1. Introduction

1.1. Overview. We first introduce two integrable systems, the KdV hierarchy and the Mumford system.

The KdV hierarchy is an infinite dimensional system given by a family of partial differential equations for the function $f = f(a_1, a_2, \ldots)$ on $\mathbb{C}^\infty$. (See [8] for example.) Here $a_1$ denotes a space coordinate, and others correspond to time coordinates. The first two equations are as follows:

\[
\begin{align*}
\frac{\partial}{\partial a_2} f &= \frac{1}{4} \frac{\partial^3}{\partial a_1^3} f + \frac{3}{2} f \cdot \frac{\partial}{\partial a_1} f, \\
\frac{\partial}{\partial a_3} f &= \frac{1}{16} \frac{\partial^5}{\partial a_1^5} f + \frac{5}{8} f \cdot \frac{\partial^3}{\partial a_1^3} f + \frac{5}{4} \frac{\partial}{\partial a_1} f \cdot \frac{\partial^2}{\partial a_1^2} f + \frac{15}{8} f^2 \cdot \frac{\partial}{\partial a_1} f,
\end{align*}
\]

where the first one is nothing but the Korteweg-de Vries (KdV) equation. The KdV hierarchy is known to have three important classes of solutions: soliton solutions, quasi-periodic solutions, and rational solutions. (See [8], [9, 5] and [3, 1, 2] respectively, and references therein.)

On the other hand, the Mumford system is a finite dimensional classical integrable system invented by Jacobi, which is studied in detail by Mumford [10]. For a fixed $g \in \mathbb{Z}_{>0}$, the phase space $M_g$ of the Mumford system is given by

\[
M_g = \left\{ \ell(x) = \begin{pmatrix} v(x) \\ u(x) \\ w(x) \end{pmatrix} \bigg| \begin{array}{c}
u(x) = x^g + u_{g-1}x^{g-1} + \cdots + u_0, \\
u(x) = v_{g-1}x^{g-1} + \cdots + v_0, \\
u(x) = w_x^g + w_{g-1}x^{g-1} + \cdots + w_0 \end{array} \right\} \simeq \mathbb{C}^{3g+1},
\]

equipped with a Poisson structure $\{ \, , \}$, where we regard the coefficients $u_i, v_i, w_i$ as the coordinates of $M_g$. We have the momentum map

\[
\Phi_g : M_g \to H_g; \quad \ell(x) \mapsto \det(\ell(x)),
\]

where

\[
H_g = \{ h(x) = x^{2g+1} + h_{2g}x^{2g} + h_{2g-1}x^{2g-1} + \cdots + h_0 \mid h_0, \ldots, h_{2g} \in \mathbb{C} \} \simeq \mathbb{C}^{2g+1}.
\]

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Through the momentum map (1.2), \( h_i \)'s becomes independent functions on \( M_g \). The \( g + 1 \) functions \( h_g, \ldots, h_{2g} \) are Casimirs, and other functions \( h_0, \ldots, h_{g-1} \) define commuting Hamiltonian vector fields \( \partial_1, \ldots, \partial_g \) on \( M_g \). Thus the system \((M_g, \{ \cdot, \cdot \}, \Phi_g)\) is integrable in Liouville’ sense. In this article we call this system the \( g \)-Mumford system. For \( h(x) \in H_g \), we have two objects of algebraic geometry:

- the algebraic curve \( C_g(h) \) given by \( y^2 = h(x) \),
- the fiber \( M_g(h) := \Phi_g^{-1}(h(x)) \).

When \( h(x) \) is generic (no multiple root), \( C_g(h) \) is the hyperelliptic curve of genus \( g \). It is shown that \( M_g(h) \) is isomorphic to the affine part of the Jacobian \( J_g(h) \) of \( C_g(h) \), and that the vector fields become the \( g \)-dimensional translation invariant vector field on \( J_g(h) \).

We remark that in [10], the explicit relation between the KdV hierarchy and the Mumford system is unveiled. By using this relation, when \( h(x) \) is generic, the quasi-periodic solutions to the KdV hierarchy is reformulated via the generic fiber \( M_g(h) \) of the Mumford system. Further, when \( C_g(h) \) has \( g \) ordinary double points, the generalized Jacobian of \( C_g(h) \) is isomorphic to \((\mathbb{C}^\times)^g\), and we obtain the \( g \)-soliton solution to the KdV hierarchy.

In this manuscript, we explain the detailed structure of the fiber for the \( g \)-Mumford system over the singular curve \( C_g = C_g(x^{2g+1}) \). The difference with the generic case is that the vector fields are degenerated on the fiber \( M_g(x^{2g+1}) \), and \( M_g(x^{2g+1}) \) is stratified into \( g + 1 \) integral manifolds of the vector fields. The biggest integral manifold is isomorphic to the affine part of the generalized Jacobian \( J_g \) of \( C_g \). Since \( J_g \) is isomorphic to \( \mathbb{C}^g \), we obtain the rational solution to the system. As a byproduct, we get an algebraic-geometrical interpretation to the rational solutions to the KdV hierarchy.

1.2. **Rational solutions to KdV hierarchy.** For the later use, we summarize the rational solutions to the KdV hierarchy [1, 2]. For a fixed \( g \in \mathbb{Z}_{>0} \), the rational solution is given by

\[
 f = \frac{\partial^2}{\partial^2 a_1} \log \tau_g(\vec{a}) \in \mathbb{C}[a_1, \ldots, a_g, \frac{1}{\tau_g(\vec{a})}], \tag{1.4}
\]

where

\[
 \tau_g(\vec{a}) = \det X_g(\vec{a}),
\]

\[
 X_g(\vec{a}) = \begin{pmatrix}
 \chi_1 & \chi_0 & 0 & \cdots \\
 \chi_3 & \chi_2 & \chi_1 & 0 & \cdots \\
 \vdots & & \ddots & \ddots & \ddots \\
 \chi_{2g-3} & \chi_{2g-4} & \cdots & \cdots & \chi_{g-2} \\
 \chi_{2g-1} & \chi_{2g-2} & \cdots & \cdots & \chi_g
 \end{pmatrix} \in M_g(\mathbb{C}[a_1, \ldots, a_g]). \tag{1.5}
\]
Here \( \chi_n \in \mathbb{C}[a_1, \ldots, a_g] \) is given by
\[
\exp(\sum_{i=1}^{g} a_i t^{2i-1}) = \sum_{n=0}^{\infty} \chi_n t^n \quad \text{in } \mathbb{C}[[t]].
\]

From now on we call (1.4) the \( g \)-rational solution. This solution depends only on \( a_1, \ldots, a_g \), and gives the non-trivial solution to the first \( g - 1 \) equations of the hierarchy.

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2. Generic Fiber of the Mumford System

2.1. Hamiltonian structure. We briefly explain the Hamiltonian structure of the Mumford system. The phase space \( M_g \) (1.1) is equipped with the Poisson structure defined by
\[
\{u(x), u(z)\} = \{v(x), v(z)\} = 0,
\]
\[
\{u(x), v(z)\} = \frac{u(x) - u(z)}{x - z},
\]
\[
\{u(x), w(z)\} = -2\frac{v(x) - v(z)}{x - z},
\]
\[
\{v(x), w(z)\} = \frac{w(x) - w(z)}{x - z} - u(x),
\]
\[
\{w(x), w(z)\} = 2\left(v(x) - v(z)\right).
\]

The functions \( h_0, \ldots, h_{2g} \) (1.3) on \( M_g \) are pairwise in involution with respect to this Poisson structure, where \( h_g, \ldots, h_{2g} \) are the Casimirs. The Hamiltonian vector fields \( \partial_1, \cdots, \partial_g \) on \( M_g \) are generated by \( h_0, \cdots, h_{g-1} \) by \( \partial_i := \{\cdot, h_{g-i}\} \). Introducing \( D(z) := \sum_{i=0}^{g-1} z^i \partial_{g-i} \), these vector fields can be simultaneously written as
\[
D(z)u(x) = 2\frac{u(x)v(z) - v(x)u(z)}{x - z},
\]
\[
D(z)v(x) = \frac{w(x)u(z) - u(x)w(z)}{x - z} - u(x)u(z),
\]
\[
D(z)w(x) = 2\left(\frac{v(x)w(z) - w(x)v(z)}{x - z} + v(x)u(z)\right).
\]

It is easy to see that the functions \( h_i \)’s are preserved by the vector fields \( \partial_i \)’s, i. e. \( \partial_j h_i = 0 \) for all \( i, j \), and that the \( g \) vector fields \( \partial_i \)’s are tangent to \( M_g(h) = \Phi_g^{-1}(h(x)) \). Thus we
call \( M_g(h) \) the isolevel set. We define the integral manifold \( M_g(h)_0 \) of the vector fields \( \partial_1, \ldots, \partial_g \) by
\[
M_g(h)_0 = \{ l(x) \in M_g(h) \mid \text{rank}_{l(x)}(\partial_1, \ldots, \partial_g) = g \}.
\]

2.2. Generic fiber. When \( h(x) \) has no multiple root, \( C_g(h) \) is a hyperelliptic curve of genus \( g \) with one smooth point at infinity \( \infty \in C_g(h) \). We write \( J_g(h) \) for the Jacobian variety of \( C_g(h) \), and \( \Theta_g(h) \) for the theta divisor. The vector fields do not degenerate on \( M_g(h) \), thus we have \( M_g(h) = M_g(h)_0 \).

**Theorem 2.1.** [10]

(i) There is an isomorphism \( \phi : M_g(h) \rightarrow J_g(h) \setminus \Theta_g(h) \).

(ii) The isomorphism \( \phi \) induces the \( g \)-dimensional translation invariant vector field on \( J_g(h) \) from the vector fields \( \partial_1, \ldots, \partial_g \).

(iii) Riemann’s theta function gives the solution of the Mumford system.

(iv) The vector fields \( \partial_1, \ldots, \partial_g \) correspond to the partial differentials with respect to the first \( g \) coordinates \( a_1, \ldots, a_g \) of the KdV hierarchy. Moreover, Riemann’s theta function gives the quasi-periodic solution to the KdV hierarchy.

**Remark 2.2.** The map \( \phi \) is called the eigenvector map, and \( \phi(l(x)) \) corresponds to the line bundle over \( C_g(h) \) determined by the eigenvector of \( l(x) \). This is given by a composition of two maps:

\[
l(x) = \begin{pmatrix}
v(x) & w(x) \\
u(x) & -v(x)
\end{pmatrix}
\begin{array}{c}
\rightarrow \\
\nu
\end{array}
\begin{array}{c}
\text{Div}_\text{eff}^g(C_g(h)) \\
\rightarrow \\
\sum_{i=1}^g P_i \\
\rightarrow \\
O_{C_g(h)}(\sum_{i=1}^g P_i - g \cdot \infty).
\end{array}
\]

Here \( P_i = (x_i, v(x_i)) \in C_g(h) \) is determined by the zeros of \( u(x) = \prod_{i=1}^g (x - x_i) \). One sees that the image of \( \nu \) is
\[
\nu(M_g(h)) = \left\{ \sum_{i=1}^g P_i \mid P_i \neq \infty \text{ for all } i, \text{ and } \iota(P_i) \neq P_j \text{ for all } i \neq j \right\} \subset \text{Div}_\text{eff}^g(C_g(h)),
\]
where \( \iota \) is the hyperelliptic involution on \( C_g(h) \) given by \( \iota : (x, y) \mapsto (x, -y) \).

3. Degenerated Fiber and Generalized Jacobian

3.1. Integral manifold and generalized Jacobian. When \( h(x) = x^{2g+1} \), we write \( C_g = C_g(x^{2g+1}) \) and \( M_g(0) = M_g(x^{2g+1}) \) for simplicity. The curve \( C_g \) has a unique singular point at the origin. We write \( \pi \) for the normalization map
\[
\pi : \mathbb{P}^1 \rightarrow C_g(0); \ t \mapsto (t^2, t^{2g+1}).
\]
The generalized Jacobian $J_g$ of $C_g$ and the theta divisor $\Theta_g$ are defined by:

$$J_g := \{ L \mid \text{local free } O_{C_g}\text{-module of rank}(L) = 1 \text{ and } \deg(L) = 0 \},$$

$$\Theta_g := \{ L \in J_g \mid \dim H^0(L \otimes O_{C_g}((g - 1) \cdot \infty)) \neq 0 \}.$$

The following properties are important:

**Proposition 3.1.** We have the following isomorphisms:

$$J_g \simeq \mathbb{C}^g \simeq \mathbb{C}[t]^{\times} / \mathbb{C}[t^2, t^{2g+1}]^{\times},$$

$$\Theta_g \simeq \{ \vec{a} = (a_1, \ldots, a_g) \in \mathbb{C}^g \mid \tau_g(\vec{a}) = 0 \},$$

where $\tau_g(\vec{a})$ is given by (1.5).

It is natural to have the function $\tau_g(\vec{a})$ here, which is related to the rational analogue of the hyperelliptic functions [4]. Note that $J_g$ is not compact any more. Now the vector fields degenerate on $M_g(0)$, i.e. $M_g(0) \neq M_g(0)$, and $M_g(0)$ is stratified as follows:

**Proposition 3.2.** (i) We have $M_g(0) \setminus M_g(0)_{0} \simeq M_{g-1}(0)$, where $M_{g-1}(0)$ is the isolevel set of the $(g-1)$-Mumford system, $M_{g-1}(0) = \Phi_{g-1}^{-1}(x^{2g-1})$.

(ii) The embedding map $\iota_g : M_{g-1}(0) \to M_g(0)$ is given by $\iota_g(l(x)) = xl(x)$.

(iii) On $M_g(0) \setminus M_g(0)_{0}$, we have $\partial_1 = 0$.

In the following we write $xM_{g-1}(0)$ for $\iota_g(M_{g-1}(0))$. We find that the integral manifold $M_g(0)_0$ is an essential object to be related to the curve $C_g$:

**Theorem 3.3.** [7]

(i) There is an isomorphism $\phi : M_g(0)_{0} \sim J_g \setminus \Theta_g$.

(ii) The isomorphism $\phi$ induces the $g$-dimensional translation invariant vector field on $J_g$ from the vector fields $\partial_1, \ldots, \partial_g$.

(iii) The function $\tau_g(\vec{a})$ gives the rational solution to the Mumford system.

**Remark 3.4.** (i) For a generic element $l(x)$ in $M_g(0)$, the map $\phi$ is given by $\phi(l(x)) = \vec{a} = (a_1, \ldots, a_g) \in \mathbb{C}^g$ where $\vec{a}$ is determined by

$$\prod_{i=1}^{g}(1 - \frac{x_i^g}{v(x_i)})^{\tau_g(\vec{a})} \equiv \exp\left(\sum_{i=1}^{g} a_it^{2i-1}\right) \mod \mathbb{C}[t^2, t^{2g+1}]^{\times}.$$

Here $x_i$’s are the zeros of $u(x)$.

(ii) The inverse map $\phi^{-1}$,

$$\phi^{-1} : \vec{a} = (a_1, \ldots, a_g) \mapsto l(x) = \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix},$$
is constructed as follows: We set \( \rho_g(\vec{a}) = \frac{\partial^2}{\partial a_1^2} \log \tau_g(\vec{a}) \), and \( \rho_g(\vec{a})^{(k)} = \frac{\partial^k}{\partial a_1^k} \rho_g(\vec{a}) \). Then we recursively determine the coefficients of \( u(x), v(x), w(x) \) starting by

\[
u_{g-1} = \rho_g(\vec{a}), \quad \nu_{g-1} = \frac{1}{2} \rho_g(\vec{a})^{(1)}, \quad \nu_g = -\rho_g(\vec{a}),
\]

using a relation \( v(x)^2 + u(x)w(x) = x^{2g+1} \) and the action of \( \partial_1 = \frac{\partial}{\partial a_1} \). Actually the coefficients are obtained in \( \mathbb{C}[\rho_g, \rho_g^{(1)}, \ldots, \rho_g^{(2g)}] \).

(iii) The function \( \rho_g(\vec{a}) \) is nothing but the \( g \)-rational solution (1.4) to the KdV hierarchy.

**Example 3.5.** (i) \( g = 2 \) case:

\[
\tau_2(\vec{a}) = \frac{a_1^3}{3} - a_2, \quad \rho_2(\vec{a}) = \frac{-3a_1(a_1^3 + 6a_2)}{(a_1^3 - 3a_2)^2}.
\]

(ii) \( g = 3 \) case:

\[
\tau_3(\vec{a}) = \frac{a_1^6}{45} - \frac{a_1^4a_2}{3} - a_2^3 + a_1a_3,
\]

\[
\rho_3(\vec{a}) = \frac{-3(24a_1^{10} + 675a_1^4a_2^2 - 1350a_1a_3^2 - 270a_1^2a_3 + 675a_3^2)}{(a_1^6 - 15a_1^4a_2 - 45a_2^2 + 45a_1a_3)^2}.
\]

### 3.2. Degenerated loci and compactification of \( J_g \).

As seen in Proposition 3.2 and Theorem 3.3, \( M_g(0) \) is stratified into \( g + 1 \) integral manifolds:

\[M_g(0) = M_g(0)_0 \sqcup xM_{g-1}(0)_0 \sqcup x^2M_{g-2}(0)_0 \sqcup \cdots \sqcup x^{g-1}M_1(0) \sqcup x^gM_0(0),\]

where \( x^kM_{g-k}(0)_0 \simeq J_{g-k} \setminus \Theta_{g-k} \) for \( k = 0, \ldots, g - 1 \) and \( x^gM_0(0) \simeq \{ \text{pt.} \} \). We have \( \partial_1 = \cdots = \partial_k = 0 \) on \( x^kM_{g-k} \).

Let \( \mathcal{J}_g \) be the compactification of \( J_g \) and \( \overline{\Theta}_g \) be the theta divisor [6, 11]:

\[
\mathcal{J}_g := \{ \mathcal{L} \mid \mathcal{L} : \text{torsion free } O_{C_g} \text{-module of rank}(\mathcal{L}) = 1 \text{ and deg}(\mathcal{L}) = 0 \} \supset J_g,
\]

\[
\overline{\Theta}_g := \{ \mathcal{L} \in \mathcal{J}_g \mid \dim H^0(\mathcal{L} \otimes O_{C_g}((g - 1) \cdot \infty)) \neq 0 \} \supset \Theta_g.
\]

Using these, we finally discribe the total space \( M_g(0) \) as follows:

**Theorem 3.6.** [7]

(i) The isolevel set \( M_g(0) \) is isomorphic to \( \mathcal{J}_g \setminus \overline{\Theta}_g \).

(ii) Via the isomorphism of (i), the vector fields \( \partial_1 \)'s on \( M_g(0) \) becomes those on \( \mathcal{J}_g \) induced by the action of \( J_g \) to \( \mathcal{J}_g \).

### References


