (up to permutation of columns) realised by the picture deformations of all the decorated germs with the given topology.

The above corollary captures all the Milnor fibers associated with all the analytic structures supported by (the cone over) $M$. Notice that it might happen that some Milnor fibers of one of the analytic structure cannot be realized by another analytic structure.

5. Milnor open books

5.1. Notations. As we already mentioned in (2.4), $(M, \xi_{\text{can}})$ can be studied via Milnor open books. On the other hand, Milnor open books can also be studied via the combinatorics of the resolutions. In order to explain this second statement we set first some notation.

Let $\pi : \hat{X} \to X$ be any good resolution of $X$. We will denote by $E_1, \ldots, E_n$ the smooth irreducible components of the exceptional curve $E := \pi^{-1}(0)$ and by $\Gamma$ its dual graph. In this section we will assume that the link $M$ is a rational homology sphere, i.e. $\Gamma$ is a tree, and each $E_i$ is rational. Consider $L := H_2(X, \mathbb{Z})$, it is the free group generated by the irreducible components of $E$ with its natural intersection pairing $(\cdot, \cdot)$ and a natural partial ordering: $\sum_i m'_i E_i \leq \sum_i m''_i E_i$ if and only if $m'_i \leq m''_i$ for all $i$. We denote the Lipman cone (semi-group) by

$$\mathcal{E}_1^+ = \{ D \in L \mid (D, E_i) \leq 0 \, \text{for any} \, i \}.$$ 

It is known that if $D = \sum m_i E_i \in \mathcal{E}_1^+$ then $m_i \geq 0$ for all $i$, and $m_i > 0$ for all $i$ whenever $D \in \mathcal{E}_1^+ \setminus \{0\}$. Moreover, $\mathcal{E}_1^+ \setminus \{0\}$ admits a unique minimal element
also be obtained by a combinatorial algorithm (of Laufer, cf. [24]).

The definition of $\mathcal{E}_\Gamma^+$ is motivated by the following fact. Let $f : (\mathcal{X}, o) \rightarrow (\mathbb{C}, 0)$ be a germ of an analytic function. Then the divisor $(\pi^*(f))$ in $\tilde{\mathcal{X}}$ of $f \circ \pi$ can be written as $D_\Gamma(f) + S(f)$, where $D_\Gamma(f)$ is supported on $E$ (is called the compact part of $(\pi^*(f))$), and $S(f)$ is the strict transform by $\pi$ of $\{ f = 0 \}$. The collection of compact parts (when $f$ runs over $\mathcal{O}_{\mathcal{X},0}$) forms a semi-group too, it will be denoted by $\mathcal{A}_\Gamma^+$. It is a sub-semi-group of $\mathcal{E}_\Gamma^+$ (since $(\pi^*(f)) \cdot E_i = 0$ and $(S(f) \cdot E_i) \geq 0$ for all $i$). $\mathcal{A}_\Gamma^+ \setminus \{0\}$ also has a unique minimal element $Z_{\text{max}}$, the maximal ideal divisor. It is the divisor of the generic hyperplane section. By definitions $Z_{\text{min}} \leq Z_{\text{max}}$.

For rational singularities $\mathcal{A}_\Gamma^+ = \mathcal{E}_\Gamma^+$ (hence $Z_{\text{max}} = Z_{\text{min}}$ too). But, in general, these equalities do not hold: the structure of $\mathcal{A}_\Gamma^+$ (hence the identification of $Z_{\text{max}}$ too) can be very difficult, it depends essentially on the analytic structure of $(\mathcal{X}, o)$. (Maybe the most general result in this direction is the combinatorial description of $\mathcal{A}_\Gamma^+$ for any splice-quotient singularity, a family which includes all the weighted homogeneous singularities as well, cf. [33].)

5.2. (Milnor) open books. One has the following facts for the Milnor open book decomposition of $M$, cf. (2.4).

1. For any such $f \in \mathcal{O}_{\mathcal{X},o}$, consider an embedded good resolution $\pi$ of the pair $(\mathcal{X}, f^{-1}(0))$. Then the strict transform $S(f)$ intersects $E$ transversally, and the number of intersection points $(S(f), E_i)$ is exactly $-(D_\Gamma(f), E_i)$. Since the intersection form is negative definite, the collection of binding components $\{(S(f), E_i)\}_{i=1}^n$ and $D_\Gamma(f) \in \mathcal{A}_\Gamma^+$ determine each other.

Moreover, by classical results of Stallings and Waldhausen, the (topological type of the) binding $L_f \subset M$ determines completely the open book up to an isotopy, provided that $M$ is a rational homology sphere. (For counterexamples for these statement in the general case, see e.g. [32].)

In particular, all the Milnor open book decompositions associated with a fixed analytic type $(\mathcal{X}, o)$ are completely described by the semi-groups $\{\mathcal{A}_\Gamma^+\}_\Gamma$ (associated with all the possible resolutions and natural identifications of elements of them). (Hence, the classification of all the Milnor open books associated with a fixed analytic type $(\mathcal{X}, o)$ is, in fact, as difficult as the determination of the semi-groups $\mathcal{A}_\Gamma^+$ of $\mathcal{E}_\Gamma^+$.)

2. Therefore, from topological points of view, it is more natural to consider the open books of all the analytic germs associated with all the analytic structures supported by the topological type of some $(\mathcal{X}, o)$.

As we already mentioned, for a fixed topological type of $(\mathcal{X}, o)$, in any (negative definite) plumbing graph $\Gamma$ of $M$ one can also define the cone $\mathcal{E}_\Gamma^+$. The point is that for any non-zero element $D$ of $\mathcal{E}_\Gamma^+$ there is a convenient analytic structure on $(\mathcal{X}, o)$ and an analytic germ $f$, such that the plumbing graph can be identified with a dual embedded resolution graph.
(for the pair \((X, f^{-1}(0))\)), and \(D\) is the compact part \(D_f(f)\). Hence, modifying the analytic structure of \((X, o)\), we fill by the collections \(A^+_{\Gamma}\) all the semi-group \(E^+_{\Gamma}\).

In particular, the collection of all the Milnor open book decompositions associated with a fixed topological type \(M\) is described by the collection of semi-groups \(\{E^+_{\Gamma}\}\) (considered in all the possible resolution graphs with a natural identifications of elements of them).

(3) For any fixed analytic type \((X, o)\), the open book associated with \(Z_{\text{max}}\) is the Milnor fibration of the generic hyperplane section, in particular this open book is (resolution) graph-independent. Similarly, for a fixed topological type of \((X, o)\), the open book associated with \(Z_{\text{min}}\) is also graph-independent, it depends only on the topology of the link.

5.3. Invariants of Milnor open books. From the above correspondence, any property defined for elements of \(E^+_{\Gamma}\) (or \(A^+_{\Gamma}\)) can be translated in the language of open books. This is true also in the opposite direction, invariants of open books can be studied via the lattice \(L\). The aim of this section is to emphasise exactly this second direction applied for some key invariants of open books.

Let us fix \(M\), a plumbing (or, a dual resolution) graph \(\Gamma\). Let us consider a Milnor open book associated with an element \(Z \in E^+_{\Gamma} \setminus \{0\}\), cf. (5.2)(2). In the sequel we will consider the following numerical invariants of the open book and also their description in terms of \(L\):

1. The number of binding components \(\text{bn}(Z)\) is given by \(- (Z, E)\).
2. Let \(F_f\) be the fiber of the open book. It is an oriented connected surface with \(- (Z, E)\) boundary components. Let \(g(Z)\) be its genus (the so-called page-genus of the open book) and \(\mu(Z)\) be the first Betti-number of \(F_f\) (the so-called Milnor number). Clearly \(\mu(Z) = 2g(Z) - 1 - (Z, E) \geq 2g(Z)\).

What is less obvious is the following identity, which connects open books with the Riemann-Roch formula. First consider the ‘canonical cycle’ \(K \in L \otimes \mathbb{Q}\) defined by the (adjunction formulas) \((K + E_i, E_i) + 2 = 0\) for all \(i\). Then define the (holomorphic) Euler-characteristic of any element \(D \in L\) by

\[
\chi(D) := -(D, D + K) / 2 \in \mathbb{Z}.
\]

Then for any \(Z \in E^+ \setminus \{0\}\) one has

\[
g(Z) = 1 + (Z, E) + \chi(-Z), \quad \text{and}\]

\[
\mu(Z) = 1 + (Z, E) + 2 \cdot \chi(-Z) = g(Z) + \chi(-Z).
\]

5.4. The ‘monotoneity’ property. This description allows to prove the next ‘monotoneity’ property of these invariants.

**Definition 5.4.1.** Assume that for any resolution \(\pi\) with graph \(\Gamma\) one has a map \(I_\Gamma : E^+_{\Gamma} \setminus \{0\} \to \mathbb{Z}_{\geq 0}\). We say that \(I = \{I_\Gamma\}\) is monotone if for any two cycles \(Z_i \in E^+_{\Gamma} \setminus \{0\}\) \((i = 1, 2)\) with \(Z_1 \leq Z_2\) one has \(I_\Gamma(Z_1) \leq I_\Gamma(Z_2)\) for any \(\Gamma\).
Remark 5.4.2. Assume that the collection of invariants \( \{I_\Gamma\}_\Gamma \) can be transformed into (or comes from) an invariant \( I \) which associates with any Milnor open book \( m \) of the link an integer. For any fixed analytic type, let \( m_{\text{max}} \) be the Milnor open book associated with \( Z_{\text{max}} \) (considered in any resolution). Similarly, for any topological type, let \( m_{\text{min}} \) be the Milnor open book associated with \( Z_{\text{min}} \) (in any resolution of an analytic structure conveniently chosen); cf. (5.2)(3).

Then, whenever \( \{I_\Gamma\}_\Gamma \) is monotone, one has automatically the next consequences:

1. Fix an analytic singularity \((X, o)\) and consider all the Milnor open books associated with all isolated holomorphic germs \( f \in \mathcal{O}_{X,o} \). Then the minimum of integers \( I(m) \) of all these Milnor open books \( m \) is realized by the generic hyperplane section, i.e. by \( I(m_{\text{max}}) \).

2. Fix a topological type of a normal surface singularity, and consider the open books associated with all the isolated holomorphic germs of all the possible analytic structures supported by the fixed topological type. Then the minimum of all integers \( I(m) \) of all these Milnor open books \( m \) is realized by the open book associated with the Artin cycle, i.e. by \( I(m_{\text{min}}) \).

The above definition is motivated by the following result:

Theorem 5.4.3. [41] Both invariants \( Z \mapsto g(Z) \) and \( Z \mapsto \mu(Z) \) are monotone.

5.5. Invariants of the canonical contact structure. In [14] Etnyre and Ozbagci consider three invariants associated with fixed contact structure defined in terms of all open book decomposition supporting it:

- the support genus \( \text{sg}(\xi) \) is the minimal possible genus for a page of an open book that supports \( \xi \);
- the binding number \( \text{bn}(\xi) \) is the minimal number of of binding components for an open book supporting \( \xi \) and that has pages of genus \( \text{sg}(\xi) \);
- the norm \( n(\xi) \) of \( \xi \) is the negative of the maximal (topological) Euler characteristic of a page of an open book that supports \( \xi \).

Now, we consider the above invariants restricted on the realm of Milnor open books (i.e. for all those open books which might appear in the analytic context). In particular, \( \xi \) will be the canonical contact structure \( \xi_{\text{can}} \). Let us denote the corresponding invariants by \( \text{sg}_{\text{an}}(\xi_{\text{can}}) \), \( \text{bn}_{\text{an}}(\xi_{\text{can}}) \) and \( n_{\text{an}}(\xi_{\text{can}}) \). Then (5.4.3) has the following consequence:

\[
\text{sg}_{\text{an}}(\xi_{\text{can}}) = g(Z_{\text{min}}); \\
\text{bn}_{\text{an}}(\xi_{\text{can}}) = \text{bn}(Z_{\text{min}}); \\
n_{\text{an}}(\xi_{\text{can}}) = \mu(Z_{\text{min}}) - 1.
\]

In particular,

\[
n_{\text{an}}(\xi_{\text{can}}) - \text{bn}_{\text{an}}(\xi_{\text{can}}) = 2 \cdot \text{sg}_{\text{an}}(\xi_{\text{can}}) - 2.
\]

These facts answer some of the questions of [14], section 8, at least in the realm of Milnor open books.

Since \( \chi(-Z) + \chi(Z) + Z^2 = 0 \), one also has

\[
g(Z_{\text{min}}) = 1 + (Z_{\text{min}}, E - Z_{\text{min}}) - \chi(Z_{\text{min}}).
\]
Therefore, if $(X, o)$ is rational (i.e. $\chi(Z_{\min}) = 1$) then $g(Z_{\min}) = (Z_{\min}, E - Z_{\min})$ (this can be strict positive, e.g. for the $E_8$-singularity it is 1); if $(X, o)$ is elliptic (i.e. $\chi(Z_{\min}) = 0$) then $g(Z_{\min}) = 1 + (Z_{\min}, E - Z_{\min}) \geq 1$. In general, 
\[ g(Z_{\min}) \geq 1 - \chi(Z_{\min}), \]
hence it can be arbitrary large.

References


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