Relations in the canonical ring of a surface

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Abstract

The degree bound for primitive generators and relations of the canonical ring of a minimal surface of general type are studied via Green’s Koszul cohomology, assuming that the fixed part of the canonical linear system does not contain any Francia cycles. Slight refinements of the results due to Ciliberto and Reid are given.

0 Introduction

Let $S$ be a non-singular, projective, minimal surface of general type defined over the complex number field $\mathbb{C}$. The canonical ring of $S$ is the graded $\mathbb{C}$-algebra

$$R(S, K_S) = \bigoplus_{m=0}^{+\infty} H^0(S, mK_S).$$

This naive object has been studied by many authors in order to see what can be expected on the analogy of Max Noether’s and Enriques-Petri’s theorems for curves. We recall here some important results obtained so far. Ciliberto [4] showed, among other things, that $R(S, K_S)$ is generated in degrees $\leq 5$ under some reasonable conditions. Green [6] considered the case that the canonical linear system is free from base points in any dimension by applying his theory of Koszul cohomology groups (see also [7]). Reid [17] showed that it has the 1-2-3 property, that is, it is generated in degrees $\leq 3$ and related in degrees $\leq 6$, when $S$ is a regular surface with $K_S^2 \geq 3$, $p_g(S) \geq 2$ which has an irreducible canonical curve on the canonical model. Mendes Lopes [16] considered surfaces with vanishing geometric genera and showed that $R(S, K_S)$ is generated in degrees $\leq 4$ provided that $2K_S$ is free.

In this article, we study the degree bound for primitive homogeneous generators and relations of $R(S, K_S)$. The key point is the well-known result essentially due to Francia [8] and Reider [20] that $2K_S$ is free except possibly when $p_g(S) = 0$ and $K_S^2 \leq 4$. We apply the machinery of Green’s Koszul cohomology groups ([6], [7]) and show the following:

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\textbf{Theorem 0.1} Let $S$ be a minimal algebraic surface of general type such that $2K_S$ is free. Then $R(S, K_S)$ is generated in degrees $\leq 5$ and related in degrees $\leq 10$. If furthermore $q(S) = 0$, then $R(S, K_S)$ is generated in degrees $\leq 4$ and related in degrees $\leq 8$ except when $(p_a(S), K_S^2) = (2, 1)$.

This is nothing more than what one naturally expects after Ciliberto’s results on generators \cite{4}. Recall that $R(S, K_S)$ needs a generator in degree 4 if the fixed part of $|K_S|$ contains a special configuration of curves called a \textit{Francia cycle}, that is, an effective divisor $E$ with $p_a(E) = -E^2 = 1$ or $p_a(E) = 2$, $E^2 = 0$, and many such surfaces are constructed in \cite{4}. Hence the bound in Theorem 0.1 is sharp for regular surfaces in this sense.

On the other hand, it is an interesting problem \cite{17} to study whether the canonical ring has the 1-2-3 property if the fixed part of $|K_S|$ does not contain any Francia cycles. We consider the problem for a regular surface whose canonical map is not composed of a pencil. However, partly because we do not know much about the fixed loci, we have to impose some extra restrictions modeled on fibred surfaces.

\textbf{Theorem 0.2}. Let $S$ be a minimal algebraic surface of general type with $p_a(S) \geq 2$, $q(S) = 0$ and $K_S^2 \geq 3$. Let $|K_S| = |M| + Z$ be the decomposition into its variable and fixed parts. Suppose that $|M|$ has an irreducible member. If one of the following conditions (1) and (2) is satisfied, then $R(S, K_S)$ is generated in degrees $\leq 3$ and related in $\leq 6$.

\begin{enumerate}
\item[(1)] $H^0(Z, K_Z) = 0$ (possibly $Z = 0$).
\item[(2)] $Z$ does not contain a Francia cycle and decomposes as $Z = \Delta + \Gamma_1 + \cdots + \Gamma_n$, where
\begin{enumerate}
\item[(a)] $\Delta$ is an effective divisor with $K_S \Delta = 0$ (possibly $\Delta = 0$), $\text{Supp}(\Delta) \cap \text{Supp}(Z - \Delta) = \emptyset$, and
\item[(b)] for each $i \in \{1, \ldots, n\}$, $\Gamma_i$ is a chain connected curve such that $K_S \Gamma_i > 0$, $\mathcal{O}_{\Gamma_i}(-\Gamma_i)$ is nef, and $\Gamma_i^2 \leq 0$ holds for any subcurve $\Gamma \preceq \Gamma_i$. Furthermore, $\mathcal{O}_{\Gamma_i}(-\Gamma_j)$ is numerically trivial whenever $j \neq i$.
\end{enumerate}
\end{enumerate}

If the fixed part supports at most exceptional sets of rational singular points, then we have $h^0(Z, K_Z) = 0$. Hence (1) can be regarded as a slight generalization of Reid’s theorem \cite{17}. A curve $D$ is \textit{chain connected}, if either it is reducible, or for any proper subcurve $\Delta \preceq D$ there exists an irreducible component $A \preceq \Delta$ such that $(D - \Delta)A > 0$. We remark that the numerical cycle on a connected bunch of curves with negative semi-definite intersection form serves an example of $\Gamma_i$ in (b). The restrictions in (2) are for technical reasons and should be replaced by a simpler assumption; for example, the intersection form is negative semi-definite on $\text{Supp}(Z)$.

The proof of Theorem 0.2 is based on the hyperplane section principle as in \cite{17} and a result for chain connected curves similar to those in \cite{14} and \cite{12}.

We recall in \S 1 some important notions treating the connectedness of effective divisors. We consider the surjectivity of multiplication maps for semi-negative chain connected curves, and show Theorem 1.5 as a generalization of \cite{14}, which is crucial in the proof of Theorem 0.2. In \S 2, we show Theorem 0.1 by studying the bi-canonical maps. Theorem 0.2 will be shown in \S 3, 4. In Appendix (\S 5), we give a few remarks on the relative canonical algebras for fibrations and singularities, in order to supplement the results in \cite{14} and \cite{12} about Reid’s 1–2–3 conjecture.
1 Chain connected curves

A non-zero effective divisor on a non-singular surface will be called a curve. In this section, we recall basic notions about the connectedness of curves for the later use. Let $D$ be a curve.

For an integer $k$, $D$ is numerically $k$-connected if $D_1D_2 \geq k$ holds for any decomposition $D = D_1 + D_2$ where $D_1$, $D_2$ are curves. Note that a nef and big curve is necessarily 1-connected. In particular, so is a canonical curve of a minimal surface of general type.

**Definition 1.1** A numerically 2-connected curve $E$ is called a Franchetta cycle, if either (i) $p_a(E) = 1$ and $E^2 = -1$, or (ii) $p_a(E) = 2$ and $E^2 = 0$.

A curve as in (i) will be sometimes called a $(−1)$ elliptic cycle.

If $\text{Supp}(D)$ is connected and the intersection form is negative semi-definite on it, then $D$ is called the numerical cycle if $D$ is the smallest curve such that $−D$ is nef on its support. When the intersection form is negative definite, that is, $\text{Supp}(D)$ is the exceptional set of a normal surface singularity, the numerical cycle is usually called the fundamental cycle. A curve $D$ is chain connected, if either it is irreducible, or for any proper subcurve $A \leq D$ there exists an irreducible component $A \leq A$ such that $(D − A)A > 0$. It is easy to see that numerical cycles and 1-connected curves are chain connected.

**Lemma 1.2** For a chain connected curve $D$, the following hold.

1. $H^0(D, O_D) = 1$.
2. If $L$ is a nef line bundle on $D$, then $H^0(D, −L) \neq 0$ if and only if $O_D(L) \cong O_D$.

**Proof.** We first show (2). Let $s \in H^0(D, −L)$ be a non-zero element. If $s$ vanishes identically on some components of $D$, we let $A$ be the biggest subcurve of $D$ on which $s$ vanishes identically. Then $s$ induces a non-zero element of $H^0(D − Δ, −L − Δ)$ which does not vanish identically on any component of $D − Δ$. This implies that $−L − Δ$ is nef on $D − Δ$. On the other hand, by the chain connectedness of $D$, there is an irreducible component $A \leq A$ such that $ΔA > 0$. Then $\text{deg}(−L − Δ)_A \leq −ΔA < 0$, which contradicts what we have just seen above. Hence $s$ does not vanish identically on any component of $D$. Then $−L$ is numerically trivial, since $L$ is nef, and the nowhere vanishing section $s$ induces an isomorphism $O_D \cong O_D(−L)$.

To show (1), we assume that $H^0(D, O_D) > 1$. Then we can find a non-zero element $s \in H^0(D, O_D)$ which vanishes identically on some components of $D$. Take the biggest subcurve $Δ$ on which $s$ vanishes identically and copy the first half of the proof of (2) with $−L$ being replaced by $O_D$. Then it will immediately lead us to a contradiction. □

Let $D$ be a chain connected curve on a non-singular surface $S$. Take an arbitrary subcurve $A_1$ of $D$ and put $D_1 = A_1$. If $D_1 \neq D$, then there is an irreducible component $A_2$ of $D − D_1$ such that $D_1A_2 > 0$. We put $D_2 = D_1 + A_2$. If $D_1$ is defined and $D_1 \neq D$, then we take an irreducible component $A_{i+1}$ of $D − D_i$ such that $D_iA_{i+1} > 0$, and put $D_{i+1} = D_i + A_{i+1}$. In this way, we get a composition series $\{A_i\}_{i=1}^N$ such that $D = \sum_{i=1}^N A_i$. It is said to be irreducible, if the first curve $A_1$ is irreducible. We put $X_i = D − D_{i+1}$ for $0 \leq i \leq N$.

The existence of a composition series enables us to generalize some known results on fundamental cycles (see, e.g., [13], [14]) to chain connected curves. Here, we collect some of them. Proofs are also given for the convenience of readers.
Lemma 1.3 Let $L$ be a line bundle on a chain connected curve $D$ such that $L - K_D$ is nef. Assume that there is an irreducible composition series $\{A_i\}_{i=1}^N$ for $D$ satisfying the following conditions.

(1) $\mathcal{O}_{A_i}(L - K_D) \not\cong \mathcal{O}_{A_i}$.

(2) For $1 \leq i \leq N$, either $\deg (L - K_D)|_{A_i} + D_{i-1}A_i \geq 2$ or $L|_{A_i} \not\cong \mathcal{O}_{A_i}$.

Then $|L|$ has no base points.

Proof. Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{A_i}(L - X_{N-i}) \to \mathcal{O}_{X_{N-i}}(L) \to \mathcal{O}_{X_{N-i}}(L) = 0.$$ 

We have

$$\deg (L - X_{N-i})|_{A_i} = (L - D + D_i)A_i = \deg (L - K_D)|_{A_i} + \deg K_{A_i} + D_{i-1}A_i.$$ 

Since $\mathcal{O}_{A_i}(L - K_D) \not\cong \mathcal{O}_{A_i}$ and $D_{i-1}A_i > 0$ for $i > 1$, we have $H^1(A_i, L - X_{N-i}) = 0$ for all $i$. It follows that the restriction map $H^0(X_{N-i+1}, L) \to H^0(X_{N-i}, L)$ is surjective.

We show that $|L|$ has no base points on $X_j$ by induction on $j$. We have $X_1 = A_N$. We have

$$\deg L|_{A_N} = \deg (L - K_D)|_{A_N} + \deg K_D|_{A_N} = \deg (L - K_D)|_{A_N} + A_N(D - A_N) + \deg K_{A_N}.$$ 

It follows from (2) that $L|_{A_N}$ is free from base points. Assume that $|L|$ has no base points on $X_{N-i}$. It suffices to show that $|L|$ has no base points on $A_j \setminus \text{Supp}(X_{N-i})$ in order to see that $|L|$ has no base points on $X_{N-i+1}$. If $\deg (L - K_D)|_{A_i} + D_{i-1}A_i \geq 2$, then $\mathcal{O}_{A_i}(L - X_{N-i})$ is free. Otherwise if $L|_{A_i} \not\cong \mathcal{O}_{A_i}$, then the restriction map $H^0(X_{N-i+1}, L) \to H^0(A_i, L)$ is non-zero, because $L|_{X_{N-i}}$ has no base points on $A_i \cap \text{Supp}(X_{N-i}) \neq 0$. This shows that $|L|$ has no base points on $X_{N-i+1}$.

\[\square\]

Corollary 1.4 Let $D$ be a chain connected curve on a smooth surface $S$. Assume that $K_S$ and $-D$ are both nef on $D$ and $D^2 < 0$. If $L$ is a line bundle on $D$ such that $L - 2K_S$ is nef on $D$, then $L$ is non-special and $|L|$ is free from base points.

Proof. We have $H^1(D, L) \cong H^0(D, -(L - K_S - D))$ by duality. Since $L - 2K_S$, $D$, and $-D$ are nef and $D^2 < 0$, we see that $L - K_S - D = (L - 2K_S) + K_S - D$ is nef and non-trivial. It follows $H^0(D, -(L - K_S - D)) = 0$ and $L$ is non-special.

Note that we have $A^2 < 0$ for any irreducible component of $D$, because $0 \geq DA = A^2 + (D - A)A$ and $(D - A)A > 0$ by the chain connectedness of $D$. Since $D^2 < 0$, we can take an irreducible composition series $\{A_i\}_{i=1}^N$ whose initial curve $A_1$ satisfies $DA_1 < 0$. We have $L - K_D = (L - 2K_S) + 2K_S - (K_S + D) = (L - 2K_S) + K_S - D$ on $D$ and it is nef. Since $DA_1 < 0$, we have $\deg (L - K_D)|_{A_i} > 0$. For $1 \leq i \leq N$, we fail to have $\deg (L - K_D)|_{A_i} + D_{i-1}A_i \geq 2$ only when $\deg L|_{A_i} = K_S A_i = DA_i = 0$. Then $A_i$ is a $(2)$-curve and $L|_{A_i} \not\cong \mathcal{O}_{A_i}$. Therefore, $|L|$ has no base points.

\[\square\]

Theorem 1.5 Let $D$ be a chain connected curve on a non-singular surface $S$ with $K_S^2 > 0$ such that $K_S$ and $-D$ are both nef on $D$. Assume furthermore that $\Delta^2 \leq 0$ holds for any subcurve $\Delta$ of $D$. Let $L_\alpha$ be a line bundle on $D$ such that $L_\alpha - 2K_S$ is nef ($\alpha = 1, 2$). Then the multiplication map

$$H^0(D, L_1) \otimes H^0(D, L_2) \to H^0(D, L_1 + L_2)$$

is surjective, unless there exists a Francia cycle $E \leq D$ on which $L_1$ and $L_2$ are both numerically equivalent to $2K_S$.
Proof. Note that we have $D^2 \leq 0$ by the nefness of $\mathcal{O}_D(-D)$. Furthermore, for a subcurve $\Delta \leq D$ with $K_S \Delta = 0$, we have $\Delta^2 < 0$ by Hodge’s index theorem, since $K_S^2 > 0$.

If $D$ is irreducible, it is easy check that the multiplication map is surjective unless $D$ is a Francia cycle and $L_1 \equiv L_2 \equiv 2K_S$ on $D$, where the symbol $\equiv$ means the numerical equivalence. So we may assume that $D$ is reducible.

We first consider the extremal case that $D^2 = 0$. Since $-D$ is nef, this implies that $\mathcal{O}_D(D)$ is numerically trivial. Then the intersection form on Supp($D$) is negative semi-definite with the 1-dimensional kernel spanned by $D$. It follows from the chain connectedness that $D$ is the numerical cycle on its support. Then, for any proper subcurve $\Delta$ of $D$, we have $\Delta^2 < 0$ and it follows from $0 = D\Delta = \Delta^2 + \Delta(D - \Delta)$ that $\Delta(D - \Delta) > 0$. This implies that $D$ is numerically 1-connected. We have $K_D = (K_S + D)|_D$ which is nef in the present case. Hence $p_a(D) \geq 1$.

If $p_a(D) = 1$, then $K_D = D^2 = 0$, which is impossible by Hodge’s index theorem because $K_S^2 > 0$. Hence we have $p_a(D) \geq 2$. Since $L_a - 2K_D = (L_a - 2K_S) - 2D|_D$ is nef, the assertion follows from [12, Theorem 1].

We assume that $D^2 < 0$ and apply a slightly modified version of Laufer’s argument [14] as in [12]. We take an irreducible composition series $\{A_i\}_{i=1}^N$ for $D$ satisfying $DA_1 < 0$. Since $L_a - 2K_S$ and $-D$ are nef on $D$, we can show that $H^1(D, L_a) = 0$ and the restriction map $H^0(X_{N-i+1}, L_a) \rightarrow H^0(X_{N-i}, L_a)$ is surjective. This in particular implies that $H^0(D, L_a) \rightarrow H^0(X_1, L_a)$ is surjective for $0 < k < N$, $\alpha = 1, 2$. Furthermore, $|L_a|$ is free from base points by Corollary 1.4. We have the exact sequence

$$0 \rightarrow H^0(A_i, L - X_{N-i}) \rightarrow H^0(X_{N-i+1}, L) \rightarrow H^0(X_{N-i}, L) \rightarrow 0$$

for $L = L_1, L_2, L_1 + L_2$. Then we see that $H^0(X_{N-i+1}, L_1) \otimes H^0(X_{N-i+1}, L_2) \rightarrow H^0(X_{N-i+1}, L_1 + L_2)$ is surjective provided that so are $H^0(X_{N-i}, L_1) \otimes H^0(X_{N-i}, L_2) \rightarrow H^0(X_{N-i}, L_1 + L_2)$ and

(1.1) $H^0(A_i, L_1 - X_{N-i}) \otimes W_{2,i} \oplus W_{1,i} \otimes H^0(A_i, L_2 - X_{N-i}) \rightarrow H^0(A_i, L_1 + L_2 - X_{N-i})$,

where $W_{1,i} = \text{Im}(H^0(D, L_a) \rightarrow H^0(A_i, L_a))$. By induction, in order to show that $H^0(D, L_1) \otimes H^0(D, L_2) \rightarrow H^0(D, L_1 + L_2)$ is surjective, it suffices to show that (1.1) is surjective for any $i$, $1 \leq i \leq N$. Since $L_a - 2K_S$ is nef, we have $\text{deg}(L_a - K_S)|_{A_i} \geq \text{deg} K_S|_{A_i} \geq 0$. If $L_a - K_S$ is of degree zero on $A_i$, then $A_i$ is a $(2,-2)$-curve and (1.1) is clearly surjective, because $L_a|_{A_i}$ is trivial and $\text{Bs}|L_a| = \emptyset$. This enables us to assume that $\text{deg}(L_a - K_S)|_{A_i} > 0$ for $\alpha = 1, 2$ in what follows.

We assume that $\text{deg} L_1|_{A_i} \geq \text{deg} L_2|_{A_i}$ and consider the multiplication map $\mu_{2,i} : H^0(A_i, L_1 - X_{N-i}) \otimes W_{2,i} \rightarrow H^0(A_i, L_1 + L_2 - X_{N-i})$. We shall show that either $\mu_{2,i}$ is surjective or there is a Francia cycle $\Delta$ containing $A_i$ and $L_2 \equiv 2K_S$ on $\Delta$. By the duality and the vanishing theorems in [7], $\mu_{2,i}$ is surjective if $h^0(A_i, K_A + L_2 - L_1 + X_{N-i}) \leq \text{dim} W_{2,i} - 2$. Hence we only have to check that

(1.2) $\dim W_{2,i} \geq \begin{cases} p_a(A_i) + 2, & \text{if } K_A + L_2 - L_1 + X_{N-i} \text{ is special} \\ p_a(A_i) + 1 + \text{deg}(L_2 - L_1)|_{A_i} + (D - D_{i-1})A_i - A_i^2, & \text{otherwise}. \end{cases}$

Assume that $W_{2,i} \equiv h^0(A_i, L_2)$. Then $\dim W_{2,i} \equiv \text{deg} L_2|_{A_i} + 1 - p_a(A_i)$. The inequality $\dim W_{2,i} \geq p_a(A_i) + 2$ is equivalent to $\text{deg}(L_2 - K_S)|_{A_i} - A_i^2 \geq 3$, which fails only when $K_S A_i = 1$, $A_i^2 = -1$ and $\text{deg} L_2|_{A_i} = 2$, that is, $A_i$ is a Francia cycle of arithmetic genus one on which
$L_2 \equiv 2K_S$. The second inequality in (1.2) is equivalent to $\deg(L_1 - K_S)|_{A_i} - DA_i + D_{i-1}A_i \geq 2$, which always holds under our assumptions.

Assume that $W_{2,i} \neq H^0(A_i, L_2)$ and let $\Delta_i = \Delta_{2,i}$ be the smallest subcurve of $D$ containing $A_i$ such that $H^0(D, L_2) \rightarrow H^0(\Delta_i, L_2)$ is surjective. Since $H^0(D, L_2) \rightarrow H^0(X_{N-i+1}, L_2)$ is surjective, we have $\Delta_i \leq X_{N-i+1}$. In particular, we have $\Delta_i = D$ only when $i = 1$. It follows from [12, Lemma 2.2.1] that we have either (i) $\Delta_i - (L_2 - K_S)$ is nef on $\Delta_i$, or (ii) $A_i$ is of multiplicity one in $\Delta_i$, $\Delta_i - (L_2 - K_S)$ is nef on $\Delta_i - A_i$ and $\mathcal{O}_{\Delta_i}(L_2 - (\Delta_i - A_i))$ is non-special; furthermore,

\begin{equation}
(1.3) \quad \dim W_i \geq 1 + h^0(A_i, L_2 - (\Delta_i - A_i)) = p_a(A_i) + \deg(L_2 - K_S)|_{A_i} + (D - \Delta_i)A_i - DA_i,
\end{equation}

since $[L_2]$ has no base points.

We exclude the possibility (i). Assume that $\Delta_i - (L_2 - K_S)$ is nef on $\Delta_i$. Then we have $\Delta_i A \geq \deg(L_2 - K_S)|_{A_i} \geq K_S A \geq 0$ and, hence, $(D - \Delta_i)A \leq 0$ holds for any irreducible component $A$ of $\Delta_i$. Since $D$ is chain connected, this is possible only if $\Delta_i = D$. If $\Delta_i = D$, then we have $(D - (L_2 - K_S))D \geq 0$ and it would follow $D^2 \geq 0$, which is absurd. Hence (i) is impossible, and we are in the case (ii). In particular, we must have $(D - \Delta_i)A_i > 0$ except when $i = 1$, $\Delta_1 = D$.

We use (1.3) to examine (1.2). If $K_A + L_2 - L_1 + X_{N-i}$ is special on $A_i$, it suffices to check that $\deg(L_2 - K_S)|_{A_i} + (D - \Delta_i)A_i - DA_i \geq 2$, which is direct. If it is non-special on $A_i$, then a sufficient condition for (1.2) is that $2p_a(A_i) + (D - \Delta_i)A_i = 2DA_i + D_{i-1}A_i + \deg(L_2 - K_S)|_{A_i} \geq 3$. This does not hold in the following cases:

(a) $i = 1, \Delta_1 = D, DA_1 = -1, p_a(A_1) = \deg(L_1 - 2K_S)|_{A_1} = 0$.

(b) $i \geq 2, p_a(A_i) = DA_i = \deg(L_1 - 2K_S)|_{A_i} = 0, (D - \Delta_i)A_i = D_{i-1}A_i = 1$.

In either case, $L_1, L_2$ and $2K_S$ are of the same degree on $A_i$.

Consider the case (a). Recall that $D - (L_2 - K_S)$ is nef on $D - A_1$. Since $L_2 - 2K_S$ and $-D$ are both nef, we see that every component of $D - A_1$ is a $(-2)$-curve, $(D(D - A_1)) = 0$ and $L_2 \equiv 2K_S$ on $D$. Consider the cohomology long exact sequence for

\[ 0 \rightarrow \mathcal{O}_{D-A_1}(-A_1) \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_{A_1} \rightarrow 0. \]

We have $h^0(D, \mathcal{O}_D) = 1$ and $h^0(D, \mathcal{O}_D) \rightarrow h^0(A_1, \mathcal{O}_{A_1})$ is non-trivial. This shows that $H^0(D - A_1, \mathcal{O}(-A_1)) = 0$. Recall that we have $\mathcal{O}_{D-A_1}(L_2) \equiv \mathcal{O}_{D-A_1}$. It follows from the exact sequence

\[ 0 \rightarrow H^0(D - A_1, L_2 - A_1) \rightarrow H^0(D, L_2) \rightarrow H^0(A_1, L_2) \]

and $H^0(D - A_1, L_2 - A_1) \cong H^0(D - A_1, \mathcal{O}_{D-A_1}(-A_1)) = 0$ that $\dim W_1 = h^0(D, L_2) = 2K_SD - D(K_S + D)/2 = (3/2)K_SA - D^2/2$. Hence the inequality in (1.2) becomes $(1/2)(KSA_1 - D^2) - DA_1 \geq 3$. This does not hold only when $KSA_1 = -D^2 = -DA_1 = 1$, that is, $D$ is a $(1)$-elliptic cycle on which $L_2 \equiv 2K_S$. We show that such a $(1)$ elliptic cycle $D$ is numerically 2-connected. Let $D = A + B$ be any effective decomposition of $D$. We can assume that $A_1$ is contained in $A$. Then $B$ consists of $(-2)$-curves and it follows that $B^2$ is a negative even integer. Recall that $D$ is numerically trivial on $D - A_1$. In particular, we have $0 = DB = AB + B^2$. Hence we get $AB \geq 2$ and conclude that $D$ is 2-connected.

Consider the case (b). (1.2) is now dim $W_{2,i} \geq -A_i^2 = K_S A + 2$. We have $\Delta_iA_i = -1$ and $(\Delta_i - A_i)\Delta_i \geq \deg(L_2 - K_S)|_{\Delta_i - A_i} \geq K_S(\Delta_i - A_i) \geq 0$. Since $0 \geq \Delta_i^2 = (\Delta_i - A_i)\Delta_i + A_i\Delta_i \geq 0$,
$K_S(\Delta_i - A_i) - 1 \geq -1$, we have either $\Delta_i^2 = -1$ or $\Delta_i^2 = 0$. Assume that $\Delta_i^2 = -1$. Then $\Delta_i - A_i$ consists of $(-2)$-curves and $L_2 \equiv 2K_S$ on $\Delta_i$. We can show that $\Delta_i$ is numerically 2-connected as in (a). Then $h^0(\Delta_i, \mathcal{O}_\Delta) = 1$ and we get $\dim W_{2,i} = h^0(\Delta_i, L_2) = (3/2)K_S A_i + 1/2$ similarly as in (a). The desired inequality $\dim W_{2,i} \geq K_S A_i + 2$ holds except when $K_S A_i = 1$, that is, $\Delta_i$ is a Francia cycle of arithmetic genus one on which $L_2 \equiv 2K_S$.

It remains to consider the case that $\Delta_i^2 = 0$. There is a unique irreducible component $B_i$ of $\Delta_i$ of multiplicity one such that $\Delta_i B_i = 1$. Then $\Delta_i - A_i - B_i$ consists of $(-2)$-curves at most. Furthermore, $\Delta_i$ is numerically trivial on $\Delta_i - A_i - B_i$. We have $1 = \Delta_i B_i \geq \deg (L_2 - K_S) B_i \geq K_S B_i$. It follows that $B_i$ is a $(2)$-curve, a $(3)$-curve or a $(1)$-elliptic curve. We claim that $\Delta_i$ is numerically 2-connected. Let $\Delta_i = A + B$ be any effective decomposition. We may assume that $A_i \leq A$. Then $AB + B^2 = \Delta_i B$ which is equal to 1 or 0 according to whether $B_i \leq B$ or not. If $B$ consists of $(2)$-curves, then $B^2 \leq -2$ and we get $AB \geq 2$. If $B_i$ is either a $(3)$-curve or a $(1)$-elliptic curve and $B_i \leq B$, then we have $K_S B_i = 1$ and it follows that $B^2$ is non-positive odd integer. Hence we get $AB \geq 2$. We have shown that $\Delta_i$ is numerically 2-connected. Then $h^0(\Delta_i, \mathcal{O}_\Delta) = 1$.

If $L_2 - K_S$ is trivial on $B_i$, then $B_i$ is a $(2)$-curve and we have $h^0(\Delta_i - A_i, L_2 - A_i) = h^0(\Delta_i - A_i, \mathcal{O}(\Delta_i)) = 0$ and $\dim W_{2,i} = h^0(\Delta_i, L_2) = (3/2)K_S A_i$ as in (a). Then we see that $\dim W_{2,i} \geq K_S A_i + 2$ does not hold only when $K_S A_i = 2$, that is, $\Delta_i$ is a Francia cycle of arithmetic genus 2 on which $L_2 \equiv 2K_S$.

Assume that $L_2 - K_S$ is of degree 1 on $B_i$. We apply the argument in [12, pp.189–190] to proceed further. We claim that $\Delta_i - B_i$ is numerically 0-connected and it is even 1-connected unless $B_i$ is a $(3)$-curve. Let $\Delta_i - B_i = A + B$ be any effective decomposition. Since $\Delta_i$ is numerically 2-connected, we have $(A + B_i) B \geq 2$ and $(B + B_i) A \geq 2$. Then $AB \geq 2 - \min(AB_i, BB_i)$. We have $\Delta_i - B_i, B_i = AB_i + BB_i$, which equals to 4, 3 or 2 according as $B_i$ is a $(3)$-curve, $(2)$-curve or a $(1)$-elliptic curve. Since $B_i$ is of multiplicity 1 in $\Delta_i, AB_i$ and $BB_i$ are both non-negative. It follows that one of them is not greater than 2, and we get $AB \geq 0$ when $B_i$ is a $(3)$-curve. Similarly we have $AB > 0$ when $B_i$ is either a $(2)$-curve or a $(1)$-elliptic curve. This shows that $\Delta_i - B_i$ is at least numerically 0-connected and it fails to be 1-connected only when $B_i$ is a $(3)$-curve. Suppose that $\Delta_i - B_i$ is not 1-connected and $h^0(\Delta_i - B_i, \mathcal{O}(\Delta_i - B_i)) = n \geq 2$. Then $B_i$ is a $(3)$-curve, and it follows from [15, Proposition 2.4] that there is an effective decomposition $\Delta_i - B_i = \Gamma_1 + \cdots + \Gamma_n$ such that $h^0(\Gamma_k, \mathcal{O}(\Gamma_k)) = 1$ and $(\Delta_i - B_i - \Gamma_k) \Gamma_k = 0$ for each $k \in \{1, 2, \ldots, n\}$. The last implies that $(\Delta_i - \Gamma_k) \Gamma_k = B_i \Gamma_k \geq 2$, because $\Delta_i$ is 2-connected. Then it follows from $4 = (\Delta_i - B_i) B_i = B_i (\Gamma_1 + \cdots + \Gamma_n)$ that we have $n = 2$ and $B_i \Gamma_1 = B_i \Gamma_2 = 2$. We have shown that $h^0(\Delta_i - B_i, \mathcal{O}(\Delta_i - B_i)) \leq 2$. We have $\dim W_{2,i} = h^0(\Delta_i, L_2) - h^0(\Delta_i - A_i, L_2 - A_i)$. Consider the cohomology long exact sequence for

$$0 \rightarrow \mathcal{O}(L_2 - \Delta_i + B_i) \rightarrow \mathcal{O}(L_2 - A_i) \rightarrow \mathcal{O}(L_2 - A_i).$$

We have $h^0(\Delta_i - A_i, L_2 - A_i) \leq h^0(\Delta_i - A_i - B_i, L_2 - A_i) + h^0(B_i, L_2 - \Delta_i + B_i)$. Note that $\mathcal{O}(L_2 - \Delta_i + B_i)$ is of degree 2 if $B_i$ is a $(2)$-curve or a $(3)$-curve and is of degree zero if $B_i$ is a $(1)$-elliptic curve. Since $\Delta_i - A_i - B_i$ consists of $(-2)$-curves and we have $L_2 \equiv 2K_S$ on it, we get $h^0(\Delta_i - A_i - B_i, L_2 - A_i) = h^0(\Delta_i - A_i - B_i, \mathcal{O}(\Delta_i)) = h^0(\Delta_i - B_i, \mathcal{O}(\Delta_i - B_i)) - 1$. In sum, we get the inequality $\dim W_{2,i} \geq h^0(\Delta_i, L_2) - \epsilon$, where $\epsilon = 0, 1$ and we have $\epsilon = 1$ only when $B_i$ is a $(1)$ elliptic curve or a $(3)$-curve with $\Delta_i - B_i$ being 1-disconnected. Hence we fail to have $\dim W_{2,i} \geq K_S A_i + 2$ only when $K_S A_i = 1$ and $\epsilon = 1$. In the exceptional case, $\Delta_i$ is a Francia cycle of arithmetic genus two on which $L_2 \equiv 2K_S$. 7
Now we exchange $L_1$ and $L_2$, and consider the multiplication map $\mu_{1,i} : W_{1,i} \otimes H^0(A_i, L_2 - X_{N-i}) \to H^0(A_i, L_1 + L_2 - X_{N-i})$ for each index $i$ for which we failed to show the surjectivity of $\mu_{2,i}$. Since we can assume $\Delta_{1,i} = \Delta_{2,i}$ by the following lemma, the assertion follows. \hfill \Box

**Lemma 1.6** Let $S$ and $D$ be as in the previous theorem. If $E$ is a subcurve with $E^2 = 0$ or $-1$, then the restriction map $H^0(D, L) \to H^0(E, L)$ is surjective for any line bundle $L$ on $D$ such that $L - 2K_S$ is nef.

**Proof.** We assume that the restriction map is not surjective and show that this leads us to a contradiction. Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{D-E}(L-E) \to \mathcal{O}_{D}(L) \to \mathcal{O}_{E}(L) \to 0.$$ 

Since $H^1(D, L) = 0$, we must have $H^1(D-E, L-E) \neq 0$. By the duality, we get $H^1(D-E, L-E)^\vee = H^0(D-E, D-(L-K_S))$. Let $s \in H^0(D-E, D-(L-K_S))$ be a non-zero element. If $Z$ is the biggest subcurve such that $s \equiv 0$ on $Z$ (possibly $Z = 0$), then $D - (L - K_S) - Z$ is nef on $D - E - Z$. In particular, we have $D - Z)(D - E - Z) \geq \deg(L - K_S))_{D-E-Z} \geq 0$. We have

$$0 \leq (D-Z)(D-E-Z) = (D-Z)^2 - E(D-Z) = (D-E-Z)^2 + E(D-Z) - E^2$$

Since $(D-Z)^2 \leq 0$ and $(D-E-Z)^2 \leq 0$, we get $E^2 \leq E(D-Z) \leq 0$. Then it is easy to see that $(D-Z)(D-E-Z) = 0$ because $E^2 = 0$, $-1$. This implies that $L-K_S$ is of degree zero on $D-E-Z$ and it follows that $D-E-Z$ consists of $(-2)$-curves, because $L - 2K_S$ is nef. Thus $(D-E-Z)^2$ is a negative even integer. But then we have $(D-Z)(D-E-Z) \leq -2 + E(D-Z) - E^2 \leq -1$, a contradiction. \hfill \Box

## 2 General degree bounds

Let $S$ be a minimal surface of general type. It is known that $2K_S$ is free when $p_g(S) > 0$ or $p_g = 0$, $K_S^2 \geq 5$ (see, [8], [20], [2], [3]). Throughout the paper, we assume that $2K_S$ is free. Put $r = P_2(S) - 1 = K_S^2 + \chi(\mathcal{O}_S) - 1$. Note that we have $r = 1$ if and only if $p_g = q = 0$, $K_S^2 = 1$, that is, $S$ is a numerical Godeaux surface. Since we have assumed that $2K_S$ is free, we have $r \geq 2$ and therefore exclude Godeaux surfaces from our considerations.

Green developed the theory of Koszul cohomology groups in [6] and [7] which gives us a powerful tool for studying graded rings. We start with recalling a general lemma which will be used frequently.

**Lemma 2.1** (Green) Let $L$ be an invertible sheaf on a projective scheme $X$ over $\mathbb{C}$ and let $k$ be the smallest positive integer such that $kL$ is generated by global sections. Assume that the graded ring $R(X, L) = \bigoplus_{m \geq 0} H^0(X, mL)$ is generated in degrees $\leq d$. Put

$$d_0 = \min_{n \in \mathbb{Z}_{>0}} \{H^0(X, kL) \otimes H^0(X, (m-k)L) \to H^0(X, mL) \text{ is surjective for any } m > n\}$$

$$d_1 = \min_{n \in \mathbb{Z}_{>0}} \{2\ H^0(X, kL) \otimes H^0(X, (m-2k)L) \to H^0(X, kL) \otimes H^0(X, (m-k)L) \to H^0(X, mL) \text{ is exact at the middle term for any } m > n\}$$

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Then \( R(X, L) \) is related in degrees \( \leq \max(d + d_0, d_1) \). If \( d_0 \) is not a multiple of \( d \), then \( R(X, L) \) is related in degrees \( \leq \max(d + d_0 - 1, d_1) \).

Proof. Though this is contained in the proof of [6, Theorem 3.11], we recall the argument for the later use.

Choosing a minimal set of homogeneous generators of \( R(X, L) \), we have a surjective homomorphism \( \Phi : \Omega \to R(X, L) \), where \( \Omega \) is a polynomial ring over \( \mathbb{C} \). We let \( \Omega = \oplus_{m \geq 0} \Omega_m \) be the decomposition into the graded pieces such that \( \Phi \) becomes a homomorphism of graded algebras. The kernel \( I \) of \( \Phi \) is a homogeneous ideal and we denote it as \( I = \oplus_{m \geq 0} I_m \). We let \( \{\xi_1, \ldots, \xi_N\} \) be a basis for \( \Omega_k \).

Assume that \( n > \max(d + d_0, d_1) \) and let \( F \in I_n \) be a relation in degree \( n \). We shall show that \( F \equiv 0 \) modulo relations in lower degrees. Let \( G \) be any monomial appearing in \( F \). Since \( n > d + d_0 \), we see that \( G \) can be divided by a monomial \( G_1 \) satisfying \( d_0 < \deg G_1 \leq d + d_0 \). Then, since \( \deg G_1 > d_0 \), \( \Phi(G_1) \) can be expressed as a sum of products of degree \( k \) and degree \( n - k \) elements in \( R(X, L) \). Therefore, we can write \( F = \sum_{i=1}^{N} \xi_i F_i \) modulo relations in degrees \( \leq \max(n - k, d + d_0) \), where the \( F_i \)'s are homogeneous polynomials of degree \( n - k \). Then, since \( n > d_1 \), we can find polynomials \( F_{ij} \) of degree \( n - 2k \) satisfying \( F_i = \sum_{j} \xi_j F_{ij} \) and \( F_{ij} = -F_{ji} \), modulo relations in degrees \( < n \). Then we obtain \( F = \sum_{i} \xi_i F_i = \sum_{i \leq j} \xi_i \xi_j (F_{ij} + F_{ji}) = 0 \) modulo relations in lower degrees.

If \( d_0 \) is not a multiple of \( d \), then the above argument works also when \( n = d_0 + d \). \( \square \)

The following two lemmas are easy applications of the results in [7] such as the duality theorem, the vanishing theorem and \( K_p \) theorem, which combined with Lemma 2.1 can cover a major part of Theorem 0.1.

**Lemma 2.2** Suppose that \( 2K_S \) is free. Then the multiplication map
\[
H^0(S, 2K_S) \otimes H^0(S, (m - 2)K_S) \to H^0(S, mK_S)
\]
is surjective in the following cases:

1. \( m > 7 \).
2. \( m > 6 \) and \( K_S^2 + p_g(S) - q(S) \geq 3 \).
3. \( m > 5 \) and \( K_S^2 \geq q(S) + 2 \).
4. \( m > 4 \), \( q(S) = 0 \) and the bi-canonical image of \( S \) is not a surface of minimal degree.

Proof. Since \( S \) is a minimal surface of general type, we have \( H^1(S, iK_S) = 0 \) when \( i \neq 0, 1 \). By the duality theorem [7, Theorem (2.c.6)], the multiplication map is surjective for \( m \geq 5 \) if and only if the Koszul sequence
\[
\begin{align*}
\bigwedge^{r-1} H^0(S, 2K_S) \otimes H^0(S, (5 - m)K_S) &\to \bigwedge^{r-2} H^0(S, 2K_S) \otimes H^0(S, (7 - m)K_S) \\
&\to \bigwedge^{r-3} H^0(S, 2K_S) \otimes H^0(S, (9 - m)K_S)
\end{align*}
\]

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is exact at the middle term, except when $m = 5$, $q(S) > 0$. Hence we get the assertion when $m \geq 8$; $m = 7$ and $r \geq 3$; $m = 6$ and $p_g \leq r - 2$; $m = 5$, $q(S) = 0$ and the bi-canonical image is not a surface of minimal degree, by the vanishing theorem and the $K_p,1$ theorem [7, Theorem (3.a.1), Theorem (3.c.1)].

Quite similarly, we can show the following:

**Lemma 2.3** Assume that $2K_S$ is free. Then the Koszul sequence

$$\bigwedge^2 H^0(2K_S) \otimes H^0((m - 4)K_S) \to H^0(2K_S) \otimes H^0((m - 2)K_S) \to H^0(mK_S)$$

is exact at the middle term in the following cases:

1. $m > 9$,
2. $m > 8$ when $K_S^2 + p_g(S) \geq q(S) + 4$,
3. $m > 7$ when $K_S^2 \geq q(S) + 3$.

Another strategy is the hyperplane section principle. Assume that $S$ is a regular surface with $p_g(S) > 0$ and let $D \in [K_S]$ be a general member. Put $L = K_S|_D$. Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_S(mK_S - D) \to \mathcal{O}_S(mK_S) \to \mathcal{O}_D(mL) \to 0$$

for any non-negative integer $m$. Since $S$ is a minimal regular surface of general type, we have $H^1(S, mK_S - D) = H^1(S, (m - 1)K_S) = 0$ and the restriction map $H^0(S, mK_S) \to H^0(D, mK_S)$ is surjective. Therefore, the restriction maps induce a surjective homomorphism of graded $\mathbb{C}$-algebras

$$(2.1) \quad R(S, K_S) = \bigoplus_{m \geq 0} H^0(S, mK_S) \to R(D, L) = \bigoplus_{m \geq 0} H^0(D, mL).$$

The kernel is the homogeneous ideal of $R(S, K_S)$ generated by a single element in degree one, that is, a section defining $D$. Therefore, for example, in order to show that $R(S, K_S)$ is generated in degrees $\leq 3$ and related in degrees $\leq 6$, it is sufficient to show the same things for $R(D, L)$.

Now, we are going to show Theorem 0.1. Our basic assumption is that $2K_S$ is free. Though a considerable part inevitably overlaps with [4] and [22], we do not exclude it for the convenience of readers, unless it involves too much. Among other things, we shall freely use basic inequalities in the surface geography: $K_S^2 > 0$, $\chi(\mathcal{O}_S) > 0$, $K_S^2 \geq 2p_g(S) - 4$, and $K_S^2 \geq 2p_g(S)$ when $q(S) > 0$ (see [5]).

We study the exceptional cases in (2), (3), (4) of Lemma 2.2.

(a) Suppose that $K_S^2 + p_g - q = 2$. Then we have $K_S^2 = 2$, $p_g = q$; $K_S^2 = 1$, $p_g = q + 1$. If $q(S) = 0$, then $(p_g, K_S^2) = (0, 2), (1, 1)$. If $q(S) > 0$, then $K_S^2 = 2$ and $p_g = q = 1$, because $K_S^2 \geq 2p_g(S)$ holds for irregular surfaces of general type.
Lemma 2.4 Assume that \( (p_q, q, K_S^2) = (0, 0, 2), (1, 0, 1), (1, 1, 2) \). Then the following hold:

1. \( \text{corank} H^0(S, 2K_S) \otimes H^0(S, 5K_S) \to H^0(S, 7K_S) = 1 \).
2. \( \text{corank} H^0(S, 2K_S) \otimes H^0(S, 4K_S) \to H^0(S, 6K_S) = p_q(S) \).
3. \( \text{corank} H^0(S, 2K_S) \otimes H^0(S, 3K_S) \to H^0(S, 5K_S) = 3q(S) \).

Furthermore, the Koszul sequence

\[
\bigwedge^2 H^0(S, 2K_S) \otimes H^0(S, (m - 4)K_S) \to H^0(S, 2K_S) \otimes H^0(S, (m - 2)K_S) \to H^0(S, mK_S)
\]

is exact at the middle term for \( m \geq 8 \) when \( q(S) = 1 \); for arbitrary \( m \) when \( q(S) = 0 \).

Proof. Let

\[
0 \to \mathcal{E} \to H^0(S, 2K_S) \otimes \mathcal{O}_S \to \mathcal{O}_S(2K_S) \to 0
\]

be the exact sequence obtained by the evaluation map. If \( m \geq 4 \), then we have \( H^1(S, (m - 2)K_S) = 0 \) and we see that the cokernel of \( H^0(S, 2K_S) \otimes H^0(S, (m - 2)K_S) \to H^0(S, mK_S) \) is isomorphic to \( H^1(S, \mathcal{E} \otimes \mathcal{O}_S((m - 2)K_S)) \) which is dual to \( H^1(S, \mathcal{E} \otimes \mathcal{O}_S((3 - m)K_S)) \). Since \( \bigwedge^2 \mathcal{E} \cong \mathcal{O}_S(-2K_S) \), we have only to calculate \( H^1(S, \mathcal{E} \otimes \mathcal{O}_S((5 - m)K_S)) \). It follows from the exact sequence

\[
H^0(S, 2K_S) \otimes H^0(S, (5 - m)K_S) \to H^0(S, (7 - m)K_S) \to H^1(S, \mathcal{E}((5 - m)K_S))
\]

\[
\to H^0(S, 2K_S) \otimes H^1(S, (5 - m)K_S) \to H^1(S, (7 - m)K_S)
\]

that we have

\[
H^1(S, \mathcal{E}(5 - m)K_S) \cong \begin{cases} H^0(S, (7 - m)K_S) & \text{when } m = 6, 7 \\ H^0(S, 2K_S) \otimes H^1(S, \mathcal{O}_S) & \text{when } m = 5. \end{cases}
\]

Hence we get (1)–(3). As to the last assertion, we have only to check whether \( H^1(S, (m - 6)K_S) \) vanishes or not, since \( \bigwedge^2 \mathcal{E} \cong \mathcal{O}_S(-2K_S) \). \( \square \)

When \( K_S^2 = p_g = 1 \) and \( q = 0 \), it is known [1] that the canonical model of \( S \) is isomorphic to a complete intersection of type \( (6, 6) \) in \( \mathbb{P}(1, 2, 2, 3, 3) \). Therefore, \( R(S, K_S) \) is generated in degrees \( \leq 3 \) and related in degrees \( \leq 6 \). We study the other two cases.

Lemma 2.5 Assume that \( K_S^2 = 2 \) and \( p_g = q = 0 \). If \( 2K_S \) is free, then \( R(S, K_S) \) is generated in degrees \( \leq 4 \) and related in degrees \( \leq 8 \).

Proof. We shall show that \( H^0(S, 3K_S) \otimes H^0(S, 4K_S) \to H^0(S, 7K_S) \) is surjective. By the duality theorem, it suffices to show that

\[
\bigwedge^5 H^0(S, 3K_S) \to \bigwedge^4 H^0(S, 3K_S) \otimes H^0(S, 3K_S) \to \bigwedge^3 H^0(S, 3K_S) \otimes H^0(S, 6K_S)
\]

is exact at the middle term. By \( K_{p_g} \)-theorem, we have the claim unless the tri-canonical image of \( S \) is a surface of degree 5 in \( \mathbb{P}^5 \). Since we have \( (3K_S)^2 = 18 \) which is not a multiple of 5, we are done. This and Lemma 2.4 show that \( R(S, K_S) \) is generated in degrees \( \leq 4 \). Then
putting \( d = 4, \ d_0 = 7, \ d_1 = 0 \) and \( k = 2 \), Lemma 2.1 shows that it is related in degrees \( \leq 10 = 4 + 7 - 1 \). Since \( p_q = 0 \), we have no generators in degree 1. If there is a relation of degree 9 or 10, then any monomial appearing the relation can be divided by a monomial of degree 5 or 6. Then, using Lemma 2.4, the proof of Lemma 2.1 shows that such a relation can be reduced to relations in degrees \( \leq 8 \). □

**Lemma 2.6** Assume that \( K_S^2 = 2, \ p_g = q = 1 \). Then \( R(S, K_S) \) is generated in degrees \( \leq 5 \) and related in degrees \( \leq 10 \).

Proof. Surfaces with such numerical invariants are studied in [11]. We freely use the results there. The Albanese map \( f : S \to B = \text{Alb}(S) \) is a fibration of genus 2. Then \( f_*\omega_S \) is an indecomposable, nef locally free sheaf of rank two and of degree one. Since it has no locally free quotient of degree zero, we have an exact sequence

\[
0 \to O_B \to f_*\omega_S \to O_B(P) \to 0,
\]

for a point \( P \in B \). Note that the relative canonical model of \( S \) is isomorphic to the finite double covering of \( \mathbb{P} = \mathbb{P}(f_*\omega_S) \) with branch locus in \([6\Delta - 2\Gamma]\), where \( \Delta \) is a tautological divisor and \( \Gamma \) is the fibre of \( \mathbb{P} \to B \) over \( P \). \( K_S \) is induced by \( \Delta \). Furthermore, we have the decomposition \( H^0(S, mK_S) \to H^0(\mathbb{P}, m\Delta) \oplus H^0(\mathbb{P}, (m - 3)\Delta + \Gamma) \) into the \((\pm 1)\)-eigen spaces under the action of the covering transformations.

We first claim that the invariant part \( \bigoplus_{m \geq 0} H^0(\mathbb{P}, m\Delta) \) is generated in degrees \( \leq 4 \). We show that \( H^0(\mathbb{P}, 2\Delta) \oplus H^0(\mathbb{P}, (m - 2)\Delta) \to H^0(\mathbb{P}, m\Delta) \) is surjective for \( m \geq 5 \). By the duality theorem, it suffices to show that \( H^1(\mathbb{P}, (6 - m)\Delta + K\mathbb{P}) = 0 \). Since the canonical bundle of \( \mathbb{P} \) is given by \(-2\Delta + \Gamma\), we are done. As to the anti-invariant part, we consider

\[
\bigoplus_{i=1}^{m-3} H^0(\mathbb{P}, i\Delta) \oplus H^0(\mathbb{P}, (m - 3 - i)\Delta + \Gamma) \to H^0(\mathbb{P}, (m - 3)\Delta + \Gamma).
\]

Note that the image of \( H^0(\Delta) \otimes H^0((m - 4)\Delta + \Gamma) \) consists of those sections vanishing on \( \Delta \), whilst that of \( H^0((m - 3)\Delta) \otimes H^0(\Gamma) \) consists of sections vanishing on \( \Gamma \). In particular, when \( m = 5 \), this implies that we need a new generator in degree 5 which does not vanish at \( \Delta \cap \Gamma \). If \( m \geq 6 \), then the image of \( H^0(2\Delta) \otimes H^0((m - 5)\Delta + \Gamma) \) contains a section not vanishing at \( \Delta \cap \Gamma \).

This and Lemma 2.4 imply that \( R(S, K_S) \) is generated in degrees \( \leq 5 \).

It remains to show that \( R(S, K_S) \) is related in degrees \( \leq 10 \). By Lemma 2.1 with \( d = 5, \ d_0 = d_1 = 7 \) and \( k = 2 \), we see that it is related in degrees \( \leq 11 = 5 + 7 - 1 \). Assume that we have a relation in degree 11. Let \( G \) be a monomial of degree 11 appearing in the relation. Then it is easy to see that \( G \) can be divided by a monomial \( G_1 \) of degree 8 or 9. Hence, as in the proof of Lemma 2.1, we can show that any relations in degree 11 are induced by relations in lower degrees. □

(b) Suppose that \( K_S^2 \leq q(S) + 1 \) but \( K_S^2 + p_g - q \geq 3 \). By Noether’s inequality and the fact that \( K_S^2 \geq 2p_g \) holds for irregular surfaces, we get \( p_g = 2, q = 0, K_S^2 = 1 \). It is known [10] that the canonical model of a surface with \( p_g = 2, q = 0, K_S^2 = 1 \) is isomorphic to a hypersurface of degree 10 in \( \mathbb{P}(1, 1, 2, 5) \). In particular, \( R(S, K_S) \) is generated in degrees \( \leq 5 \) and related in degrees \( \leq 10 \).
(c) Suppose that $q(S) = 0$, $K_S^2 + p_g \geq 3$, $K_S^2 \geq 2$ and that the bi-canonical image of $S$ is a surface of minimal degree $r - 1$ in $\mathbb{P}^d$. By the classification [19], a surface of minimal degree is either (a) $\mathbb{P}^2$ ($r = 2$), or (b) a quadric surface in $\mathbb{P}^3$ ($r = 3$), or (c) the Veronese surface in $\mathbb{P}^5$ ($r = 5$), or (d) a rational normal surface scroll. In particular, it is ruled by straight lines except when it is the Veronese surface. If $k$ denotes the degree of the bi-canonical map (onto the image), then

$$4K_S^2 = k(K_S^2 + p_g - 1).$$

We divide the case into subcases according to $k$.

(c1) If $k = 2$, then $K_S^2 = p_g - 1$ and we have $(K_S^2, p_g) = (2, 3)$. Then it is shown in [9] that the canonical model of $S$ is isomorphic to a hypersurface of degree 8 in $\mathbb{P}(1, 1, 1, 4)$. In particular, $R(S, K_S)$ is generated in degrees $\leq 4$ and related in degrees $\leq 8$.

(c2) If $k = 3$, then we have $K_S^2 = 3p_g - 3$. Assume that the bi-canonical image is ruled by straight lines and let $D$ be the pull-back to $S$ of a line. Then, since the bi-canonical map is of degree 3, we must have $2K_SD = 3$, which is absurd. Assume that the bi-canonical image is the Veronese surface in $\mathbb{P}^5$. Then we have $p_g(S) = 2$. $K_S^2 = 3$. Furthermore, we have a divisor $H$ with $2K_S = 2H$ coming from a line on $\mathbb{P}^5$. Note that we have $H \neq K_S$, because $h^0(S, K_S) = 2$ but $h^0(S, H) \geq h^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1)) = 3$. In particular, $S$ is not simply connected and the unramified double covering $\tilde{S}$ of $S$ associated to the 2-torsion divisor $K_S - H$ is a surface with $p_g = 5$ and $K^2 = 6$. Now, $\tilde{S}$ is on the Noether line and, therefore, has a unique pencil of curves of genus 2 (see, [9]). By the uniqueness, the pencil is preserved by the action of the covering transformation group of $\tilde{S} \to S$. This implies that $\tilde{S}$ also has a pencil of curves of genus 2. Then the bi-canonical map of $S$ cannot be of odd degree, since it factors through the relative bi-canonical map which is of degree 2. Hence such a surface does not exist.

(c3) If $k = 4$, then we have $p_g = 1$. We shall show that $K_S^2 \leq 4$ with the aid of Xiao’s theorem [21, Théorème 5.6] which in particular implies that $S$ does not have a pencil of curves of genus 2 when $K_S^2 \geq 5$ and the degree of the bi-canonical map is 4. We assume that $K_S^2 \geq 5$ and show that this leads us to a contradiction. Assume first that $K_S^2 = 5$ and the bicanonical image is the Veronese surface. Then we have a line bundle $H$ such that $2K_S = 2H$. Since $h^0(S, H) \geq 3$, we have $H \neq K_S$. Let $\tilde{S}$ be the unramified double covering of $S$ associated to $K_S - H$. Then $\tilde{S}$ is a surface with $K_S^2 = 10$ and $\chi(O_{\tilde{S}}) = 4$, because $K_S^2 = 2K_S^2$ and $\chi(O_{\tilde{S}}) = 2\chi(O_S)$. On the other hand, we have $p_g(\tilde{S}) = h^0(S, K_S) + h^0(S, H) \geq 1 + 3 = 4$. Thus $q(\tilde{S}) = 1$ and $p_g(\tilde{S}) = 4$. Since $K_S^2 < (8/3)\chi(O_S)$, it follows from [11] that the Albanese map of $\tilde{S}$ is a pencil of curves of genus 2. Then $S$ also has a pencil of curves of genus 2, which is impossible. We next assume that the bicanonical image is ruled by lines. Then we have a pencil $|D|$ of curves with $2K_SD = 4$, that is, $K_SD = 2$. Hodge’s index theorem shows that $4 = (K_SD)^2 \geq K_S^2D^2$. Since $K_S^2 \geq 5$, we get $D^2 = 0$. Then $|D|$ is a pencil of curves of genus 2, which is impossible when $K_S^2 \geq 5$. Therefore, we have $K_S^2 \leq 4$.

(c4) If $k \geq 5$, then $p_g = 0$ and $(k, K_S^2) = (5, 5)$ or $(6, 3)$. As Mendes Lopes showed in [16], both cases are impossible.

In sum, we have $p_g = 3$, $K_S^2 = 2$ or $p_g(S) = 1$, $2 \leq K_S^2 \leq 4$; if $q(S) = 0$, $K_S^2 + p_g \geq 3$, $K_S^2 \geq 2$ and the bi-canonical image is a surface of minimal degree.
**Theorem 2.7** Let $S$ be a minimal surface of general type for which $2K_S$ is free. Then the canonical ring $R(S, K_S)$ is generated in degrees $\leq 5$ and related in degrees $\leq 10$. If furthermore $q(S) = 0$, then $R(S, K_S)$ is generated in degrees $\leq 4$ and related in degrees $\leq 8$ except when $(p_g(S), K^2_S) = (2, 1)$.

Proof. The first assertion follows from Lemmas 2.2, 2.3 and what we showed above. As to the second, we are left the possibility that $p_g = 1$ and $2 \leq K^2_S \leq 4$. In these exceptional cases, we can argue as follows. Since $p_g = 1$, we have a unique canonical curve $D \in [K_S]$. We show that the multiplication map $H^0(S, 2K_S) \otimes H^0(S, 3K_S) \oplus H^0(S, K_S) \otimes H^0(S, 4K_S) \to H^0(S, 5K_S)$ is surjective. It is sufficient to show that $H^0(D, 2K_S|D) \otimes H^0(D, 3K_S|D) \to H^0(D, 5K_S|D)$ is surjective (see the argument around (2.1)). By the duality theorem, it is surjective when

\[
\bigwedge^{k^2-1} H^0(D, 2K_S|D) \otimes H^0(D, K_S|D) \to \bigwedge^{k^2-2} H^0(D, 2K_S|D) \otimes H^0(D, 3K_S|D)
\]

is injective. Since $p_g = 1$, we have $h^0(D, K_S|D) = 0$. Hence we get the second assertion for regular surfaces. □

**Remark 2.8** The assertion for generators were already shown by Ciliberto [4]. See also [16] for surfaces with $p_g = q = 0$.

### 3 The case of rational fixed part

In this and the next sections, we shall show Theorem 0.2.

We let $S$ be a minimal surface of general type with $p_g(S) \geq 2$ and $q(S) = 0$. Let $K_S = |M| + Z$ be the decomposition of $|K_S|$ into the variable part $|M|$ and the fixed part $Z$. We denote by $C \in |M|$ a general member. Then $C$ is a reduced Gorenstein curve of arithmetic genus $M^2 + 1 +MZ/2$. We put $D = C + Z$. Then $D \in |K_S|$. Since $K_S$ is nef and big, and since $K_SA + A^2$ is an even integer for any curve $A$ on $S$, we see that $D$ is a numerically 2-connected curve.

**Lemma 3.1** $MZ$ is a non-negative even integer. Furthermore, $MZ = 0$ if and only if $Z = 0$.

Proof. Since $K_SZ + Z^2$ is even and $K_SZ + Z^2 = MZ + 2Z^2$, we see that $MZ$ is a non-negative even integer. Assume that $MZ = 0$. Then we have $0 \leq K_SZ = MZ + Z^2 = Z^2$. If $M^2 > 0$, then it follows from Hodge’s index theorem that $Z^2 \leq 0$. Hence $Z^2 = 0$ which implies that $Z = 0$. If $M^2 = 0$, then the canonical map of $S$ is composed of a pencil and it follows from $MZ = 0$ that every component of $Z$ is vertical. Since the intersection form is negative semi-definite on fibres by Zariski’s lemma, we have $Z^2 \leq 0$ implying $Z^2 = 0$. Since $K_SZ = 0$, we get $Z = 0$ by Hodge’s index theorem. □

We put $L = K_S|D$. Then $L$ is nef and we have $2L = K_D$ by the adjunction formula. The restriction map $H^0(S, mK_S) \to H^0(D, mL)$ is surjective for any non-negative integer $m$, since $q(S) = 0$.  

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Lemma 3.2 If $K_S^2 \geq 2$, then $|2L|$ and $|3L|$ are free from base points.

Proof. Since $p_a(S) > 0$, the bi-canonical system $|2K_S|$ is free from base points by Francia’s theorem. Hence $|K_D| = |2L|$ is also free from base points. We have $p_a(D) = K_S^2 + 1$ and $\deg L = K_S^2$. It follows that $3L$ may have base points only when $K_S^2 = 1$ (see, e.g., [12]). □

Note that the restriction map $H^0(D, L) \to H^0(Z, L)$ is the zero map, because $Z$ is the fixed part of $|K_S|$. Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(Z) \to \mathcal{O}_Z(Z) \to 0.$$

Since we have $h^0(S, \mathcal{O}_S(Z)) = 1$ and $h^1(S, \mathcal{O}_S) = 0$, we obtain $h^0(Z, \mathcal{O}_Z(Z)) = 0$.

Lemma 3.3 Let $|K_S| = |M| + Z$ be the decomposition as above. If $\Gamma \leq Z$ is a subcurve satisfying $M\Gamma = 2$, $(Z - \Gamma)\Gamma = 0$ and $\Gamma^2 \leq 0$, then either $\Gamma$ is a fundamental cycle of a rational double point, or $\Gamma$ contains a Francia cycle.

Proof. Since $D$ is numerically 2-connected and $(D - \Gamma)\Gamma = (C + (Z - \Gamma))\Gamma = 2$, we see that $Z$ is numerically 1-connected. We have $0 \leq K_S\Gamma = M\Gamma + \Gamma^2 = 2 + \Gamma^2$. Assume that $\Gamma^2 = -2$. Then $p_a(\Gamma) = 0$ and $\Gamma$ consists of $(-2)$-curves because $K_S\Gamma = 0$. It follows that $\Gamma$ is a fundamental cycle of a rational double point. Assume that $\Gamma^2 = -1$. Then $p_a(\Gamma) = 1$. If $\Gamma$ is numerically 2-connected, then $\Gamma$ is a Francia cycle. If $\Gamma$ is not 2-connected, we get an effective decomposition $\Gamma = A + B$ with $AB = 1$. If $A$ is chosen so that it is minimal with respect to the condition $A(\Gamma - A) = 1$, then $A$ is numerically 2-connected and $p_a(A) = 1$. Then $K_S A + A^2 = 0$. If $K_S A = 0$, then $A^2 = 0$ and Hodge’s index theorem shows that $A = 0$, a contradiction. Hence $K_S A > 0$. On the other hand, we have $1 = K_S\Gamma \geq K_S A$. It follows $K_S A = 1$ and $A$ is a Francia cycle. Assume finally that $\Gamma^2 = 0$. Then $p_a(\Gamma) = 2$ and $\Gamma$ is a Francia cycle if it is numerically 2-connected. If $\Gamma$ is not 2-connected, letting $\Gamma = A + B$ be an effective decomposition such that $AB = 1$, we get $p_a(A) = p_a(B) = 1$. Furthermore, we can assume that $A$ is numerically 2-connected. Then $A$ is a Francia cycle of arithmetic genus one. □

The following is an easy consequence of [15, Theorem 4.2].

Lemma 3.4 With the above notation, assume that $C$ is an irreducible curve and that $Z$ does not contain a Francia cycle. If the bi-canonical map of $S$ does not map $C$ birationally onto the image, then $C$ is a hyperelliptic curve, that is, $C$ has a $g_2^1$. If furthermore either $M^2 \geq 5$ or $h^0(Z, K_Z) \leq 2$ or $\mathcal{O}_Z(-Z)$ is nef, then $Z$ consists of at most the fundamental cycles of rational double points.

Proof. We have $p_a(C) \geq 2$ because $S$ is of general type and $C$ moves in $|M|$. Recall that $D$ is 2-connected.

Suppose that there are points $x, y \in C$ which are non-singular points of $D$ (possibly $x = y$) and such that $x, y$ are not separated by $|K_D|$. Then, by [15, Theorem 4.2], we have the following possibilities:

(a) $D$ is an irreducible hyperelliptic curve and $|\mathcal{O}_D(x + y)|$ is the unique $g_2^1$ on $D$.
(b) $D$ is reducible, $C$ is a hyperelliptic curve, and $D$ decomposes as a sum $C + F_1 + \cdots + F_m$ satisfying:

(i) $F_1, \ldots, F_m$ are curves such that $CF_i = 2$ for every $i \in \{1, \ldots, m\}$.

(ii) $\partial_i^C(F_i) = \partial^C(x + y)$ for every $i \in \{1, \ldots, m\}$.

(iii) $|\partial_i^C(x + y)|$ is a $g^2$ on $C$.

(iv) $F_i F_j = 0$ for $i \neq j$.

(v) $\partial_i^F(F_k) = \partial_i^F$, for all $k < i$.

If we are in case (a), then we have nothing to prove. Suppose that we are in case (b). We have $F_i \leq Z$ and $(Z - F_i)F_i = 0$ for each $i$. Since $Z$ does not contain a Francia cycle, it suffices to show that $F_i^2 \leq 0$ by the previous lemma. If $M^2 \geq 5$, then we get $F_i^2 \leq 0$, because we have $4 = (MF_i)^2 \geq M^2 F_i^2$ by Hodge’s index theorem. If $h^0(Z, K_Z) \leq 2$, then we have $p_g(F_i) \leq 2$ and it follows that $F_i^2 \leq 0$. If $\partial_Z(-Z)$ is nef, then we automatically have $F_i^2 \leq 0$.

We remark here that there are at most one index $i$ such that $F_i^2 > 0$ in this case: Suppose that $F_i^2 > 0$. If $j \neq i$, then it follows from $F_i F_j = 0$ that $F_j^2 \leq 0$ by Hodge’s index theorem. □

Note that the condition that $C$ is irreducible is equivalent to that the canonical map of $S$ is generically finite onto the image, when $p_g(S) \geq 3$. From now on, we assume that $C$ is an irreducible curve.

**Lemma 3.5** Assume that $C$ is an irreducible curve. For a positive integer $p$, the Koszul map

$$\bigwedge^p H^0(D, 2L) \otimes H^0(D, L) \to \bigwedge^{p-1} H^0(D, 2L) \otimes H^0(D, 3L)$$

is injective if $h^0(D, L) \leq p - h^0(Z, K_Z)$.

**Proof.** Recall that we have $h^0(Z, \partial_Z(Z)) = 0$. Then, it follows from the exact sequence

$$0 \to H^0(Z, \partial_Z(Z)) \to H^0(D, L) \to H^0(C, L)$$

that the restriction map $H^0(D, L) \to H^0(C, L)$ is injective.

Put $r = h^0(D, 2L) - 1 = K_Z^2$ and let $W$ be the image of the restriction map $H^0(D, L) \to H^0(C, L)$. From the exact sequence

$$0 \to H^0(Z, K_Z) \to H^0(D, 2L) \to H^0(C, 2L) \to H^1(Z, K_Z) \to H^1(D, 2L) \to 0,$$

we get $\dim W = r + 1 - h^0(Z, K_Z)$. We choose a basis $\{s_0, s_1, \ldots, s_r\}$ of $H^0(D, 2L)$ such that $s_i$ vanishes identically on $C$ for $r + 1 - h^0(Z, K_Z) \leq i \leq r$ and the rest induce a basis for $W$. We take sufficiently general points $P_0, \ldots, P_{r - h^0(Z, K_Z)}$ on $C$ so that $s_i(P_j) = \delta_{i,j}$. We consider when the Koszul map

$$\bigwedge^p H^0(D, 2L) \otimes H^0(D, L) \to \bigwedge^{p-1} H^0(D, 2L) \otimes H^0(D, 3L)$$

is injective. The map can be identified with

$$\bigwedge^{r+1-p} H^0(D, 2L) \otimes H^0(D, L) \to \bigwedge^{r+2-p} H^0(D, 2L) \otimes H^0(D, 3L).$$

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Put $t = r + 1 - p$ and let $\{e_0, e_1, \ldots, e_r\}$ be the dual basis for $H^0(D, 2L)^\vee$. Then the last map is given by
\[ \sum \alpha_{i_0, \ldots, i_t} e_{i_0} \wedge \cdots \wedge e_{i_t} \rightarrow \sum (-1)^j \alpha_{i_0, \ldots, i_{j-1}, \ldots, i_{t+1}} s_{i_j} e_{i_0} \wedge \cdots \wedge e_{i_t} \]
Therefore, an element in the kernel can be identified with a collection $\{\alpha_{i_0, \ldots, i_t}\} \subset H^0(D, L)$ such that
\[ \alpha_{i_{t+1}} \wedge s_{i_j} + \cdots + \alpha_{i_0, \ldots, i_{t+1}} s_{i_t} = 0 \]
Evaluating at $P_j$, we see that $\alpha_{i_0, \ldots, i_t}(P_j) = 0$ for any $j \in \{0, \ldots, r - h^0(Z, K_Z)\} \setminus \{i_1, \ldots, i_t\}$. Hence, if $\dim \text{Im} H^0(D, L) \rightarrow H^0(C, L) \leq r + 1 - h^0(Z, K_Z) - t = p - h^0(Z, K_Z)$, then $\alpha_{i_0, \ldots, i_t}$ vanishes identically on $C$. Since the restriction map $H^0(D, L) \rightarrow H^0(C, L)$ is injective, we conclude that $\alpha_{i_0, \ldots, i_t} = 0$ in $H^0(D, L)$. Therefore, the Koszul map in question is injective when $h^0(D, L) \leq p - h^0(Z, K_Z)$. □

**Lemma 3.6** Assume that $p_g(S) \geq 2$ and $C$ is an irreducible curve. If $K_S^2 \geq p_g(S) + 1 + h^0(Z, K_Z)$, then
\[ H^0(D, 2L) \otimes H^0(D, (m - 2)L) \rightarrow H^0(D, mL) \]
is surjective for $m \geq 5$, and
\[ \bigwedge^{r-1} H^0(D, 2L) \otimes H^0(D, (6 - m)L) \rightarrow \bigwedge^{r-2} H^0(D, 2L) \otimes H^0(D, (8 - m)L) \]
is exact at the middle term for $m \geq 7$.

**Proof.** By the duality theorem (see, e.g., [12, Lemma 1.2.1]), we only have to check that
\[ \bigwedge^{r-1} H^0(D, 2L) \otimes H^0(D, (6 - m)L) \rightarrow \bigwedge^{r-2} H^0(D, 2L) \otimes H^0(D, (8 - m)L) \]
and
\[ \bigwedge^{r-2} H^0(D, 2L) \otimes H^0(D, (8 - m)L) \rightarrow \bigwedge^{r-3} H^0(D, 2L) \otimes H^0(D, (10 - m)L) \]
are injective for $m \geq 5$ and $m \geq 7$, respectively, where $r = K_S^2$. Since $r \geq 3$, this is clear except in the cases $m = 5$ and $m = 7$, respectively. The rest follows from the previous lemma. □

**Lemma 3.7** If $H^0(Z, K_Z) = 0$ and the bi-canonical map of $S$ maps $C$ birationally onto the image, then $H^0(D, 2L) \otimes H^0(D, 2L) \rightarrow H^0(D, 4L)$ is surjective.

**Proof.** By the duality theorem, we only have to check that
\[ \bigwedge^{r-1} H^0(D, 2L) \rightarrow \bigwedge^{r-1} H^0(D, 2L) \otimes H^0(D, 2L) \rightarrow \bigwedge^{r-2} H^0(D, 2L) \otimes H^0(D, 4L) \]
is exact at the middle term, where $r = K_S^2 = h^0(D, 2L) - 1$. Recall that $K_D = 2L$. Consider the exact sequence
\[ 0 \rightarrow H^0(Z, K_Z) \rightarrow H^0(D, K_D) \rightarrow H^0(C, K_D) \rightarrow H^1(Z, K_Z) \]
If $H^0(Z, K_Z) = 0$, then the restriction map $H^0(D, K_D) \to H^0(C, K_D)$ is injective. This implies that the image $C'$ of $C$ under the canonical map of $D$ (or, the bi-canonical map of $S$) is an irreducible non-degenerate curve in $\mathbb{P}H^0(D, 2L)$. Therefore, if the above Koszul complex is not exact at the middle term, we can apply [6, Theorem 1.13] (or $K_{p,1}$-theorem) to conclude that $C'$ is a rational normal curve. This is absurd, because $C$ is not a rational curve. □

Assume that $Z$ supports at most rational double points (possibly $Z = 0$). Then, by contracting $Z$, we have an irreducible canonical curve on the canonical model of $S$. In this case, it has already been shown by Reid [17] that $R(S, K_S)$ is generated in degrees $\leq 3$ and related in degrees $\leq 6$ provided that $p_g \geq 2$ and $K_S^2 \geq 3$. The main ingredient of his proof is to apply the following fact to that curve:

**Lemma 3.8** (See [17, p. 246, Theorem]) Let $X$ be an irreducible Gorenstein curve of genus $g \geq 2$, and $E$ a Cartier divisor on $X$ such that $2E \sim K_X$. If $g \geq 4$, then the ring $\bigoplus_{m \geq 0} H^0(X, mE)$ is generated in degrees $\leq 3$ and related in degrees $\leq 6$.

We now state our main result in this section.

**Theorem 3.9** Let $S$ be a minimal surface of general type with $q(S) = 0$, $p_g(S) \geq 2$ and $K_S^2 \geq 3$. Assume that the variable part of $|K_S|$ contains an irreducible curve and the fixed part $Z$ of $|K_S|$ satisfies $h^0(Z, K_Z) = 0$. Then the canonical ring is generated in degrees $\leq 3$ and related in degrees $\leq 6$.

**Proof.** We can assume that the bi-canonical map of $S$ maps $C$ birationally onto the image. In fact, if it is not the case, then $D$ induces on the canonical model of $S$ an irreducible, hyperelliptic canonical curve by Lemma 3.4 and we can apply the above result of Reid.

We may also assume that $Z \neq 0$. Then $MZ$ is a positive even integer. We are going to show that $K_S^2 \geq p_g(S) + 1$, which is clear when $p_g(S) = 2$ since $K_S^2 \geq 3$. If $p_g(S) \geq 3$, then $M^2 \geq 2p_g - 4$ and

$$K_S^2 = K_S M + K_S Z = M^2 + MZ + K_S Z \geq 2p_g(S) - 4 + MZ + K_S Z.$$  

Hence we get $K_S^2 \geq p_g(S) + 1$. Then we get the assertion by Lemmas 3.7 and 3.8. □

We close the section stating a similar result for irregular surfaces which is obtained analogously as above by relacing $H^0(D, 2L)$ by $V = \text{Im}(H^0(S, 2K_S) \to H^0(D, 2L))$.

**Theorem 3.10** Let $S$ be a minimal irregular surface of general type with $K_S^2 \geq \min\{p_g(S) + 2q(S), 2q(S) + 4\}$ (e.g., $\chi(O_S) \geq 3$). Assume that the variable part of $|K_S|$ contains an irreducible curve and the fixed part supports at most exceptional sets of rational singular points. Then the canonical ring is generated in degrees $\leq 4$ and related in degrees $\leq 8$.

**Proof.** Let $D = C + Z \in |K_S|$ be a general member as above. The exact sequence

$$0 \to H^0(S, K_S) \to H^0(S, 2K_S) \to H^0(D, 2L) \to H^1(S, K_S) \to 0$$

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shows us that \( \dim V = K_Z^2 + 1 - q(S) \). Since \( Z \) supports at most exceptional sets of rational singular points, we have \( H^0(Z, \mathcal{O}_Z) = H^0(Z, K_Z) = 0 \). Hence, as in the proof of Lemma 3.5, we can show for \( m \geq 5 \), the multiplication map

\[
V \otimes H^0(D, (m-2)L) \to H^0(D, mL)
\]

is surjective when \( h^0(D, L) \leq \dim V - 2 = K_Z^2 - q(S) - 1 \). We have \( h^0(D, L) \leq p_g(S) + q(S) - 1 \) and \( h^0(D, L) \leq h^0(C, L) \leq K_S C/2 + 1 \) by Clifford’s theorem. Hence we get \( h^0(D, L) \leq \dim V - 2 \) provided that \( K_Z^2 \geq \min(p_g(S) + 2q(S), 2q(S) + 4 - K_S Z) \). Since we always have \( K_Z^2 \geq 2p_g(S) \) for irregular surfaces, we get \( K_Z^2 \geq 2q(S) + 4 \) when \( \chi(\mathcal{O}_S) \geq 3 \). Furthermore, for \( m \geq 9 \)

\[
\bigwedge^2 V \otimes H^0(D, (m-4)L) \to V \otimes H^0(D, (m-2)L) \to H^0(D, mL)
\]

is exact at the middle term, because \( H^0(D, (8-m)L) = 0 \). □

### 4 The case of semi-negative fixed part

In this section, we complete the proof of Theorem 0.2 by studying the case \( h^0(Z, K_Z) > 0 \) when \( Z \) is a numerically disjoint sum of chain connected curves which are semi-negative:

**Theorem 4.1** Let \( S \) be a minimal algebraic surface of general type with \( p_g(S) \geq 2 \), \( q(S) = 0 \) and \( K_Z^2 \geq 3 \). Let \( |K_S| = |M| + Z \) be the decomposition into its variable and fixed parts. Suppose that \( |M| \) contains an irreducible member and \( Z \) does not contain a Francia cycle. If there is a decomposition \( Z = \Delta + \Gamma_1 + \cdots + \Gamma_n \) satisfying the following conditions, then \( R(S, K_S) \) is generated in degrees \( \leq 3 \) and related in degrees \( \leq 6 \):

1. \( \Delta \) is a curve consisting of \((-2)\)-curves (possibly \( \Delta = 0 \)), \( \text{Supp}(\Delta) \cap \text{Supp}(Z - \Delta) = \emptyset \) and \( Z - \Delta \neq 0 \).

2. For each \( i \in \{1, 2, \ldots, n\} \), \( \Gamma_i \) is a chain connected curve such that \( \mathcal{O}_{\Gamma_i}(-\Gamma_i) \) is nef, \( \Gamma_i^2 \leq 0 \) holds for any subcurve \( \Gamma \leq \Gamma_i \), and \( K_S \Gamma_i > 0 \). Furthermore, \( \mathcal{O}_{\Gamma_i}(\Gamma_j) \) is numerically trivial when \( i \neq j \).

The key point is as usual to show the following:

**Proposition 4.2** With the same notation and assumptions in Theorem 4.1, \( H^0(S, 4K_S) \) is generated by products of elements of lower degrees.

We first show Theorem 4.1 assuming Proposition 4.2.

**Proof of Theorem 4.1.** By Lemma 1.1 and Proposition 4.2, it suffices to check that \( K_S^2 \geq p_g(S) + 1 + h^0(Z, K_Z) \) in view of Lemma 3.6. Let \( Z = \Delta + \Gamma_1 + \cdots + \Gamma_n \) be the decomposition as in the statement. We have

\[
h^0(Z, K_Z) = \frac{1}{2}Z(K_S + Z) + v, \quad v = h^0(Z, \mathcal{O}_Z)
\]
Then the condition $K^2_S \geq p_g(S) + 1 + h^0(Z, K_Z)$ is equivalent to

$$K^2_S + M^2 + MZ \geq 2p_g(S) + Z^2 + 2v + 2.$$ 

We have $v = h^0(Z, \mathcal{O}_Z) = h^0(\Delta, \mathcal{O}_\Delta) + h^0(Z - \Delta, \mathcal{O}_{Z - \Delta})$ by (1). We have $h^0(\Delta, \mathcal{O}_\Delta) = -\Delta^2/2$, because $h^1(\Delta, \mathcal{O}_\Delta) = 0$ and $\chi(\Delta, \mathcal{O}_\Delta) = -\Delta(K_S + \Delta)/2$. It follows from $K_S \Delta = (Z - \Delta)\Delta = 0$ that $M\Delta = -\Delta^2$. By (2), we have $\Gamma^2_i \leq 0$ and $\Gamma_i \Gamma_j = 0 (i \neq j)$. Furthermore, $\Gamma_i$ is not a fundamental cycle of a rational double point by the assumption that $K_S \Gamma_i > 0$. Since $Z$ does not contain a Franchetta cycle, it follows from Lemma 3.4 that $M\Gamma_i$ is a positive even integer not less than 4. Hence we have $M(Z - \Delta) \geq 4n \geq 4v - h^0(\Delta, \mathcal{O}_\Delta)) = 4(v + \Delta^2/2) = 4v + 2\Delta^2$. Summing up, we get $MZ \geq 4v + \Delta^2$. By the assumption that $Z - \Delta \neq 0$, we have $v \geq h^0(\Delta, \mathcal{O}_\Delta) + 1$ which is equivalent to $\Delta^2 \geq 2 - 2v$. Hence $MZ \geq 2v + 2$. Then we have

$$K^2_S = K_S M + K_SZ = M^2 + MZ + K_SZ \geq 2v + 2 \geq 4.$$ 

From this and $M^2 \geq 2p_g(S) - 4$, we get $K^2_S + M^2 \geq 2p_g(S)$. Then

$$K^2_S + M^2 + MZ \geq K^2_S + M^2 + 2v + 2 \geq 2p_g(S) + 2v + 2 \geq 2p_g(S) + 2v + 2 + Z^2,$$

which is what we want. \(\square\)

In order to show Proposition 4.2, we slightly change the situation as follows. Let $\sigma : S \to X$ be the contraction of $\Delta$. Then $X$ is a normal Gorenstein surface with (at most) rational double points and we have $\sigma^* \omega_X = \mathcal{O}_S(K_S)$. By (1) of Theorem 4.1, we can harmlessly push everything on $S$ down to $X$, which enables us to assume that $\Delta = 0$ with the cost that we cannot stay on a non-singular surface. We shall indeed work on such $X$. However, we use the same symbols as before for simplicity: $S$ is now a normal Gorenstein surface with at most rational double points, $Z = \Gamma_1 + \cdots + \Gamma_Z$ and $|\omega_S| = |M| + Z$ is the decomposition into the variable and fixed parts. $C \in |M|$ is irreducible, $D = C + Z$ and $L = \omega_S \otimes \mathcal{O}_D$. Note that $\text{Supp}(Z)$ is entirely contained in the regular locus of $S$. Similarly as in the previous section, it suffices for our purpose to show that $H^0(D, 4L)$ is generated by products of lower degree elements.

We put $W_m = \text{Im}[H^0(D, mL) \to H^0(C, mL)]$ for a non-negative integer $m$.

**Lemma 4.3** $W_m = H^0(C, mL)$ when $m \geq 3$, and $\dim W_2 = h^0(C, 2L) - h^0(Z, \mathcal{O}_Z) + 1$.

**Proof.** For integers $m \geq 2$, consider the exact sequence

$$H^0(D, mL) \to H^0(C, mL) \to H^1(Z, mL - C) \to H^1(D, mL).$$

We have $\mathcal{O}_Z(mL - C) = \mathcal{O}_Z(K_Z + (m - 2)K_S)$. Since $Z$ is a numerically disjoint sum of chain-connected curves $\Gamma_i$ with $K_S \Gamma_i > 0$, we have $H^1(Z, mL - C) = 0$ for $m \geq 3$ by Lemma 1.2. When $m = 2$, we have $h^1(Z, 2L - C) = h^0(Z, \mathcal{O}_Z)$ by duality, $h^1(D, 2L) = 1$ and $h^1(C, 2L) = 0$. Hence the assertion follows. \(\square\)

Similarly as above, using the exact sequence

$$0 \to \mathcal{O}_C(mL - Z) \to \mathcal{O}_D(mL) \to \mathcal{O}_Z(mL) \to 0,$$
we can show that the restriction map $H^0(D, mL) \to H^0(Z, mL)$ is surjective for $m \geq 2$. Then, by a diagram chasing, we see that the multiplication map

$$H^0(D, L) \otimes H^0(D, 3L)$$

(4.1) $$\oplus \quad \longrightarrow \quad H^0(D, 4L)$$

$$H^0(D, 2L) \otimes H^0(D, 2L)$$

is surjective provided that

$$H^0(Z, 2L) \otimes H^0(Z, 2L) \to H^0(Z, 4L)$$

(4.2)

and

$$H^0(C, L - Z) \otimes H^0(C, 3L)$$

(4.3) $$\oplus \quad \longrightarrow \quad H^0(C, 4L - Z)$$

$$H^0(C, 2L - Z) \otimes W_2$$

are both surjective. Recall that $\mathcal{O}_C(L - Z) = \mathcal{O}_C(M)$ and $\mathcal{O}_C(2L - Z) = \omega_C$.

**Lemma 4.4** The multiplication map (4.2) is surjective.

**Proof.** Recall that $\Gamma_i$ is chain connected. Since $2K_S|_{\Gamma_i} - K_{\Gamma_i} = (K_S - \Gamma_i)|_{\Gamma_i}$ which is nef and non-trivial by the assumption, we have $H^1(\Gamma_i, \mathcal{L}) = 0$ for any invertible sheaf $\mathcal{L}$ on $\Gamma_i$ numerically equivalent to $\mathcal{O}_{\Gamma_i}(2K_S)$, by Lemma 1.2. By the assumption, $\mathcal{O}_{\Gamma_i}(\Gamma_j)$ is numerically trivial for $i \neq j$. From the inductive argument using the exact sequences

$$0 \to H^0(\Gamma_k, 2K_S - \Gamma_1 - \cdots - \Gamma_{k-1}) \to H^0(\Gamma_1 + \cdots + \Gamma_k, 2K_S) \to H^0(\Gamma_1 + \cdots + \Gamma_{k-1}, 2K_S) \to 0,$$

it follows that the restriction map $H^0(Z, 2K_S) \to H^0(\Gamma_i, 2K_S)$ is surjective. At the same time, we see that, in order to show that (4.2) is surjective, it suffices to show that $H^0(\Gamma_i, L_1) \otimes H^0(\Gamma_i, 2K_S) \to H^0(\Gamma_i, L_1 + 2K_S)$ is surjective for each $\Gamma_i \leq Z$ and for any line bundle $L_1$ on $\Gamma_i$ which is numerically equivalent to $2K_S$ on $\Gamma_i$. But it is nothing but a special case of Theorem 1.5. □

We next consider the map (4.3). We let $\zeta$ denote the fixed part of $|M_C|$, that is, the smallest effective Cartier divisor $\zeta$ on $C$ such that $|\mathcal{O}_C(M - \zeta)|$ is free from base points. We put $N = M_C - \zeta$.

**Lemma 4.5** The image of the multiplication map

$$H^0(C, M) \otimes H^0(C, 3L) \to H^0(C, 4L - Z)$$

is the subspace spanned by elements vanishing on $\zeta$. 21
Proof. The kernel of the restriction map $H^0(C, 4L - Z) \to H^0(\zeta, 4L - Z)$ is isomorphic to $H^0(C, 4L - Z - \zeta)$, and we have an injection $H^0(C, N) \approx H^0(C, M - \zeta) \to H^0(C, M)$ by multiplying the section defining $\zeta$. Hence it suffices to show that the multiplication map

$$H^0(C, N) \otimes H^0(C, 3L) \to H^0(C, 4L - Z - \zeta)$$

is surjective. We have $B_s[N] = \emptyset$. By Castelnuovo’s theorem, it suffices to show that $H^1(C, 3L - N) = 0$. Since $O_C(3L - N) \approx O_C(K_C + 2Z + \zeta)$, we indeed have $h^1(C, 3L - N) = 0$. □

The following completes the proof of Proposition 4.2.

Lemma 4.6 The multiplication map (4.3) is surjective.

Proof. By the previous lemma, it is sufficient to show that the composite of the multiplication map $H^0(C, K_C) \otimes W_2 \to H^0(C, 4L - Z)$ and the restriction map $H^0(C, 4L - Z) \to H^0(\zeta, 4L - Z)$ is surjective. We claim that the restriction map $H^0(C, K_C) \to H^0(\zeta, K_C)$ is surjective. This can be seen as follows. Note that we have $h^0(C, O_C(\zeta)) = 1$, because $\zeta$ cannot move. Then it follows from the Riemann-Roch theorem that $h^1(C, \zeta) = h^0(C, K_C - \zeta) = p_a(C) - \deg \zeta$. By the dimension count, this implies that $H^0(C, K_C) \to H^0(\zeta, K_C)$ is surjective. Then, since $|2K_C|$ is free from base points, the multiplication map $H^0(\zeta, K_C) \otimes W_{2k} \to H^0(\zeta, 4L - Z)$ is clearly surjective. This show the claim and we see that (4.3) is surjective. □

5 Appendix.

In this appendix, we state some results about canonical algebras of curves on a smooth surface in order to supplement [12].

Let $D$ be a numerically 1-connected curve on a smooth surface such that $p_a(D) \geq 2$ and $K_D$ is nef. In [12], we studied the canonical ring $R(D, K_D) = \bigoplus_{m \geq 0} H^0(D, mK_D)$ and showed that it is generated in degrees $\leq 3$.

Theorem 5.1 Let $D$ be a numerically 1-connected curve on a smooth surface such that $p_a(D) \geq 2$ and $K_D$ is nef. Then the canonical ring of $D$ is generated in degrees $\leq 3$ and related in degrees $\leq 6$.

Proof. Let the notation be as in Lemma 2.1. As we remarked above, we have $d \leq 3$ by [12, Theorem II]. By [12, Corollary 1.2.3], we have $d_0 = 4$ and $d_1 = 6$. Now, apply Lemma 2.1. □

This also applies to the relative canonical algebra of a relatively minimal fibration $f : S \to B$ over a non-singular curve $B$ whose general fibre is of genus $\geq 2$.

Theorem 5.2 The relative canonical algebra for a relatively minimal fibration of curves of genus $\geq 2$ is generated in degrees $\leq 4$ and related in degrees $\leq 8$. If furthermore there are no multiple fibres whose canonical system has a ($-1$)-elliptic cycle as a fixed component, then the relative canonical algebra is generated in degrees $\leq 3$ and related in degrees $\leq 6$. 

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A further application goes over the relative canonical algebra for a normal surface singularity. Let \((V, p)\) be a normal surface singularity and \(\pi : S \to V\) the minimal resolution. We regard \(S\) as a sufficiently small, strongly pseudoconvex neighbourhood of \(A = \pi^{-1}(p)\).

Let \(D\) be the fundamental cycle on \(A\). We are interested in the canonical ring \(R(S, K_S) = \bigoplus_{m=0}^{\infty} H^0(S, mK_S)\). Laufer [14] showed that it is generated in degrees \(\leq 3\). The following can be found in [13] and [14]:

**Theorem 5.3** Let \(D\) be the fundamental cycle on \(A\) and \(n\) a positive integer. Then \(H^1(nD, L) = 0\) for any line bundle \(L\) on \(S\) such that \(L - K_S\) is nef on \(D\). Furthermore, the ring \(R(nD, K_S) = \bigoplus_{m=0}^{\infty} H^0(nD, mK_S)\) is generated in degrees \(\leq 3\).

Let \(n\) be a positive integer. We claim that the Koszul sequence

\[
\bigwedge^2 H^0(nD, 2K_S) \otimes H^0(nD, (m-4)K_S) \to H^0(nD, 2K_S) \otimes H^0(nD, (m-2)K_S) \to H^0(nD, mK_S)
\]

is exact at the middle term for \(m \geq 7\). By the duality theorem, it suffices to show that

\[
\bigwedge^{r_n-2} H^0(nD, 2K_S) \otimes H^0(nD, K_{nD} - (m-6)K_S) \to \bigwedge^{r_n-3} H^0(nD, 2K_S) \otimes H^0(nD, K_{nD} - (m-8)K_S)
\]

is injective for \(m \geq 7\), where \(r_n = h^0(nD, 2K_S) - 1\). By Serre duality, we have \(H^0(nD, K_{nD} - (m-6)K_S) \cong H^1(nD, (m-6)K_S)\), which vanishes for any integer \(m\) with \(m \geq 7\) by the above theorem. Hence we conclude that \(R(nD, K_S)\) is related in degrees \(\leq 6\).

Note that, for any positive cycle \(A\) supported on \(A\), we can find a positive integer \(n\) such that \(\Delta \leq nD\). We have \(\pi_* \mathcal{F} = \lim_{\Delta} H^0(\pi^{-1}(p), \mathcal{F} \otimes \mathcal{O}_A)\) for a coherent sheaf \(\mathcal{F}\) on \(S\). Therefore, we get the following as conjectured by Reid:

**Theorem 5.4** The relative canonical algebra for a normal surface singularity is generated in degrees \(\leq 3\) and related in degrees \(\leq 6\).

**References**


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