Branched coverings of projective varieties

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Let \( d \) be an integer greater than 1. The symmetric group \( S_d \) of degree \( d \) acts on \( (\mathbb{P}^1)^d \). Let \( \lambda_d : (\mathbb{P}^1)^d \to \mathbb{P}^d \) be the quotient map by this action. Then the covering \( \theta_d : \mathbb{P}^1 \times \mathbb{P}^{d-1} \to \mathbb{P}^d \) defined by

\[
([u_0 : u_1], [w_0 : w_1 : \cdots : w_{d-1}]) \mapsto [u_0w_0 : u_0w_1 + u_1w_0 : \cdots : u_0w_{d-1} + u_1w_{d-2} : u_1w_{d-1}]
\]
satisfies \( \lambda_d = \theta_d \circ (\text{id} \times \lambda_{d-1}) \), where \( \lambda_1 = \text{id} \). Let \( Y \) be a projective variety and let \( \nu : Y_0 \to \mathbb{P}^d \) be a holomorphic map from a Zariski open set \( Y_o \) of \( Y \) to \( \mathbb{P}^d \). If the image \( \nu(Y_o) \) of \( \nu \) is not contained in the branch locus of \( \theta_d \) and the normalization \( X_o \) of the fiber product \( Y_o \times_{\mathbb{P}^d} (\mathbb{P}^1 \times \mathbb{P}^{d-1}) \) of \( \nu \) and \( \theta_d \), is irreducible, then by Grauert-Remmert Theorem[2], there exists a finite branched covering \( \pi : X \to Y \) of degree \( d \) extending the projection \( X_o \to Y_o \). Conversely, the following theorem holds.

**Theorem 1.** Let \( \pi : X \to Y \) be a finite covering of degree \( d \) of a projective variety \( Y \). Then there exist meromorphic maps \( \mu : X \to \mathbb{P}^1 \times (\mathbb{P}^1)^{d-1} \) and \( \nu : Y \to \mathbb{P}^d \) satisfying the following three conditions.

(i) \( \theta_d \circ \mu = \nu \circ \pi \).

(ii) The image \( \nu(Y_o) \) of \( Y_o \) under \( \nu \) is not contained in the branch locus of \( \theta_d \), where \( Y_o \) is a Zariski open set of \( Y \) to which the restriction of \( \nu \) is holomorphic.

(iii) \( (\pi \times \mu)(\pi^{-1}(Y_o)) = Y_o \times_{\mathbb{P}^d} (\mathbb{P}^1 \times \mathbb{P}^{d-1}) \).

We obtain the following theorems 2, 4 and 5 applying the above theorem. Let \( \pi : X \to \mathbb{P}^n \) be a covering of \( \mathbb{P}^n \). We define \( g(\pi) = g(\pi^{-1}(L)) \) for a generic line \( L \) on \( \mathbb{P}^n \). We consider the sets

\[
\mathcal{M}(d, g, n) = \{ \pi : X \to \mathbb{P}^n \mid \pi \text{ is a covering, } \deg(\pi) = d, \ g(\pi) = g \}/\sim
\]

and \( \mathcal{M}^\circ(d, g, n) = \{ [\pi : X \to \mathbb{P}^n] \in \mathcal{M}(d, g, n) \mid X \text{ is non-singular} \} \)

for an integer \( d \) greater than 2, a non-negative integer \( g \) and a positive integer \( n \). Here we say that two coverings \( \pi_1 : X_1 \to Y_1 \) and \( \pi_2 : X_2 \to Y_2 \) are equivalent and denote by \( \pi_1 \sim \pi_2 \), if there exist two isomorphisms \( \phi : X_1 \simeq X_2 \) and \( \psi : Y_1 \simeq Y_2 \) with \( \pi_2 \circ \phi = \psi \circ \pi_1 \). Namba[4] constructed \( \mathcal{M}(d, g, 1) \) for positive integers \( g \). Here we note that the above coverings \( \lambda_d \) and \( \theta_d \) are \( \text{Aut}(\mathbb{P}^1) \)-equivariant maps.

**Theorem 2.** Let \( d \) be an integer greater than 2. Let \( H_1 \) and \( H_2 \) be planes on \( \mathbb{P}^d \) with the same dimension. Assume that \( X_i = \theta_d^{-1}(H_i) \) are irreducible. Then \( \pi_1 \sim \pi_2 \), where \( \pi_i = \theta_d|_{X_i} \) (\( i = 1, 2 \)), if and only if \( gH_1 = H_2 \) for an element \( g \) in \( \text{Aut}(\mathbb{P}^1) \).

Let \( L \) be a generic line on \( \mathbb{P}^d \). Then as we see in §3, the restriction \( p_1|_{\theta_d^{-1}(L)} \) to \( \theta_d^{-1}(L) \) of the projection \( p_1 : \mathbb{P}^1 \times \mathbb{P}^{d-1} \to \mathbb{P}^1 \), is one to one. Hence \( \theta_d^{-1}(L) \) is a rational curve. Therefore, \( g(\theta_d|_{\theta_d^{-1}(H)}) = 0 \) for a plane \( H \) on \( \mathbb{P}^d \), if \( \theta_d^{-1}(H) \) is irreducible. Let \( n \) be an integer greater than \( \dim H \) and let \( \lambda : \mathbb{P}^n \to H \) be a projective cone. Then \( g(\lambda) = 0 \), where \( p : \mathbb{P}^n \times_H \theta_d^{-1}(H) \to \mathbb{P}^n \) is the projection. Hence we see by the above theorem that

\[
\mathcal{M}(d, 0, n) \supset \bigcup_{1 \leq m \leq \min(n, d)} \text{Aut}(\mathbb{P}^1) \setminus G_{d+1,m+1}(\mathbb{C}) .
\]
where $G^o_{d+1,m+1}(C)$ is the open set of the Grassman manifold $G_{d+1,m+1}(C)$ consisting of $m$-planes on $\mathbb{P}^d$ whose inverse images under $\theta_d$ are irreducible. Let $\pi : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map from $\mathbb{P}^2$ onto itself with $\deg(\pi) > 1$. Then $\pi$ is a Veronese map of degree 2, if and only if $g(\pi) = 0$ and then $\deg(\pi) = 4$. Hence $M(4,0,n) \supset \{\pi : \mathbb{P}^2 \to \mathbb{P}^2 \mid \deg(\pi) = 4\}/\sim$

for $n \geq 2$. Let $B$ be an irreducible component of the branch locus $B_\pi$ of $\pi$ and let $p$ be a point of $B - \text{Sing}(B_\pi)$. Let $U$ be a small neighborhood of $p$ homeomorphic to an open ball and let $U_1, U_2, \ldots, U_l$ be the connected components of $\pi^{-1}(U)$. We say that $B$ is of type $\{d_1, d_2, \ldots, d_l\}$, if $\deg(\pi(U_i)) = d_i$. Note that $\sum_{i=1}^l d_i = \deg(\pi)$.

**Theorem 3.** Let $\pi : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map with $\deg(\pi) = 4$. Then the branch locus $B_\pi$ of $\pi$ is one of the followings. In each case, all irreducible components which are lines, are of type $\{2, 2\}$.

The others are of type $\{2, 1, 1\}$

(i) three lines which does not meet at a point
(ii) a conic and its two tangents
(iii) a 2 cuspidal quartic and its bitangent
(iv) a 9 cuspidal sextic

The map $\pi$ in the above theorem is expressed as

$$(z^2_0 : z^2_1 : z^2_2), \quad (z^2_0 : z^2_1 : z_0 z_1 + z^2_2) \quad \text{and} \quad (z^2_0 : z^2_1 + z_0 z_2 : z^2_2 + z_0 z_1)$$

in the case (i), (ii) and (iii), respectively. On the other hand, (ii) is a small deformation of (i), and (iii) is a small deformation of (i) and (ii). Hence $M(4,0,n)$ is not a Hausdorff space for $n \geq 2$.

**Theorem 4.** If $d > n \geq 1$ and $(d,n) \neq (4,2)$, then

$$M^o(d,0,n) = \text{Aut}(\mathbb{P}^1) \backslash G_{d+1,n+1}(C)$$

and

$$M^o(4,0,2) = \text{Aut}(\mathbb{P}^1) \backslash G_{3,3}(C) \bigcup \{\pi : \mathbb{P}^2 \to \mathbb{P}^2 \mid \deg(\pi) = 4\}/\sim.$$ 

$M^o(d,0,d)$ consists of one point. If $2 < d < n$, then $M^o(d,0,n) = \emptyset$.

There exists much study on triple coverings (see [3] for references). We add the following.

**Theorem 5.** Let $\pi : X \to \mathbb{P}^2$ be a triple covering with $g(\pi) = 1$.

(I) If $\pi$ is a cyclic covering, then $B_\pi$ is a cubic curve.

(II) If $\pi$ is not a cyclic covering, then $B_\pi$ is one of the following five.

(i) a 9 cuspidal sextic
(ii) the $(2,3)$ torus $(g_3^2 - 4g_2^3)$ for a cubic $g_3$ and a quadratic $g_2$
(iii) the quartic $(h_2^2 - 4mh_1^2)$ and the line $(m)$ for linear equations $h_1, m$ and a quadratic $h_2$
(iv) the two conics $(g_2)$ and $(h_2^2 - 4g_2)$ for a linear equation $h_1$ and a quadratic $g_2$
(v) the two conics $(h_2 + 2m^2)$, $(h_2 - 2m^2)$ and the line $(m)$ for a linear equation $m$ and a quadratic $h_2$

In the above theorem, $B_\pi$ is a sextic only in the cases II (i) and (ii). Let $\pi : X \to \mathbb{P}^2$ be a triple covering branched at a sextic whose all irreducible components are of type $\{2,1\}$. Then $\pi$ is not a cyclic covering and $g(\pi) = 1$. Hence we obtain the same result as Theorem 0.1 in [3].
1 Proof of Theorem 1

Let \( \varpi : Z \to Y \) be a Galois closure of \( \pi \) and let \( G = \Gal(\varpi) \). Then there exists a Galois covering \( \varpi_0 : Z \to X \) whose Galois group is a subgroup \( H \) of \( G \) with the index \( d \) and \( \varpi = \pi \circ \varpi_0 \).

\[
\begin{array}{c}
Z \\
\varpi_0 \\
\varpi \\
X \\
\pi \\
Y
\end{array}
\]

Let \( H, H\sigma_2, \ldots, H\sigma_d \) be the right cosets of \( H \), i.e., \( \sigma_i \) are elements in \( G \) and

\[
G = H \sqcup H\sigma_2 \sqcup \cdots \sqcup H\sigma_d.
\]

There exists a rational function \( g_0 \) on \( X \) which distinguishes the points in the fiber \( \pi^{-1}(y) = \{ x_1, x_2, \ldots, x_d \} \) of \( \pi \) over a point \( y \) on \( Y \), i.e., \( g_0(x_i) \neq g_0(x_j) \) if \( i \neq j \). Then there exist a positive divisor \( D_0 \) and elements \( \xi_0 \) and \( \eta_0 \) in \( H^0(X, \mathcal{O}(D_0)) \) with \( g_0 = \xi_0/\eta_0 \). Let \( D_1 = \varpi_*^* D_0 \), \( D_i = \sigma_i^* D_1 \) and let \( \xi_1 = \varpi_0^* \xi_0 \), \( \eta_1 = \varpi_0^* \eta_0 \), \( \xi_i = \sigma_i^* \xi_1 \), \( \eta_i = \sigma_i^* \eta_1 \) \((i = 2, \ldots, d)\). Then \( \xi_i, \eta_i \in H^0(Z, \mathcal{O}(D_i)) \). Since \( \xi_1 \) and \( \eta_1 \) are invariant under the action of \( H \), \( \sigma^* \xi_1 = \xi_i \), \( \sigma^* \eta_1 = \eta_i \) for any element \( \sigma \) in \( H\sigma_i \). Moreover, if \( \sigma, \sigma \in H\sigma_j \) for an element \( \sigma \) in \( G \), then \( \sigma^* \xi_i = \sigma^* (\sigma_i^* \xi_1) = (\sigma \sigma_i)^* \xi_1 = \xi_j \). Let

\[
\alpha_0 = \eta_1 \eta_2 \cdots \eta_d, \quad \alpha_i = \left( \begin{array}{c} \xi_1 \eta_2 \cdots \eta_d \\ \xi_2 \eta_3 \cdots \eta_d \\ \vdots \\ \xi_d \eta_1 \cdots \eta_{d-1} \end{array} \right) \alpha_0 \quad (i = 1, 2, \ldots, d).
\]

Then \( \alpha_i \in H^0(Z, \mathcal{O}(D_1 + D_2 + \cdots + D_d)) \). Let \( g_1 = \varpi_0^* g_0 \) and let \( g_i = \sigma_i^* g_1 \). Then \( g_i = \xi_i/\eta_i \) and \( \alpha_i/\alpha_0 \) is the \( i \)-th symmetric equation of \( g_1, g_2, \ldots, g_d \). Hence they are rational functions on \( Y \). Moreover, there exists a divisor \( D \) on \( Y \) such that \( \varpi^* D = D_1 + D_2 + \cdots + D_d \) and that \( \alpha_i \) are elements in \( H^0(Y, \mathcal{O}(D)) \).

\[
\beta_0 = \eta_2 \eta_3 \cdots \eta_d, \quad \beta_i = \left( \begin{array}{c} \xi_2 \eta_3 \cdots \eta_d \\ \xi_3 \eta_4 \cdots \eta_d \\ \vdots \\ \xi_d \eta_2 \cdots \eta_{d-1} \end{array} \right) \beta_0 \quad (i = 1, 2, \ldots, d - 1).
\]

Then \( \beta_i \in H^0(Z, \mathcal{O}(D_2 + D_3 + \cdots + D_d)) \). Since \( \beta_i/\beta_0 \) are symmetric equations of \( g_2, g_3, \ldots, g_d \), they are rational functions on \( X \). Moreover, there exists a divisor \( D' \) on \( X \) such that \( \varpi_0^* D' = D_2 + D_3 + \cdots + D_d \) and that \( \beta_i \) are elements in \( H^0(X, \mathcal{O}(D')) \). We define rational maps \( \mu : X \to \mathbb{P}^1 \times \mathbb{P}^{d-1} \) and \( \nu : Y \to \mathbb{P}^d \) by

\[
([\xi_0 : \eta_0], [\beta_{d-1} : \cdots : \beta_0]) \quad \text{and} \quad [\alpha_d : \alpha_{d-1} : \cdots : \alpha_0],
\]

respectively. Then \( \theta_d \circ \mu = \nu \circ \pi \), because \( \alpha_i = \xi_1 \beta_{i-1} + \eta_1 \beta_i \) \((\beta_{-1} = \beta_d = 0)\). Hence the conditions (i) and (iii) are satisfied. The condition (ii) follows from that on \( g_0 \).

2 Branch loci

The covering map \( \lambda_d : (\mathbb{P}^1)^d \to \mathbb{P}^d \) is expressed as

\[
([\xi_1 : \eta_1], [\xi_2 : \eta_2], \ldots, [\xi_d : \eta_d]) \mapsto [\alpha_d : \alpha_{d-1} : \cdots : \alpha_0],
\]

where \([\xi_i : \eta_i]\) are homogenous coordinates of \( \mathbb{P}^1 \) and \( \alpha_i \) are the same as above. Since \( \lambda_d \) is a Galois closure of \( \theta_d \), the branch loci of \( \theta_d \) and \( \lambda_d \) coincide. Also the branch loci of \( \varpi \) and \( \pi \) coincide. Now we consider the defining equation of the branch locus \( B_{\lambda_d} \) of \( \lambda_d \). Since

\[
\prod_{1 \leq i < j \leq d} (\xi_i \eta_j - \xi_j \eta_i)^2 = (\eta_1 \eta_2 \cdots \eta_d)^{2(d-1)} \prod_{1 \leq i < j \leq d} \left( \frac{\xi_i}{\eta_i} - \frac{\xi_j}{\eta_j} \right)^2,
\]

the equation of the branch locus is given by
there exists a polynomial $P$ of the degree $2(d - 1)$ with $d + 1$ variables satisfying

$$
\prod_{1 \leq i < j \leq d} (\xi_i \eta_j - \eta_i \xi_j)^2 = P(\alpha_d, \alpha_{d-1}, \ldots, \alpha_0).
$$

Then $B_{d}$ is defined by $P(x_0, x_1, \ldots, x_d) = 0$. When $d = 3$,

$$
P(x_0, x_1, x_2, x_3) = x_1^2 x_2^2 - 4x_1 x_3 x_2^4 - 4x_0 x_2^3 + 18x_0 x_1 x_2 x_3 - 27x_0^2 x_3^2.
$$

When $d = 4$,

$$
P(x_0, x_1, x_2, x_3, x_4) = x_1^2 x_2^2 x_3^2 - 4x_1 x_3 x_2^4 - 4x_0 x_2^3 x_4 - 4x_0 x_2^3 x_3^2 + 18x_1 x_2 x_3 x_4 + 18x_0 x_1 x_2 x_3^2
\quad - 27x_0^2 x_3^4 - 6x_0 x_2^2 x_3^4 - 27x_0^2 x_3^4 - 80x_0 x_1 x_2^2 x_3 x_4 + 16x_0 x_2^4 x_4
\quad + 144x_0 x_2^2 x_3^2 x_4^2 + 144x_0^2 x_2^2 x_3^2 x_4 - 192x_0^2 x_1 x_3 x_4^2 - 128x_0^2 x_2^2 x_3^2 + 256x_0^3 x_3^3.
$$

Next, we consider the singularities of $B_{d}$. Let

$$
S = \{(s, s, s, t_4, \ldots, t_d) \mid s, t_4, \ldots, t_d \in \mathbb{P}^1\}, \quad T = \{(s_1, s_1, s_2, s_2, t_5, \ldots, t_d) \mid s_1, s_2, t_5, \ldots, t_d \in \mathbb{P}^1\}.
$$

Then $\lambda_d(S)$ and $\lambda_d(T)$ are singularities of $B_{d}$. Moreover, $\lambda_d(S)$ and $\lambda_d(T)$ are locally isomorphic to the products of a cusp and a node, respectively with $d - 2$-dimensional manifolds, where

$$
S_0 = \{(s, s, s, t_4, \ldots, t_d) \in S \mid s \neq t_i, \ t_i \neq t_j \text{ if } i \neq j\},
$$

$$
T_0 = \{(s_1, s_1, s_2, s_2, t_5, \ldots, t_d) \in T \mid s_1 \neq s_2, \ s_i \neq t_j, \ t_i \neq t_j \text{ if } i \neq j\}.
$$

### 3 Proof of Theorem 2

$	ext{Aut}(\mathbb{P}^1)$ acts on $(\mathbb{P}^d)^d$ as

$$
g : (x_1, x_2, \ldots, x_d) \mapsto (gx_1, gx_2, \ldots, gx_d).
$$

This action commutes with that of $S_d$. Hence Aut$(\mathbb{P}^1)$ acts also on $\mathbb{P}^1 \times \mathbb{P}^{d-1}$ and $\mathbb{P}^d$. Moreover, $\lambda_d, \text{id} \times \lambda_{d-1}$ and $\theta_d$ are Aut$(\mathbb{P}^1)$-equivariant maps. Hence $\pi_1 \sim \pi_2$, if $gH_1 = H_2$ for an element $g$ in Aut$(\mathbb{P}^1)$.

Now assume that $\pi_1 \sim \pi_2$. Let $[w_0 : u_1]$ and $[w_0 : w_1 : \cdots : w_{d-1}]$ be homogeneous coordinates of $\mathbb{P}^1$ and $\mathbb{P}^{d-1}$, respectively. Then $\theta_d^{-1}(H)$ for any hyperplane $H$ on $\mathbb{P}^d$ is defined by $u_0h_0 + u_1h_1 = 0$, where $h_0$ and $h_1$ are linear combinations of $w_0, w_1, \ldots, w_{d-1}$. Hence if $\theta_d^{-1}(H)$ is irreducible, then the intersection of $\theta_d^{-1}(H)$ and any fiber of the projection $p_1 : \mathbb{P}^1 \times \mathbb{P}^{d-1} \to \mathbb{P}^1$, is a hyperplane on $\mathbb{P}^{d-1}$.

Let $J$ be an $l$-dimensional plane on $\mathbb{P}^d$ and assume that $J = \theta_d^{-1}(J)$ is irreducible. Since the intersection of $J$ and any fiber of $p_1$ is an $(l - 1)$-dimensional plane on $\mathbb{P}^{d-1}$, there exist no holomorphic maps from $\tilde{J}$ onto $\mathbb{P}^1$ except $p_{1|\tilde{J}}$, if $l \neq 2$. When $l = 2$, $\tilde{J}$ is a Hirzebruch surface. Hence if there exists a holomorphic map from $\tilde{J}$ onto $\mathbb{P}^1$ except $p_{1|\tilde{J}}$, then $\tilde{J} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Note that $\tilde{L} = \theta_d^{-1}(L)$ for a generic line $L$ on $J$, is a section of $p_{1|\tilde{J}}$ and that $\tilde{J}^2 = d$. Hence if $d > 2$, there exist no holomorphic maps from $\tilde{J}$ onto $\mathbb{P}^1$ such that $\tilde{L}$ is a section, except $p_{1|\tilde{J}}$. Therefore, there exists an element $g$ in Aut$(\mathbb{P}^1)$ such that $g \circ p_{1|\tilde{X}_i} = p_{1|\tilde{X}_2} \circ \phi$, because $\theta_d^{-1}(L)$ is a section of $p_{1|\tilde{X}_i}$ for a generic line $L$ on $H_1$. Then $gH_1 = H_2$.

### 4 Proof of Theorem 3

We note that $\pi^{-1}(L)$ for a line $L$ on $\mathbb{P}^2$, is a smooth conic, a double line or consists of two lines. Let $B_1, B_2, \ldots, B_s$ be the irreducible components of $B_\pi$. We define $r(B_i) = \sum_{j=1}^s d_j - 1$, if $B_i$ is of type...
\{d_1, d_2, \ldots, d_l\}. Then by Riemann-Hurwitz formula we obtain
\[
\sum_{i=1}^s r(B_i) \deg B_i = 6,
\]
because \( \deg(\pi) = 4 \) and \( g(\pi) = 0 \). If \( B_i \) is of type \( \{2, 1, 1\}, \{2, 2\}, \{3, 1\} \) or \( \{4\} \), then the defining equation of \( \pi^{-1}(B_i) \) is equal to \( Q^2 R, Q^2, Q^3 R \) or \( Q^4 \), respectively, for square free nonconstant polynomials \( Q \) and \( R \). Assume that \( B_1 \) is a line. Then it is of type \( \{2, 2\} \) and \( \pi^{-1}(B_1) \) is a double line, because it is quadratic. Moreover, the restriction \( \pi|_{\pi^{-1}(B_1)} \) to \( \pi^{-1}(B_1) \) of \( \pi \) is a double covering. Since it branches at just two points, \( B_1 \) intersects at just two points with the other irreducible components of \( B_\pi \). Let \( y_0 \) be a point on \( B_1 \) at which \( B_1 \) intersects with another irreducible component \( B_j \) of \( B_\pi \) and let \( U \) be a small neighborhood of \( y_0 \). Let \( \pi_1: V \to U \) be the double covering branched at \( B_1 \cap U \). Then the projection \( \overline{U} \times_U \pi^{-1}(U) \to \pi^{-1}(U) \) does not ramify, because \( B_1 \) is of type \( \{2, 2\} \). Hence there exists a holomorphic map \( \pi_2: \pi^{-1}(U) \to \overline{U} \) with \( \pi_1 \circ \pi_2 = \pi|_{\pi^{-1}(U)} \). Since \( \deg(\pi) = 4 \) and \( \pi|_{\pi^{-1}(U)} \) branches at \( (B_1 \cup B_j) \cap U \), \( \pi_2 \) is a double covering whose branch locus \( B_{\pi_2} \) is contained in \( \pi_1^{-1}(B_j) \). Since \( \pi^{-1}(U) \) and \( \overline{U} \) are both non-singular, \( B_{\pi_2} \) is also non-singular. Moreover, \( B_{\pi_2} \) intersects with \( \pi_1^{-1}(B_1) \) transversally, because \( \pi^{-1}(B_1) \) is a line. Hence \( B_1 \) is non-singular at \( y_0 \) and the intersection number of \( B_1 \) and \( B_j \) at \( y_0 \) is not greater than 2. We easily see that if \( B_1 \) intersect with \( B_j \) transversally, then \( B_j \) is of type \( \{2, 2\} \). The number of the irreducible components of \( B_\pi \) which are lines, is not greater than 3, because \( r(B_k) = 2 \) for any irreducible component \( B_k \) of \( B_\pi \) which is a line. Moreover, if the number is just equal to 3, then \( B_\pi \) consists of three lines which does not meet at a point.

Next, we consider the case that \( B_1 \) is not a line. Let \( L \) be a tangent at a generic point of \( B_1 \) and assume that \( L \) is not an irreducible component of \( B_\pi \). If \( B_i \) is of type \( \{2, 2\}, \{3, 1\} \) or \( \{4\} \), then \( \pi^{-1}(L) \) must have two nodes, a cusp or a tacnode, respectively. Hence \( B_i \) is of type \( \{2, 1, 1\} \). Suppose that \( B_\pi \) has two irreducible components \( B_1 \) and \( B_2 \) which are not lines. Then \( \pi^* B_k = 2C_k + D_k \) \( (k = 1, 2) \) for certain positive divisors \( C_k \) and \( D_k \). Let \( x_0 \) be a point at which \( C_1 \) intersects with \( C_2 \) and let \( y_0 = \pi(x_0) \). Let \( U \) be a small neighborhood of \( y_0 \) and let \( V \) be the connected component of \( \pi^{-1}(U) \) containing \( x_0 \). Then \( \deg(\pi|_V) \geq 3 \). Since there exist no irreducible components of \( B_\pi \) which are lines passing through \( y_0 \), \( \pi^{-1}(L) \) for any line \( L \) passing through \( y_0 \), is a smooth conic or consisting of two lines meeting at \( x_0 \). Suppose that there exist two lines passing through \( y_0 \) whose inverse images under \( \pi \) both consist of two lines. Then \( \pi|_V \) is expressed as \( (z_1, z_2) \mapsto (z_1 z_2, (z_1 + z_2)(c_1 z_1 + c_2 z_2)) \) by a local coordinate \( (z_1, z_2) \) of \( V \), where \( c_i \) are non-zero constants. Then \( B_1 \) and \( B_2 \) must be of type \( \{2, 2\} \). Suppose that there exist lines \( L_1 \) and \( L_2 \) passing through \( y_0 \) such that \( \pi^{-1}(L_1) \) consists of two lines \( M_1 \) and \( M_2 \) and that \( \pi^{-1}(L_2) \) is a smooth conic. Then \( M_1 \) or \( M_2 \) is a tangent of \( \pi^{-1}(L_2) \) at \( x_0 \), because \( \deg(\pi|_V) \geq 3 \). Hence \( \pi|_V \) is expressed as \( (z_1, z_2) \mapsto (z_1 z_2, z_2 + h(z_1, z_2)) \) by a local coordinate of \( V \), where \( h \) is a holomorphic function on \( V \). Therefore, the ramification divisor of \( \pi|_V \) must be non-singular. We easily see that there exists a line \( L \) passing through \( y_0 \) such that \( \pi^{-1}(L) \) is not a smooth conic. Hence there exists at most one irreducible component of \( B_\pi \) which is not a line.

Let \( B_1 \) be an irreducible component of \( B_\pi \) which is not a line. If \( L \) is a bitangent of \( B_1 \) and is not an irreducible component of \( B_\pi \), then \( \pi^{-1}(L) \) must have two nodes. If \( L \) is a tangent at a reflection point of \( B_1 \), then \( \pi^{-1}(L) \) must have a cusp. If \( B_\pi \) does not contain lines, then it is an irreducible sextic. Then \( B_\pi \) is the dual curve of a non-singular cubic, because it does not have bitangents and reflection points. If \( B_\pi \) contains just one line \( L \), then the other irreducible component \( B_1 \) of \( B_\pi \) is a quartic which contains no reflection points and only one bitangent which is equal to \( L \). Hence \( B_1 \) is the dual curve of a cubic with a node. If \( B_\pi \) contains two lines \( L_1 \) and \( L_2 \), then the other irreducible component of \( B_\pi \) is a conic of which \( L_1 \) and \( L_2 \) are tangents.
5 Proof of Theorem 4

Let $\pi : X \to \mathbb{P}^n$ be a covering with $g(\pi) = 0$ and assume that $X$ is non-singular. Let $H$ be a hyperplane on $\mathbb{P}^n$ and let $\mathcal{M} = \mathcal{O}(\pi^*H)$. Then the sectional genus $g(X, \mathcal{M})$ is equal to 0. Hence by [1], $(X, \mathcal{M}) \simeq (\mathbb{P}^n, \mathcal{O}(1)), (Q, \mathcal{O}(1)), (\mathbb{P}^2, \mathcal{O}(2))$ or $(\mathcal{E}, \mathcal{O}(1))$, where $Q$ is a quadratic hypersurface on $\mathbb{P}^{n+1}$ and $\mathcal{E}$ is a vector bundle on $\mathbb{P}^1$. In the first and the second cases, $\deg(\pi) = 1$ and 2, respectively. In the third case, $\deg(\pi) = 4$ and $n = 2$. In the last case, there exists a surjective holomorphic map $f : X \to \mathbb{P}^1$ such that the restriction of $f$ to $\pi^{-1}(L)$ is one to one for a generic line $L$ on $\mathbb{P}^n$. Take $f$ as $g_0$ in Section 1. Then

$$(D_1 + D_2 + \cdots + D_d) \cdot \varpi^{-1}(L) = dD_1 \cdot \varpi^{-1}(L) = d\deg(\varpi_0)D_0 \cdot \pi^{-1}(L) = \deg(\varpi).$$

Hence $D \cdot L = 1$. It implies that $\nu$ is defined by linear equations. If $\nu$ is not holomorphic, then the normalization of the fiber product $\mathbb{P}^n \times_{\mathbb{P}^{\deg(\pi)}} (\mathbb{P}^1 \times \mathbb{P}^{\deg(\pi)-1})$ must have singularities. Hence $\nu$ is an immersion. Therfore, $n \leq \deg(\pi)$. So the assertion of the theorem follows from Theorem 2.

6 Proof of Theorem 5

Let $L$ be a generic line on $\mathbb{P}^2$. Assume that $\pi$ is a cyclic covering. Then $\pi_{|\pi^{-1}(L)}$ branches at three points with the ramification index 3, because $\pi^{-1}(L)$ is an elliptic curve. Hence $B_\pi$ is a cubic curve. Now assume that $\pi$ is not a cyclic covering. Let $\varpi : Z \to \mathbb{P}^2$ be a Galois closure of $\pi$. First, we consider the case that $Z$ is an Abelian surface. Since $Z$ is non-singular and $\varpi$ is a $D_6$-covering, $B_\pi$ has only cusps as the singularities. Hence $B_\pi$ is irreducible. If $B_\pi$ is of type $\{3\}$, then $\pi$ must be a cyclic covering. Hence $B_\pi$ is of type $\{2, 1\}$. Therefore, we see by Riemann-Hurwitz formula that it is a sextic. It is well-known that if a $D_6$-cover branched at an irreducible sextic, is an Abelian surface, then the branch locus is a 9 cuspidal sextic.

**Lemma 6.** If $Z$ is not an Abelian surface, then $\dim H^0(X, \mathcal{O}(\pi^*L)) > 3$.

**Proof.** Let $\lambda : \tilde{X} \to X$ be a resolution. Let $L$ be a generic line on $\mathbb{P}^2$ and let $\tilde{L} = (\pi \circ \lambda)^{-1}(L)$. Then $\tilde{L}^2 = 3$, because $\pi$ is a triple covering and $L^2 = 1$. Since $\tilde{L}$ is an elliptic curve, $\tilde{X}$ is a rational surface or an elliptic ruled surface. We obtain by Riemann-Roch Theorem

$$\chi(\mathcal{O}(\tilde{L})) = \frac{1}{2} \tilde{L}(\tilde{L} - K_{\tilde{X}}) + \frac{1}{12}(c_1^2 + c_2).$$

Since $K_{\tilde{X}} \equiv 0$, we have $\tilde{L} \cdot K_{\tilde{X}} = -\tilde{L}^2 = -3$. Hence $\frac{1}{2} \tilde{L} - K_{\tilde{X}} = 3$. On the other hand, $\frac{1}{12}(c_1^2 + c_2) = 1$ or 0, accordingly as $\tilde{X}$ is rational or elliptic ruled. If $\tilde{X}$ is rational, then $\dim H^0(X, \mathcal{O}(\pi^*L)) > 3$, because $\tilde{X}$ is an elliptic surface, $\tilde{X}$ is an Abelian surface, because $\kappa(\tilde{Z}) \leq 0$. Hence there exists an exceptional curve $E$ with $g(E) \geq 1$. Then $H^1(E, \mathcal{O}(\tilde{L})) > 0$, because $\mathcal{O}_E(\tilde{L}) \cong \mathcal{O}$. Consider the exact sequence of cohomology groups followed by the short exact sequence:

$$0 \to \mathcal{O}_{\tilde{X}}(\tilde{L} - E) \to \mathcal{O}_{\tilde{X}}(\tilde{L}) \to \mathcal{O}_E(\tilde{L}) \to 0$$

Then we see that $\dim H^1(X, \mathcal{O}(\pi^*L)) > 0$, because $\dim H^2(\tilde{X}, \mathcal{O}(\tilde{L} - E)) = \dim H^0(\tilde{X}, \mathcal{O}(K_{\tilde{X}} - \tilde{L} + E)) = 0$. Hence $\dim H^0(X, \mathcal{O}(\pi^*L)) > 3$. \qed
Now, we consider the case that $Z$ is not an Abelian surface. Since $\dim H^0(P^2, O(L)) = 3$, there exists an element $\xi_0$ in $H^0(X, O(\pi^* L))$ which is not contained in $\pi^* H^0(P^2, O(L))$, by the above theorem. Let $l$ be a non-zero element in $H^0(P^2, O(L))$. Then $g_0 = \xi_0/\pi^* l$ is a rational function satisfying the condition of §1. Let $h_i$ be the $i$-th symmetric equation of $\xi_1, \xi_2, \xi_3$. Then $h_i \in H^0(P^2, O(iL))$ and $\alpha_i = h_i^{i-1}$. Since

$$P(h_3, h_2l, h_1^2, l^3) = l^6 \left( h_1^2 h_2^2 - 4h_1^3 h_3 - 4h_2^3 + 18h_1 h_2 h_3 - 27h_3^2 \right)$$

$$= \frac{l^6}{27} \left( -(2h_3^3 - 9h_1 h_2 + 27h_3)^2 + 4 \left( h_3^2 - 3h_2 \right)^3 \right),$$

$B_\pi$ is a $(2, 3)$ torus, if $g_3 := 2h_3^3 - 9h_1 h_2 + 27h_3 \neq 0$, $g_2 := h_3^2 - 3h_2 \neq 0$ and $g.c.d.(g_2, g_3) = 1$. Suppose that $g_3 \equiv 0$ and let $U$ be the surface on $P^4$ defined by $2x_2^3 - 9x_2x_3 + 27x_0x_3^2 = 0$. Then $\nu(P^2)$ is contained in $U$. However,

$$2(u_0w_2 + u_1w_1)^3 - 9(u_0w_1 + u_1w_0)(u_0w_2 + u_1w_1)u_1w_2 + 27u_0w_0(u_1w_2)^2$$

$$= (2u_0w_2 - u_1w_1)(u_0^2w_2^2 - u_0u_1w_1w_2 - u_1^2(2w_1^2 - 9w_0w_2)).$$

Hence $\theta_{\pi}^{-1}(U)$ is reducible. Suppose that $g_2 \equiv 0$. Then $(id \times \lambda_2)^{-1}(\mu(P^2))$ is reducible, because

$$(\xi_1 + \xi_2 + \xi_3)^2 - 3(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1) = (\xi_1 + \rho \xi_2 + \rho^2 \xi_3)(\xi_1 + \rho^2 \xi_2 + \rho \xi_3),$$

where $\rho = \exp(2\pi \sqrt{-1}/3)$. Hence $\pi$ must be a cyclic covering. Next, we consider the case that $m = g.c.d.(g_2, g_3)$ is not a constant. If $\deg m = 1$ and if $m \mid h_i$ ($i = 1, 2$), where $h_i = g_{i+1}/m$ for $i = 1, 2$, then $B_\pi = (m) + (h_2^3 - 4mh_1^3)$. If $\deg m = 2$, then we may assume that $m = g_2$ and $B_\pi = (g_2) + (h_2^3 - 4g_2)$, where $h_1 = g_3/g_2$. If $\deg m = 1$ and if $m^2 \mid g_2$, then we may assume that $g_2 = m^2$ and $B_\pi = (m) + (h_2 + 2m^2) + (h_2 - 2m^2)$, where $h_2 = g_3/m$. If $\deg m = 1$ and if $m^2 \mid g_3$, then $B_\pi = (m) + (mh_2^3 - 4h_1^3)$, where $h_0 = g_3/m^2$ and $h_1 = g_2/m$. However, then $g(\pi) = 0$.

References


[3] Hirotaka Ishida and Hiro-o Tokunaga, Triple covers of algebraic surfaces and a generalization of Zariski’s example, preprint