Sextic curves with six double points on a conic

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Abstract. Let $C_6$ be a plane sextic curve with 6 double points that are not nodes. It is shown that if they are on a conic $C_2$, then the unique possible case is that all of them are ordinary cusps. From this it follows that $C_6$ is irreducible. Moreover, there is a plane cubic curve $C_3$ such that $C_6 = C_3^2 + C_2^3$. Such curves are closely related to both the branch curve of the projection to a plane of the general cubic surface from a point outside it and canonical surfaces in $\mathbb{P}^3$ or $\mathbb{P}^4$ whose desingularizations have birational invariants $q > 0$, $p_g = 4$ or $p_g = 5$, $P_2 \leq 23$.

Introduction

In this paper, we study, a priori not necessarily irreducible, singular plane curves, paying special attentions to double points that are not nodes on them. Here, a node is a double point with two distinct singular tangent lines, e.g., $y^2 + x^2 = 0$. Therefore, a double point that is not a node has two coincident singular tangent lines, e.g., $y^2 + x^n = 0$, $n \geq 3$. In this case, the line $y = 0$ is called the singular tangent line. In the case $y^2 + x^3 = 0$ the double point is called an ordinary cusp and in the case $y^2 + x^4 = 0$ it is called a tacnode.

Let $C_6$ be a plane sextic curve that has 6 singular points each of which is either an ordinary cusp or a tacnode. If these 6 singular points are not on a single conic, then every possible combination actually occurs. Namely, for every integer $i$ with $0 \leq i \leq 6$, there exists a sextic curve with $6 - i$ cusps and $i$ tacnodes such that these 6 singular points are not on a conic (cf. [K]). However, if there is a conic passing through all the 6 singular points, then the situation changes drastically as the following theorem, our main result in this paper, shows.

Theorem. Let $C_6$ be a (not necessarily irreducible) plane sextic curve that has 6 distinct double points $K_1, \ldots, K_6$, none of which is a node, lying on a conic $C_2$. Then, there exists a cubic curve $C_3$ meeting $C_2$ transversally at the 6 points $K_i$ and the tangent line to $C_3$ at $K_i$ coincides with the singular tangent line to $C_6$, $i \in \{1, 2, \ldots, 6\}$. Therefore, the only
one case is possible: the 6 singular points are all ordinary cusps, \( C_6 \) is irreducible and the equation of \( C_6 \) is given by \( C_2^3 + C_2^3 = 0 \).

Indeed, although the authors were familiar enough with sextic curves with ordinary cusps or tacnodes as singularities (cf. [Z1], [Z2], [CPS, Section 1.1], [K], [St1], [St2] and [KS]), they did not imagine that such a surprising statement holds true. What convinced them of its validity were the explicit examples constructed in [K]:

**Examples.** Affine equations of sextic curves with only ordinary cusps and tacnodes. \((i^2 = -1)\).

1. Seven cusps:
   \[
   C_6(x, y) = 4x^6 + y^6 - 4x^3y^2 - 2y^4 - 4x^3 + 5y^2
   = (y^2 - 1)^3 + (-2x^3 + y^2 + 1)^2 = 0.
   \]

2. Six cusps and one tacnode:
   \[
   C_6(x, y) = x^6 - 18x^4y^2 - 27x^2y^4 - 12x^4 + 36x^2y^2 + 48x^2 + 37y^2 - 64
   = (x^2 + 3y^2 - 4)^3 + [i\sqrt{27}(x^2y + y^3 - 2y)]^2 = 0.
   \]

3. Eight cusps:
   \[
   C_6(x, y) = x^6 + y^6 + 2i\sqrt{3}xy(x^4 - y^4) - 5x^2y^2(x^2 + y^2)
   - 4i\sqrt{3}xy(x^2 - y^2) + 12x^2y^2 - 4
   = [\sqrt{3}(x^2 - 1)]^3 + [y^3 - \sqrt{3}i xy^2 + x^2y - \sqrt{3}ix(x^2 - 2)]^2 = 0.
   \]

4. Nine cusps:
   \[
   C_6(x, y) = x^6 + y^6 + 1 - 2x^3y^3 - 2x^3 - 2y^3
   = (\sqrt{3}xy)^3 + (x^3 + y^3 - 1)^2 = 0.
   \]

In these examples, the sextic curve \( C_6 \) has more than 6 singular points, but any of them is either a cusp or a tacnode. Since we imposed particular symmetries on the curves, we could easily realize that 6 ordinary cusps are on the conic \( C_2 \) in the expression \( C_6 = C_2^3 + C_2^3 \) given above, where and in the sequel we denote by \( C_i = 0 \) the equation of the curve \( C_i \) of degree \( i \).

As far as known examples concern, any sextic curve with only cusps or tacnodes has 6 cusps on a conic, when the number of the singular points is at least 7. Then, a natural question arises: Is this true in general? That is, does any sextic \( C_6 \), with at least 7 singular points each of which is either a cusp or a tacnode, have 6 cusps on a conic \( C_2 \)? If the answer is affirmative, as we think so, then the equation of \( C_6 \) is of the form \( C_3^3 + C_2^3 = 0 \) by the above theorem.
When the sextic $C_6$ has exactly 6 cusps on a conic (and no other singularities), our result can be applied to the branch curve of the projection to a plane of the general cubic surface from a point outside it, as discussed in [Se, Ch. IV, p. 19] and [M]. In these papers, curves $C_{6m}, C_{3m}$ and $C_{2m}$ of respective degrees $6m, 3m$ and $2m$ were considered, whereas we consider here the case $m = 1$ only. Hence, Theorem is well known in the case of 6 cusps if we assume the generality of curves. Nevertheless, their methods seem not extend to more general situations; for example, when $C_6$ has singularities other than the six cusps, and so $g \not\in C_2$, or when $C_2$ is reducible. Thus, our result is more general than theirs when $m = 1$. Furthermore, our proof is new and quite elementary at the cost of considerable length. When $C_6$ is irreducible, we have another argument to prove Theorem suggested by M. Oka: show that the Alexander polynomial is non-trivial and apply the result of Degtyarev [D] to see that $C_6$ is a $(2, 3)$ torus curve, i.e., $C_6 = C_2^3 + C_3^2$. Again, ours has an advantage that $C_6$ can be reducible.

Returning to the case that $C_6$ has 6 cusps on a conic (and no other singularities), Theorem has an interesting consequence which is closely related to irregular canonical surfaces in $\mathbb{P}^4$. To explain it, let us recall the work of O. Zariski, cf. [Z1], [Z2]. Indeed, $C_6$ gives us a normal surface $S$ in $\mathbb{P}^3$ whose affine equation is $z^6 - C_6(x, y) = 0$. A desingularization $X$ of $S$ is an irregular surface with $q = 1$ and $p_g = 5$, where $q$ and $p_g$ respectively denote the irregularity and the geometric genus of $X$. The reason why $q > 0$ is because the 6 cusps are on a conic, as shown in [Z1]. Furthermore, it is not so hard to see that the canonical transformation $\varphi|_{K_X}$ of $X$ is birational. So, the image under $\varphi|_{K_X}$, which is called a canonical surface, is a surface of general type in $\mathbb{P}^4$. Our result affirms that we cannot modify Zariski’s example by replacing some cusps by tacnodes on the sextic. This is one of the important consequences of ours. On the other hand, such a replacement is possible if the 6 cusps are not on a conic.

In fact, we constructed in [K] normal surfaces $S \subset \mathbb{P}^3$ whose affine equation is $z^6 - C_6(x, y) = 0$ and the desingularizations have the birational invariants: $q = 0, p_g = 4$ and $P_2 = 23 - i$, by giving $C_6$ with $6 - i$ cusps and $i$ tacnodes not lying on a conic ($i = 0, \ldots, 6$). The case of $P_2 = 23$, i.e., $C_6$ with 6 cusps, is again due to Zariski [Z1], [Z2]. The importance of these surfaces $S$ is in the facts that they themselves are canonical (image of the canonical transformation) and that they are the only known examples of regular, normal and canonical surfaces in $\mathbb{P}^3$ of degree $> 5$, whereas the case of non-normal, regular and canonical surfaces in $\mathbb{P}^3$ is a classical subject dates back to F. Enriques (cf. [E, Ch. VIII]). In other words, Zariski’s example can be extended, by “replacing” cusps with tacnodes, to the normal canonical surfaces in $\mathbb{P}^3$, whose desingularizations have $q = 0, p_g = 4$ and $P_2 = 22, 21, 20, 19, 18, 17$. This makes it more strange that the replacement is impossible when the six singular points are on a conic.

Although we assume that the ground field $k$ is the algebraically closed field of characteristic zero, what we actually need is only a “weak” version of Bézout’s theorem, i.e.,
if a curve of degree \(n\) and a curve of degree \(m\) have at least \(mn + 1\) points in common, then they have a common component. Such a weak version of Bézout theorem needs less hypotheses on the ground field.

The authors thank the referee for the precious suggestions which could improve the original version.

1. Proof

Let the notation and the assumption be as in Theorem in the Introduction.

**Lemma 1.1.** If the conic \(C_2\) passing through \(K_1, \ldots, K_6\) is reducible and splits in two distinct lines as \(C_2 = \ell_1 \ell_2\), then none of the \(K_i\)'s are the point \(\ell_1 \cap \ell_2\).

**Proof.** If a singular point of \(C_6\) is at \(\ell_1 \cap \ell_2\), then one of the two lines, say \(\ell_1\), contains 4 of the 6 singular points. By Bézout’s theorem, \(C_6\) splits into \(\ell_1\) and a quintic \(C_5\). Recall that none of the 4 singular points can be a node. It follows that \(C_5\) and \(\ell_1\) have to be tangent to each other at the 4 points. By Bézout’s theorem again, \(C_5\) has \(\ell_1\) as a component and, thus, \(C_6\) can be divided by \(\ell_1^2\). Then, it would follow that on \(\ell_1\) there are no double points of \(C_6\). Therefore, none of the 6 singular points of \(C_6\) are at \(\ell_1 \cap \ell_2\). \(\square\)

**Corollary 1.2.** If \(C_2\) splits in two lines \(C_2 = \ell_1 \ell_2\), then on each line \(\ell_i\) there are 3 distinct singular points of \(C_6\).

**Lemma 1.3.** \(C_6\) and \(C_2\) cannot have a common component.

**Proof.** We first assume that \(C_2\) is irreducible. If \(C_6\) has at least one cusp, then \(C_6\) cannot have \(C_2\) as a component, because \(C_2\) passes through the cusp (which has only one local analytic branch). If all the 6 singular points on \(C_2\) are different from cusps, then \(C_6 = D_2 D'_2 D''_2\) with \(D_2, D'_2, D''_2\) being two by two bitangent conics by [C1], [C2]. In this case, however, \(C_2\) can be none of the three conics \(D_2, D'_2, D''_2\), because each of them contains only 4 tacnodes. Next, we assume that \(C_2 = \ell_1 \ell_2\) and that \(\ell_1\) is a component of \(C_6\). Put \(C_6 = C_5 \ell_1\). In this case none of the 3 singular points on \(\ell_1\) (Corollary 1.2) is a cusp. This means that \(C_5\) and \(\ell_1\) have at least 6 points in common (counting multiplicities). By Bézout’s theorem, \(C_5\) has \(\ell_1\) as a component and, thus, \(C_6\) can be divided by \(\ell_1^2\), which is absurd. \(\square\)

**Corollary 1.4.** \(C_6\) cuts out on \(C_2\) the divisor \(2 \sum_{i=1}^{6} K_i\).

**Proof.** The conic \(C_2\) cannot be tangent at a double point to the singular tangent line of \(C_6\), otherwise, by Bézout’s theorem \(C_6\) and \(C_2\) have a common component, which Lemma 1.3 forbids. \(\square\)

Now, since the space of homogeneous polynomials of degree 3 in three variables is of dimension 10, we can find a cubic \(C_3\) passing through \(K_1, K_2, \ldots, K_6\) which is tangent
at $K_1, K_2, K_3$ to the singular tangent line of $C_6$. This tangency can be understood also improperly, in the sense that “improperly tangent” means that $C_3$ has $K_1$, or $K_2$, or $K_3$ as a singular point.

Furthermore, we can assume that $K_1, K_2, K_3$ are not collinear, by a suitable re-labeling if necessary.

**Lemma 1.5.** Let the choice of $K_1, K_2, K_3$ and the cubic $C_3$ be as above. Then $C_3$ and $C_2$ have no common components.

**Proof.** We first assume that the conic $C_2$ is irreducible and that $C_3 = C_2E_1$, where $E_1$ is a line. From Corollary 1.4, the conic $C_2$ has no tangents coincident with the singular tangent line of $C_6$ at the three points $K_1, K_2, K_3$. Thus $E_1$ must pass through $K_1, K_2$ and $K_3$, which is impossible because $K_1, K_2$ and $K_3$ are not collinear. Next, let us assume that $C_2$ is reducible, $C_2 = \ell_1\ell_2$, and that $C_3 = D_2\ell_1$ with a conic $D_2$. From Corollary 1.2, the conic $D_2$ meets $\ell_2$ at three points and, thus, $D_2$ can be divided by $\ell_2$. Then, $C_3 = \ell_1\ell_2F_1$ with the third line $F_1$. By the choice of $C_3$, the line $F_1$ must pass through $K_1, K_2$ and $K_3$, which is again impossible since they are not collinear. □

**Corollary 1.6.** $C_3$ cuts out on $C_2$ the divisor $\sum_{i=1}^{6} K_i$.

**Proof.** Indeed $C_3$ and $C_2$ cannot be tangent at any point $K_i$, otherwise they have a common component, which Lemma 1.5 forbids. □

Let $C_6$ and $C_3$ be the two curves considered above. In particular $C_3$ at $K_1, K_2, K_3$ is tangent to the singular tangent line of $C_6$. Let us consider the following pencil of plane sextic curves, given by the equation

$$\lambda C_6 + \mu C_3^2 = 0, \quad \forall \lambda, \mu \in k.$$  

(1.1)

It may happen that $C_6$ and $C_3$ have some common components (even, $C_3$ may be a component of $C_6$), nevertheless the equation (1.1) defines a pencil of curves because $C_6$ and $C_3^2$ are different.

**Lemma 1.7.** In the pencil (1.1), there exists a sextic curve of the form $C_2C_4$ with a suitable quartic $C_4$. That is, there are two values $\lambda$ and $\mu$ of $\lambda$ and $\mu$ such that the following identity of polynomials holds true

$$\lambda C_6 + \mu C_3^2 = C_2C_4.$$  

(1.2)

**Proof.** From Corollaries 1.4 and 1.6, the generic curve of the pencil cuts out on $C_2$ the divisor $2\sum_{i=1}^{6} K_i$. So, if we consider a curve of the pencil passing through a point $P \in C_2, P \neq K_i, \forall i$, and in the case $C_2 = \ell_1\ell_2$ is reducible we choose $P = \ell_1 \cap \ell_2$, then such a curve intersects $C_2$ in at least 13 points (counting multiplicities) if $C_2$ is irreducible and in at least 14 points if $C_2$ is reducible (cf. Lemma 1.1). In any case, by Bézout theorem,
it can be divided by $C_2$ and takes the form $C_2C_4$ with a quartic $C_4$, i.e., the identity (1.2) holds true. 

**Lemma 1.8.** The quartic $C_4$ appearing in (1.2) is equal to $C_2^2$. Thus, there exists $\sigma \in k$ such that $\overline{C}_6 = -\sigma C_3^2 + \sigma C_2^3$, where the cubic $C_3$ passes through $K_1, K_2, \ldots, K_6$ and its tangent line at each $K_i$ is the singular tangent line at $K_i$ of $C_6$.

**Proof.** Let us consider the singular point $K_i$. The $K_i$'s are singular points of both of the two generators of the pencil (1.1). It follows that $C_4$ passes through every $K_i$ ($1 \leq i \leq 6$).

Now, we consider $K_1$. We can assume that $K_1 = (0, 0)$ in the affine plane $(x, y)$ and that the singular tangent line at $K_1$ to $C_6$ is $y = 0$. With the above considerations and assumptions on $C_3$ at $K_1$, the identity (1.2) can be written in the following way

\[(1.3) \quad \overline{x}[ay^2 + (\geq 3)] + \overline{\mu}[by + (\geq 2)]^2 = [cx + dy + (\geq 2)]([ax + \beta y + (\geq 2)],\]

where $a, b, c, d, \alpha, \beta \in k$ and the symbol $(\geq m)$ indicates that it is a sum of monomials in $x$ and $y$ of degree $\geq m$.

From (1.3), considering the terms of degree 2, we obtain

\[(1.4) \quad (\overline{\lambda}a + \overline{\mu}b^2)y^2 = (cx + dy)(\alpha x + \beta y).

Then, comparing the both sides of (1.4), we deduce

\[
\begin{aligned}
\{ & c\alpha = 0, \\
& c\beta + d\alpha = 0.
\end{aligned}
\]

Since at $K_1$ the tangent line to $C_2$ is different from $y = 0$ (Corollary 1.2), we have $c \neq 0$. This implies $\alpha = 0$ and from this $\beta = 0$. This means that $C_4$ has at $K_1$ a singular point of multiplicity $\geq 2$. Repeating for $K_2$ and $K_3$ what we did for $K_1$, we see that $C_4$ has at each of the three points $K_1, K_2$ and $K_3$ a singular point of multiplicity $\geq 2$.

In addition, $C_4$ also passes through $K_1, K_5$ and $K_6$. Thus the common points of $C_4$ and $C_2$, counting multiplicities, are at least 9. Now, if $C_2$ is irreducible, then we have $C_4 = C_2E_2$ by Bézout's theorem, where $E_2$ is a suitable conic. If $C_2$ is reducible and $C_2 = \ell_1\ell_2$, then two of the three non-collinear points $K_1, K_2, K_3$ are on a line. Assume that this line is $\ell_1$ and $K_1, K_2, K_4 \notin \ell_1$. The quartic $C_4$ and $\ell_1$ have at least 5 points in common and therefore $\ell_1$ is a component of $C_4$. Put $C_4 = \ell_1D_3$. The cubic $D_3$ cuts out on $\ell_2$ the divisor $2K_3 + K_5 + K_6$. Therefore, $D_3$ contains $\ell_2$. In conclusion, we have again $C_4 = C_2E_2$.

Next, substituting $C_4 = C_2E_2$ in (1.2) and considering the terms of degree 2, we obtain an identity similar to (1.4). More precisely,

\[(1.5) \quad (\overline{\lambda}a + \overline{\mu}b^2)y^2 = (cx + dy)^2\gamma,
\]

where $\gamma$ is the constant appearing in the equation $\gamma + (\geq 1) = 0$ of $E_2$. 

\[\square\]
From (1.5), we obtain $c^2\gamma = 0$. As we already remarked, $c \neq 0$. Hence we have $\gamma = 0$. This means that the conic $E_2$ is passing through $K_1$. Similarly, we see that $E_2$ is passing through $K_2$ and $K_3$.

Let us consider the singularity $K_4$. Substituting $C_4 = C_2E_2$ in (1.2), we obtain $\overline{K}C_6 + \overline{C}C_3^3 = C_2^2E_2$. Assuming also here $K_4 = (0,0)$ with singular tangent line $y = 0$, we can rewrite it in the following way, similarly as in (1.3)

\[
(1.6) \quad \overline{K}[a''y^2 + (\geq 3)] + \overline{a}[a'x + b'y + (\geq 2)]^2 = [c'x + d'y + (\geq 2)][\gamma' + (\geq 1)],
\]

where $\gamma' + (\geq 1) = 0$, $\gamma' \in k$, is the equation of the conic $E_2$. The terms of degree 2 in (1.6) give us

\[
\overline{a}a''y^2 + \overline{a}(a'x + b'y)^2 = (c'x + d'y)^2\gamma'
\]

and we obtain

\[
\begin{cases}
\overline{a}a^2 = c^2\gamma', \\
\overline{a}a' = c'd^2\gamma'.
\end{cases}
\]

We note that $\overline{a} \neq 0$, otherwise in (1.2) $C_2$ is a component of $C_6$, contradicting Lemma 1.3. Since at $K_4$ the tangent line to $C_2$ is different from $y = 0$ by Corollary 1.4, we have $c' \neq 0$.

If $\gamma' \neq 0$, then $a' \neq 0$ and it follows from the two equalities $\frac{b'}{a'} = \frac{d'}{c'}$. This means that $a'x + b'y = a'(x + \frac{b'}{a'}y) = \frac{a'}{c'}(c'x + d'y)$. In other words, the two curves $C_3$ and $C_2$ have the same tangent at $K_4$, contradicting Corollary 1.6. This implies that $\gamma' = 0$ and, thus, $a' = 0$. Now, $a' = 0$ tells us that the cubic $C_3$ is tangent at $K_4$ to the singular tangent line at $K_4$ of the sextic $C_6$. Moreover, $\gamma' = 0$ tells us that the conic $E_2$ passes through $K_4$.

We can say the same things for $K_5$, $K_6$. Therefore, we conclude that the tangent line at any $K_i$ of the cubic $C_3$ coincides with the singular tangent line at $K_i$ to $C_6$, for any $i$. The conics $E_2$ and $C_2$ have the six points $K_i$ in common and therefore they coincide, in either case that $C_2$ is irreducible or not.

Proof of Theorem in the Introduction. Using Lemma 1.8, we can complete the proof of Theorem easily. If at least one of the six singular points $K_i$, $1 \leq i \leq 6$, is not an ordinary cusp, then the sextic $C_6$ and the cubic $C_3$ have at least 19 points in common (counting multiplicities). By Bézout’s theorem, $C_6$ and $C_3$ have a common component. Then, from the identity $\overline{K}C_6 = -\overline{a}C_3^3 + \overline{b}C_2^3$ in Lemma 1.8, we deduce that such a component is a component of $C_2^3$. However, this is impossible, since $C_3$ and $C_2$ have no common components by Lemma 1.5. Therefore, the only possible case is that the six singularities $K_1, K_2, \ldots, K_6$ on the conic are all ordinary cusps. In particular $C_6$ is irreducible, because no reducible curve of degree 6 can have 6 cusps.

The existence of six ordinary cusps of $C_6$ on the conic $C_2$ is clear from the equation $-\overline{a}C_3^3 + \overline{b}C_2^3 = [\sqrt{-\overline{a}}C_3]^2 + [\sqrt{\overline{b}C_2]^3 = 0$, where the cubic $C_3$ and the conic $C_2$ have only to satisfy the condition that they intersect at distinct 6 points.
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