Small asymptotic translation lengths of pseudo-Anosov maps on the curve complex

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Abstract
Let \( M \) be a hyperbolic fibered 3-manifold whose first Betti number is greater than 1 and let \( S \) be a fiber with pseudo-Anosov monodromy \( \psi \). We show that there exists a sequence \((R_n, \psi_n)\) of fibers and monodromies contained in the fibered cone of \((S, \psi)\) such that the asymptotic translation length of \( \psi_n \) on the curve complex \( C(R_n) \) behaves asymptotically like \( 1/|\chi(R_n)|^2 \). As applications, we can reprove the previous result by Gadre–Tsai that the minimal asymptotic translation length of a closed surface of genus \( g \) asymptotically behaves like \( 1/g^2 \). We also show that this holds for the cases of hyperelliptic mapping class group and hyperelliptic handlebody group.

Keywords: pseudo-Anosov, curve complex, asymptotic translation length, fibered 3-manifold, hyperelliptic mapping class group, handlebody group

Mathematics Subject Classification (2010). 57M99, 37E30

1 Introduction
Let \( S_{g,n} \) be an orientable surface of genus \( g \) with \( n \) punctures. We will simply denote it by \( S \). The mapping class group of \( S \), denoted Mod(\( S \)), is the group of isotopy classes of orientation-preserving homeomorphisms of \( S \). By the Nielsen–Thurston classification theorem, each element of Mod(\( S \)) is either periodic, reducible, or pseudo-Anosov.

For a non-sporadic surface \( S \), that is, a surface with \( 3g - 3 + n \geq 2 \), the curve complex \( C(S) \) is defined to be a simplicial complex whose vertex set \( C^0(S) \) is the set of homotopy classes of essential simple closed curves in \( S \), and whose \( k \)-simplices
are formed by $k + 1$ distinct vertices whose representatives can be chosen to be pairwise disjoint. We will restrict our attention to the 1-skeleton $C^1(S)$ of $C(S)$ with path metric $d_C$ by assigning each edge length 1. Then $\text{Mod}(S)$ acts on $C(S)$ by isometry and the asymptotic translation length of $f \in \text{Mod}(S)$ on $C^1(S)$ is defined by

$$\ell_C(f) = \liminf_{j \to \infty} \frac{d_C(\alpha, f^j(\alpha))}{j},$$

where $\alpha$ is an essential simple closed curve in $S$. It follows from the definition that $\ell_C(f)$ is independent of the choice of $\alpha$ and that $\ell_C(f^k) = k\ell_C(f)$ for $k \in \mathbb{N}$.

Masur and Minsky [MM99] showed that $f \in \text{Mod}(S)$ is pseudo-Anosov if and only if $\ell_C(f) > 0$, and Bowditch [Bow08] proved that there exists a constant $m > 0$ depending only on the surface $S$ such that for each pseudo-Anosov mapping class $f$ in $\text{Mod}(S)$, $f^k$ has an invariant geodesic axis on $C(S)$ for some $k \leq m$. In other words, $\ell_C(f)$ is a positive rational number with bounded denominator.

For any subgroup $H \subset \text{Mod}(S)$, let us denote by $L_C(H)$ the minimum of $\ell_C(f)$ over all pseudo-Anosov elements $f \in H$. Then $L_C(H) \geq L_C(\text{Mod}(S))$. We also write $F \asymp G$ if there exists a universal constant $C > 0$ so that $1/C \leq F/G \leq C$.

For the closed surface $S_g$ of genus $g$, Gadre and Tsai [GT11] showed that

$$L_C(\text{Mod}(S_g)) \asymp \frac{1}{g^2}.$$

In fact, using the invariant train tracks constructed by Bestvina and Handel [BH95] and the nesting lemma by Masur and Minsky [MM99], they obtained in [GT11] the lower bound of the asymptotic translation lengths in terms of the Euler characteristic $\chi(S_{g,n})$ of $S_{g,n}$. That is,

$$\ell_C(f) \geq \frac{1}{18\chi(S_{g,n})^2 + 30|\chi(S_{g,n})| - 10n}$$

for any pseudo-Anosov element $f \in \text{Mod}(S_{g,n})$. To obtain the upper bound, they use an explicit family of pseudo-Anosov mapping classes. This family was first considered by Penner [Pen91] to find small stretch factors of pseudo-Anosov maps:

$$L_C(\text{Mod}(S_g)) \leq \frac{4}{g^2 + g - 4}.$$

In this paper, we describe a way of generating a sequence of pseudo-Anosov mapping classes $\psi_n \in \text{Mod}(S_n)$ with small asymptotic translation lengths on the curve complex. We say that a sequence $\{\psi_n\}$ has a small asymptotic translation length if $\ell_C(\psi_n) \asymp 1/|\chi(S_n)|^2$, where $\chi(S_n)$ is the Euler characteristic of the corresponding surface $S_n$ such that $|\chi(S_n)| \to \infty$ as $n \to \infty$.

Let $M$ be a hyperbolic fibered 3-manifold with the first Betti number $b_1(M) \geq 2$ and let $S \subset M$ be a fiber with pseudo-Anosov monodromy $\psi$. Then the assumption $b_1(M) \geq 2$ implies that there is a primitive cohomology class $\xi_0 \in H^1(S; \mathbb{Z})$ fixed by $\psi$, that is, $\xi_0 \circ \psi_* = \xi_0$, where $\psi_* : H_1(S; \mathbb{Z}) \to H_1(S; \mathbb{Z})$. Let $p : \tilde{S} \to S$ be a $\mathbb{Z}$-covering map corresponding to $\xi_0$ whose deck transformation group is generated by $h : \tilde{S} \to \tilde{S}$ and let $\tilde{\psi}$ be a lift of $\psi$ to $\tilde{S}$. Then we have the following main theorem.
Figure 1: Simple closed curves and the standard basis for $H_1(S_2;\mathbb{Z})$.

**Theorem A.** For all sufficiently large $n$, $R_n = \tilde{S}/(h^n\tilde{\psi})$ is a fiber of $M$ with $|\chi(R_n)| \asymp n$ whose pseudo-Anosov monodromy $\psi_n$ satisfies

$$\ell_C(\psi_n) \asymp \frac{1}{|\chi(R_n)|^2}.$$  

The above family of fibers in a fibered 3-manifold was first considered by McMullen and he proved the following theorem providing short geodesics on the moduli space when $S$ is a closed surface.

**Theorem 1.1** (McMullen, Theorem 10.2 in [McM00]). For all $n$ sufficiently large, 

$$R_n = \tilde{S}/(h^n\tilde{\psi})$$  

is a closed surface of genus $g_n \asymp n$, and $h^{-1}: \tilde{S} \to \tilde{S}$ descends to a pseudo-Anosov mapping class $\psi_n \in \text{Mod}(R_n)$ with 

$$\log \lambda(\psi_n) \asymp \frac{1}{g_n},$$  

where $\lambda(\psi_n)$ is the stretch factor of $\psi_n$.

Although McMullen dealt with closed hyperbolic 3-manifolds in Theorem 1.1, we can adopt the same proof for the general case of fibers of cusped hyperbolic 3-manifolds. In such case, we have to say $\log \lambda(\psi_n) \asymp 1/|\chi(R_n)|$ and $|\chi(R_n)| \asymp n$.

As a consequence of Theorem A, we can determine the behavior of minimal asymptotic translation lengths of a few subgroups of mapping class groups. First of all, the fact that $L_C(\text{Mod}(S_g)) \asymp 1/g^2$ also follows from Theorem A by considering the genus 2 surface $S_2$ and any mapping class fixing a nontrivial cohomology class. For instance, consider the mapping class $\psi = T_{a_1}T_{a_2}T_{a_3}T_{b_1}^{-1}T_{b_2}^{-1}$ of the closed surface of genus 2 as in Figure 1, where $T_\gamma$ is the left-handed Dehn twist about a simple closed curve $\gamma$. (We apply elements of the mapping class group from right to left.) Then $\psi$ is pseudo-Anosov because it is coming from Penner’s construction (see, for instance, [FM12, Theorem 14.4]). The action of $\psi$ on the first homology $H_1(S_2;\mathbb{Z})$ with respect to the basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ as in Figure 1 is given by the matrix

$$\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$
Table 1: Minimal asymptotic translation lengths.

<table>
<thead>
<tr>
<th></th>
<th>Mod(S₀ₙ)</th>
<th>Mod(S₁₂ₙ)</th>
<th>For any fixed g ≥ 2, Mod(Sₜₙ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_C(\cdot))</td>
<td>(\frac{1}{n^2}) [Val14]</td>
<td>(\frac{1}{n^2}) [GT11]</td>
<td>(\frac{1}{n}) [Val14]</td>
</tr>
</tbody>
</table>

Therefore there is a 1-dimensional subspace of \(H_1(S_2; \mathbb{Z})\) and its dual \(\xi_0 \in H^1(S_2; \mathbb{Z})\) is given by the algebraic intersection number with an oriented simple closed curve \(c\). Hence, \(\xi_0\) is a cohomology class fixed by \(\psi\).

Valdivia [Val14] showed that fixing \(g \geq 2\) as \(n \to \infty\),

\[L_C(\text{Mod}(S_{g,n})) \approx \frac{1}{n},\]

and for the remaining cases of \(S_{0,n}\) and \(S_{1,2n}\) with even number of punctures as \(n \to \infty\), see Table 1. We will determine the minimal asymptotic translation lengths of a few other types of surfaces including the surface of genus 1 with odd number of punctures. Let \(D_n\) be the closed disk \(D\) with \(n\)-punctures and let \(\text{Mod}(D_n)\) be the mapping class group of \(D_n\) fixing the boundary \(\partial D\) of the disk \(D\) pointwise. As an application of Theorem A, we have the following results.

**Theorem B.** We have

1. \(L_C(\text{Mod}(D_n)) \approx \frac{1}{n^2}\), and
2. \(L_C(\text{Mod}(S_{1,n})) \approx \frac{1}{n^2}\).

Furthermore, we improve the upper bound for the minimal asymptotic translation length for \(S_g\). The *hyperelliptic mapping class group* \(H(S_g)\) of \(S_g\) is the subgroup of \(\text{Mod}(S_g)\) consisting of elements with representative homeomorphisms that commute with some fixed hyperelliptic involution.

**Theorem C.** For closed surfaces \(S_g\) with \(g \geq 3\),

\[L_C(H(S_g)) \leq \frac{1}{g^2 - 2g - 1},\]

and as a direct consequence, we have

\[L_C(\text{Mod}(S_g)) \leq \frac{1}{g^2 - 2g - 1}.\]

We remark that for \(g \geq 4\), this is a sharper upper bound than Gadre–Tsai’s.

As another application, we determine the asymptotes of minimal asymptotic translation lengths for handlebody groups and hyperelliptic handlebody groups. Let \(H_g\) be the handlebody of genus \(g\), that is, a 3-manifold bounded by a closed orientable surface \(S_g\) of genus \(g\). The *handlebody group* \(\text{Mod}(H_g)\) is the subgroup
of $\text{Mod}(S_g)$ consisting of elements whose representative homeomorphisms of $S_g$ can be extended to homeomorphisms of $\mathbb{H}_g$. Then the hyperelliptic handlebody group is defined by
\[ \mathcal{H}(\mathbb{H}_g) = \text{Mod}(\mathbb{H}_g) \cap \mathcal{H}(S_g). \]

**Theorem D.** We have
\[ L_C(\mathcal{H}(\mathbb{H}_g)) \asymp \frac{1}{g^2}. \]

The following is an immediate corollary of the previous Theorem D and the lower bound by Gadre–Tsai.

**Corollary E.** We have
\[ L_C(\mathcal{H}(S_g)) \asymp \frac{1}{g^2} \text{ and } L_C(\text{Mod}(\mathbb{H}_g)) \asymp \frac{1}{g^2}. \]

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## 2 Proof of Theorem A

In this section, we begin with the following simple observation.

**Lemma 2.1.** Let $f \in \text{Mod}(S)$ be a pseudo-Anosov mapping class and let $\alpha$ be any essential simple closed curve in $S$. If $d_C(\alpha, f^m(\alpha)) = 1$ for some $m \in \mathbb{N}$, then
\[ \ell_C(f) \leq \frac{1}{m}. \]

**Proof.** By the triangle inequality, we have
\[
\ell_C(f^m) = \liminf_{j \to \infty} \frac{d_C(\alpha, f^{jm}(\alpha))}{j} \\
\leq \liminf_{j \to \infty} \frac{\sum_{i=1}^{j} d_C(f^{(i-1)m}(\alpha), f^{im}(\alpha))}{j} \\
= \liminf_{j \to \infty} \frac{j \cdot d_C(\alpha, f^m(\alpha))}{j} = 1
\]

Since $\ell_C(f^m) = m \ell_C(f)$, this completes the proof. \(\square\)
Now we prove our main theorem.

**Proof of Theorem A.** Since the lower bound was established by Gadre–Tsai, it is enough to show that there exists some constant $C$ such that

$$
\ell_c(\psi_n) \leq \frac{C}{|\chi(R_n)|^2}.
$$

The proof consists of the following three steps. In step 1, we establish the structure of the $\mathbb{Z}$-cover $\tilde{S}$ of $S$ as the union of $\mathbb{Z}$-copies of $S \setminus \{c\}$, where $[c]$ is the homology class dual to the primitive cohomology $\xi_0$ fixed by $\psi$. In step 2, using the decomposition of $\tilde{S}$ in step 1, we will find an integer $r$ so that $\psi^r_n(\overline{\alpha})$ and $\alpha$ are disjoint in the quotient surface $R_n$, where $\overline{\alpha}$ is either a simple closed curve or a simple proper arc and $\psi_n$ is a pseudo-Anosov monodromy on $R_n$. In step 3, using Lemma 2.1, we deduce that the asymptotic translation length of $\psi_n$ is less than or equal to $1/r$ and we show that we can choose $r$ to be quadratic in $n$. This finishes the proof.

**Step 1.** (The decomposition of $\tilde{S}$) Let $[c]$ be a homology class in $H_1(S; \mathbb{Z})$ which is dual to the primitive cohomology $\xi_0 \in H^1(S; \mathbb{Z})$. Since $\xi_0$ is primitive, $[c]$ is also a primitive element. If $S$ is a closed surface, one can find a representative $c$ that is an oriented simple closed curve and if $S$ is a surface with punctures, $c$ can be chosen to be a simple proper arc or union of disjoint simple proper arcs (see, for instance, Proposition 6.2 in [FM12]). Let $\tilde{S}$ be the surface obtained by cutting $S$ along $c$ and concatenating $\mathbb{Z}$-copies of $S \setminus \{c\}$ together (see Figure 2 in the case of closed surfaces and see Figure 8 in the case of punctured surfaces). Then the natural projection map $p : \tilde{S} \to S$ is a covering map corresponding to $\xi_0$ because the kernel of the composition $\pi_1(S) \to H_1(S; \mathbb{Z}) \to \mathbb{Z}$ of the Hurewicz map and $\xi_0$ is equal to $p_*(\pi(\tilde{S}))$. Let $\Sigma_i$ be the copies of $S \setminus \{c\}$ on $\tilde{S}$ such that the generator $h : \tilde{S} \to \tilde{S}$ for the deck transformation group is given by $h(\Sigma_i) = \Sigma_{i+1}$ for all $i$ (See Figure 2).

**Step 2.** (Finding a positive integer $r$ such that $\psi^r_n(\overline{\alpha})$ and $\overline{\alpha}$ are disjoint) Choose a lift $\tilde{\psi}$ and take a constant $k = k(\tilde{\psi})$ such that

$$
\tilde{\psi}(\Sigma_0) \subset \Sigma_{-k} \cup \ldots \cup \Sigma_{k-1} \cup \Sigma_k.
$$
Suppose $n$ is large enough so that $n - k > 1$. (More precise condition on $n$ will be determined later.) Then $h^n \bar{\psi}(\Sigma_0)$ and $\Sigma_0$ are disjoint and no orbit of $\Sigma_0$ under the cyclic group $\langle h^n \bar{\psi} \rangle$ intersect with $\Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_{n-k-1}$. Let $\alpha$ be a simple closed curve or a simple proper arc contained in $\Sigma_0$ and let $\bar{\pi}$ be the $\langle h^n \bar{\psi} \rangle$-orbit of $\alpha$ in $\bar{S}$. (One can choose $\alpha$ to be one of parallel copies of $c$, and $\alpha$ in Figure 4 is not the case.) It follows that if a simple closed curve or a simple proper arc $\beta$ lies in $\Sigma_1 \cup \ldots \cup \Sigma_{n-k-1}$, i.e., disjoint from both $\alpha$ and $h^n \bar{\psi}(\alpha)$, then $\bar{\beta}$ and $\bar{\pi}$ are disjoint in $R_n = \bar{S}/\langle h^n \bar{\psi} \rangle$. Since $h^{-1}$ induces a pseudo-Anosov map on $R_n$, let us find a positive integer $r$ as large as possible such that one of the representative of $\bar{h}^{-r}(\alpha)$ is contained in $\Sigma_1 \cup \ldots \cup \Sigma_{n-k-1}$ (see Figure 4). Then this representative of $\bar{h}^{-r}(\alpha)$ and $\alpha$ and $h^n \bar{\psi}(\alpha)$, and because of the previous argument, $\bar{h}^{-r}(\alpha)$ and $\bar{\pi}$ are disjoint in $R_n$. By the fact that $h^{-1}$ descends to a pseudo-Anosov $\psi_n$ in $R_n$ together with Lemma 2.1, this allows us to obtain the upper bound for the asymptotic translation length of $\psi_n$. To find such $r$, first note that since $\alpha$ is in $\Sigma_0$, we have $\bar{\psi}(\alpha) \subset \Sigma_{-k} \cup \ldots \cup \Sigma_k$ and $\bar{\psi}^n(\alpha) \subset \Sigma_{-m} \cup \ldots \cup \Sigma_{m}$ for any $m \in \mathbb{N}$. Since the generator $h$ of the deck transformation group translates $\Sigma_i$'s, after applying $h^{mk+1}$, we have $h^{mk+1} \bar{\psi}^m(\alpha) \subset \Sigma_1 \cup \ldots \cup \Sigma_{2mk+1}$. In order that $h^{mk+1} \bar{\psi}^m(\alpha)$ lies in $\Sigma_1 \cup \ldots \cup \Sigma_{n-k-1}$, we require that $2km + 1 \leq n - k - 1$. Let us choose the biggest such $m$, that is

$$m = \left\lceil \frac{n-k-2}{2k} \right\rceil$$

(note that the precise assumption on $n$ is $(n-k-2)/2k \geq 1$ because we want $m$ to be positive). Since $\bar{\psi}(\alpha) = h^{-n}(\alpha)$ and hence $h^{mk+1} \bar{\psi}^m(\alpha) = h^{-n(\alpha + 1)}(\alpha)$, the desired integer for $h^{-r}(\alpha)$ and $\bar{\pi}$ being disjoint is $r = (n-k)m - 1$.

**Step 3.** (Small asymptotic translation length $\ell_C(\psi_n)$) We first remark that arc and curve complex $\mathcal{AC}(S)$ and curve complex $\mathcal{C}(S)$ are 2-bilipschitz (see, for instance, [MM00, Lemma 2.2] or [HPW15]). This implies that the asymptotic translation lengths $\ell_{\mathcal{AC}}(f)$ and $\ell_C(f)$ of a pseudo-Anosov mapping class $f$ on the 1-skeletons $\mathcal{AC}(S)$ and $\mathcal{C}(S)$, respectively, have the same asymptotic behavior, that is,

$$\ell_{\mathcal{AC}}(f) \asymp \ell_C(f).$$

So without loss of generality, we may assume that $\alpha$ is a simple closed curve and compute the asymptotic translation length on the curve complex $\mathcal{C}(R_n)$. In the case when $\alpha$ is a simple proper arc, we think of computing the asymptotic translation length of the arc and curve complex $\mathcal{AC}(R_n)$ and it gives us the same asymptotic behavior on the curve complex.

By the previous step, $\psi_n^{(n-k)m-1}(\bar{\pi})$ and $\bar{\pi}$ are disjoint in $R_n$. Then by Lemma 2.1, we have

$$\ell_C(\psi_n) \leq \frac{1}{(n-k)m - 1}.$$
3 Backgrounds for Theorems B, C, and D.

This section includes some backgrounds and basic facts for the proofs of the rest of theorems. Consider a pseudo-Anosov mapping class $\psi \in \text{Mod}(S)$. Let $\Psi : S \to S$ be any representative of $\psi$. The mapping torus $M_\psi$ is defined by

$$M_\psi = S \times [0, 1] / \sim,$$

where $\sim$ identifies $(x, 1)$ with $(\Psi(x), 0)$ for each $x \in S$. Then the manifold $M_\psi$ fibering over the circle $S^1$ is hyperbolic. Suppose that there is a primitive cohomology class $\xi_0 \in H^1(S; \mathbb{Z})$ fixed by $\psi$. This implies that $b_1(M_\psi) \geq 2$. Then Theorem A says that for $n$ sufficiently large, $R_n = \tilde{S}/\langle \psi h^n \rangle$ is a fiber of $M_\psi$ with $\chi(R_n) \approx n$ such that the pseudo-Anosov monodromy $\psi_n$ defined on the fiber $R_n$ satisfies $\ell_C(\psi_n) \approx 1/|\chi(R_n)|^2$.

3.1 Fibered 3-manifolds from braids

Let $B_n$ be the braid group with $n$ strands. In this paper braids are depicted vertically. We define the product $\beta \beta'$ of $\beta, \beta' \in B_n$ in the usual way, namely, we
Figure 5: (1) Arcs $c_i$ in the $n$-punctured disk $D_n$. (2) Generators $\sigma_i$. (3) Half twist $h_i$. ($c'_i = h_i(c_i)$ and $c'_{i+1} = h_i(c_{i+1})$.)

stack $\beta$ on $\beta'$ and concatenate the bottom $i$th end point of $\beta$ with the top $i$th end point of $\beta'$ for each $i = 1, \cdots, n$. Then we obtain $n$ strands. The product $\beta\beta'$ is the resulting $n$-braid after rescaling.

We briefly review a relation between $B_n$ and $\text{Mod}(D_n)$. To do this we assign an orientation for each $n$-braid from the bottom endpoints to the top endpoints (see Figure 5(2)). We take a natural basis $t_i \in H_1(D_n; \mathbb{Z})$, where a representative of $t_i$ is a small oriented loop in $D_n$ centered at the $i$th puncture of $D_n$ for $i = 1, \cdots, n$. Let $c_i$ be a simple proper arc in $D_n$ which connects the $i$th puncture of $D_n$ to the boundary $\partial D$ as in Figure 5(1). Then there is an isomorphism

$$\Gamma : B_n \rightarrow \text{Mod}(D_n)$$

which sends the generator $\sigma_i$ of $B_n$ to the left-handed half twist $h_i$ (see Figure 5(2)(3)). The orientation of braids as we described above induces the motion of $n$ punctures in the disk, which defines the above map $\Gamma$.

We have a natural homomorphism

$$c : \text{Mod}(D_n) \rightarrow \text{Mod}(S_{0,n+1})$$

collapsing the boundary $\partial D$ of the disk to the $(n+1)$th puncture of $S_{0,n+1}$. By definition, $c(\text{Mod}(D_n))$ is isomorphic to the subgroup of $\text{Mod}(S_{0,n+1})$ which fixes this puncture. We sometimes identify $f \in \text{Mod}(D_n)$ with $c(f) \in \text{Mod}(S_{0,n+1})$. We simply denote by $\beta$, both mapping classes $\Gamma(\beta) \in \text{Mod}(D_n)$ and $c(\Gamma(\beta)) \in \text{Mod}(S_{0,n+1})$.

The closure $\text{cl}(\beta)$ of $\beta \in B_n$ is a knot or link in the 3-sphere $S^3$. Let $A$ be a braid axis of $\beta$ which is an unknot in $S^3$. Then $\text{cl}(\beta)$ runs around the unknot $A$ in a monotone manner. We set $\text{br}(\beta) = \text{cl}(\beta) \cup A$ which is a link in $S^3$ whose number of the components is greater than or equal to 2, and let us set $M_\beta = S^3 \setminus \text{br}(\beta)$. The 3-manifold $M_\beta$ is homeomorphic to the interior of the mapping torus of the
monodromy $\beta \in \text{Mod}(D_n)$, and $b_1(M_\beta) \geq 2$. A spanning disk by the unknot $A$ has $n$ punctures in $M_\beta$, and such a disk with punctures is a fiber of $M_\beta$ with monodromy $\beta$.

### 3.2 Subgroups of mapping class groups

Let $SB_n$ be the spherical $m$-braid group. We now introduce the subgroup $SW_{2n}$ of $SB_{2n}$. Let $A_1, A_2, \cdots, A_n$ be $n$ disjoint unknotted arcs properly embedded in the 3-ball $D^3$ so that $A = A_1 \cup \cdots \cup A_n$ is unlinked as in Figure 6(1). The boundary $\partial A$ is the set of $2n$ points in the 2-sphere $\partial D^3$.

For $b \in SB_{2n}$, we stack $b$ on $A$, and concatenate the bottom endpoints of $b$ with the endpoints of $A$. As a result we obtain $n$ disjoint (knotted) arcs $b^k A$ properly embedded in $D^3$ (see Figure 6(2)). The wicket group $SW_{2n}$ is the subgroup of $SB_{2n}$ generated by braids $b$'s such that $b^k A$ is isotopic to $A$ relative to $\partial A$. It is easy to see that the braid $w \in SW_6$ as shown in Figure 6(3) is an element of $SW_6$.

There is a spherical version of the isomorphism $\Gamma : B_n \to \text{Mod}(D_n)$, namely we have a surjective homomorphism $SB_n \to \text{Mod}(S_{0,m})$ which sends the generator $\sigma_i$ of $SB_m$ to the left-handed half twist between the $i$th and $(i+1)$st punctures.
(cf. Figure 5(2)(3)). We also denote this homomorphism by
\[ \Gamma : SB_m \rightarrow \text{Mod}(S_{0,m}). \]
Its kernel is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) generated by a full twist \( \Delta^2 \in SB_m \), where \( \Delta \) is a half twist. When \( m = 2n \) the image \( \Gamma(SW_{2n}) \) of \( SW_{2n} \) under the map \( \Gamma \) is a subgroup of \( \text{Mod}(S_{0,2n}) \) which is so-called \textit{Hilden group}, denoted by \( SH_{2n} \), and
\[ SH_{2g+2} \simeq SW_{2g+2}/(\langle \Delta^2 \rangle) \]
holds (see [HK17]).

For the proof of Theorem D, we recall a connection between the wicket group and the hyperellitic handlebody group. We first state a theorem by Birman and Hilden which relates \( H(S_g) \) to \( \text{Mod}(S_{0,2g+2}) \). Each homeomorphism on \( S_g \) which commutes with some fixed hyperelliptic involution \( I : S_g \rightarrow S_g \) (Figure 7(1)) preserves the set of fixed points of \( I \) consisting of \( 2g + 2 \) points. Such a homeomorphism induces a homeomorphism on a sphere \( S_g/I \) which preserves these fixed points (Figure 7(2)). Thus we have a map
\[ q : H(S_g) \rightarrow \text{Mod}(S_{0,2g+2}) \]
by choosing a representative of each mapping class of \( H(S_g) \) which commutes with \( I \). It is shown in [BH71] that the map \( q \) is well-defined and it is a surjective homomorphism whose kernel is generated by \( \iota = [I] \in H(S_g) \). In particular we have
\[ H(S_g)/\langle \iota \rangle \simeq \text{Mod}(S_{0,2g+2}) \simeq SB_{2g+2}/(\langle \Delta^2 \rangle). \]

On the other hand, it is proved in [HK17] that there is a surjective homomorphism
\[ Q : H(H_g) \rightarrow SH_{2g+2} \]
whose kernel is generated by \( \iota \). The map \( Q \) is given by the restriction
\[ q|_{H(H_g)} : H(H_g) \rightarrow SH_{2g+2} < \text{Mod}(S_{0,2g+2}). \]
Putting all things together, we have
\[ H(H_g)/\langle \iota \rangle \simeq SH_{2g+2} \simeq SW_{2g+2}/(\langle \Delta^2 \rangle). \]
Thus an element \( f \in SH_{2g+2} \) can be described by a braid \( v \in SW_{2g+2} \), i.e., \( f = \Gamma(v) \). Moreover a lift \( \hat{f} \) of \( f \) under the map \( q|_{H(H_g)} = Q \) is an element of \( H(H_g) \). We simply denote by \( v \), the element \( \Gamma(v) \) in the Hilden group \( SH_{2g+2} \).

The following lemma is used in the proofs of the rest of theorems (other than Theorem B(2)).

**Lemma 3.1.** Let \( f \in \text{Mod}(S_{0,2g+2}) \) for \( g \geq 2 \) and let \( \hat{f} \in H(S_g) \) be a lift of \( f \) under the map \( q : H(S_g) \rightarrow \text{Mod}(S_{0,2g+2}) \). We take any \( \alpha \in \text{AC}^0(S_{0,2g+2}) \), i.e., \( \alpha \) is a homotopy class of an arc or simple closed curve in \( S_{0,2g+2} \). Suppose that \( d_{AC}(\alpha, f^m(\alpha)) = 1 \) for some \( m \in \mathbb{N} \), where \( d_{AC} \) is the path metric on \( AC(S_{0,2g+2}) \). Then
\[ \ell_C(\hat{f}) \leq \frac{1}{m}. \]
It is well-known and not hard to see that if \( f \in \text{Mod}(S_{0,2g+2}) \) is pseudo-Anosov, then \( \hat{f} \in \mathcal{H}(S_g) \) is also pseudo-Anosov.

**Proof of Lemma 3.1.** By abuse of the notation, a representative of \( \alpha \in \mathcal{AC}^0(S_{0,2g+2}) \) is denoted by the same \( \alpha \). Let \( \hat{\alpha} \subset S_g \) be the preimage \( q^{-1}(\alpha) \) of a simple arc or simple closed curve \( \alpha \) in \( S_{0,2g+2} \) under the map \( q \). If \( \alpha \) is a simple arc, then \( \hat{\alpha} \) is a non-separating simple closed curve in \( S_g \) which means \( \hat{\alpha} \) is essential. Hence \( \hat{\alpha} \in \mathcal{C}^0(S_g) \). The assumption implies that \( d_C(\hat{\alpha}, (\hat{f})^m(\hat{\alpha})) = 1 \). The claim follows from Lemma 2.1.

If \( \alpha \) is a simple closed curve, then \( \alpha \) cuts \( S_{0,2g+2} \) into two components \( S(1) \) and \( S(2) \) which are disks with punctures \( n_1 \geq 2 \) and \( n_2 \geq 2 \) respectively. Since \( n_1 + n_2 = 2g + 2 \), both \( n_1 \) and \( n_2 \) have the same parity.

We first consider the case where \( n_1 \) and \( n_2 \) are odd. Then \( n_1, n_2 \geq 3 \). Observe that \( \hat{\alpha} \) is a single simple closed curve. Since \( \hat{\alpha} \) cuts \( S_g \) into the essential surfaces \( q^{-1}(S(1)) \) and \( q^{-1}(S(2)) \) with positive genera, \( \hat{\alpha} \) is a separating and essential simple closed curve. We have \( d_C(\hat{\alpha}, (\hat{f})^m(\hat{\alpha})) = 1 \) by the assumption. Thus \( \ell_C(\hat{f}) \leq \frac{1}{m} \) holds.

Let us consider the remaining case where both \( n_1 \) and \( n_2 \) are even with \( n_1, n_2 \geq 2 \). Observe that \( \hat{\alpha} \) has two components \( \hat{\alpha}(1) \) and \( \hat{\alpha}(2) \) which are non-separating simple closed curves. Hence \( \hat{\alpha}(i) \in \mathcal{C}^0(S_g) \) for \( i = 1, 2 \). We have \( d_C(\hat{\alpha}(i), (\hat{f})^m(\hat{\alpha}(i))) = 1 \) by the assumption, and hence \( \ell_C(\hat{f}) \leq \frac{1}{m} \) holds. We complete the proof. \( \square \)

## 4 Proof of Theorem B

This section is devoted to the proof of Theorem B. In the proof of Theorem B(2), we reprove the previous result \( \text{Mod}(S_{1,2n}) \cong \frac{1}{n!} \) by Gadre–Tsai.

**Proof of Theorem B(1).** We separate the proof into two cases, depending on the parity of the number of punctures of \( D_n \). We first deal with the case where \( n \) is even. We consider the pseudo-Anosov braid \( \beta = \sigma_1^2 \sigma_2 \in B_3 \) (Figure 8(1)) and the fibered hyperbolic 3-manifold \( M_\beta \). We take a fiber \( S = D_3 \) with monodromy \( \psi = \beta \) of \( M_\beta \). Let \( \xi_0 \in H^1(S; \mathbb{Z}) \) be the primitive cohomology class which is dual to the homology class of the proper arc \( c = c_1 \) in \( S \) (see Figure 5(1) for \( c_1 \)).

From Figure 8(1), one sees that the induced \( \psi_* = \beta_* : H_1(D_3; \mathbb{Z}) \rightarrow H_1(D_3; \mathbb{Z}) \) maps \( t_1, t_2, t_3 \in H_1(D_3; \mathbb{Z}) \) to \( t_1, t_3, t_2 \) respectively, where the set of \( t_i \)'s is a natural basis of \( H_1(D_n; \mathbb{Z}) \) (see Section 3.1). This tells us that \( \xi_0 \) is fixed by \( \psi \). Figure 8(2) illustrates the \( \mathbb{Z} \)-cover \( \tilde{S} \) corresponding to \( \xi_0 \). We consider the canonical lift \( \tilde{\psi} : \tilde{S} \rightarrow \tilde{S} \) of \( \psi \) which means that \( \tilde{\psi} \) fixes the preimage \( p^{-1}(\partial D) \) of the (outer) boundary of the 3-punctured disk pointwise. (In Figure 9(1)(2), the set \( p^{-1}(\partial D) \cap \Sigma_i \) is thickened.) We set \( \tilde{\Sigma}_i = \Sigma_{i-1} \cap \Sigma_i \) which is a connected component of the preimage \( p^{-1}(c) \) of \( c \) (see Figure 8(2)). In other words, \( \tilde{\Sigma}_i \) and \( \tilde{\Sigma}_{i+1} \) bound the copy \( \hat{\Sigma}_i \). To see the image \( \tilde{\psi}(\Sigma_i) \) of \( \Sigma_i \) under \( \tilde{\psi} \), we consider \( \tilde{\psi}(\tilde{\Sigma}_i) \) and \( \tilde{\psi}(\tilde{\Sigma}_{i+1}) \) which are determined by the proper arc \( \psi(c) = \beta(c) \) (see
Figure 8: (1) $\beta = \sigma_1^{-2}\sigma_2 \in B_3$. (2) $\mathbb{Z}$-cover $\tilde{S}$ over $S = D_3$ corresponding to the dual to $c = c_1$. (3) $c$, $\beta(c)$ and $\beta^2(c)$.

Figure 9: Illustration of $h^2\tilde{\psi} : \tilde{S} \to \tilde{S}$. Shaded regions in (1)(2) and (3) are $\Sigma_i$, $\tilde{\psi}(\Sigma_i)$ and $h^2\tilde{\psi}(\Sigma_i)$ respectively.
Figure 10: (1) Shaded region descends to $R_2 \simeq S_{0,5}$. See also Figure 9. (2) Shaded region descends to $R_3 \simeq S_{0,7}$. (Note that $[\tilde{c}(i)] = [\tilde{h}^n \tilde{\psi}(\tilde{c}(i))]$ in $R_n$.)

Figure 8(3)). Observe (from Figure 9(1) and (2)) that

$$\tilde{\psi}(\Sigma_i) \subset \Sigma_i - 1 \cup \Sigma_i \quad \text{and} \quad \tilde{\psi}^{-1}(\Sigma_i) \subset \Sigma_i \cup \Sigma_{i+1}.$$  

Hence for each $n \geq 0$

$$h^n \tilde{\psi}(\Sigma_i) \subset \Sigma_{i-1+n} \cup \Sigma_{i+n} \quad \text{and} \quad (h^n \tilde{\psi})^{-1}(\Sigma_i) = h^{-n} \tilde{\psi}^{-1}(\Sigma_i) \subset \Sigma_{i-n} \cup \Sigma_{i+n+1}.$$  

For $\ell > 0$, we have

$$(h^n \tilde{\psi})^\ell(\Sigma_i) \subset \Sigma_{i-\ell+n} \cup \cdots \cup \Sigma_{i-1+\ell+n} \cup \Sigma_{i+\ell},$$  

$$(h^n \tilde{\psi})^{-\ell}(\Sigma_i) \subset \Sigma_{i-\ell} \cup \Sigma_{i-\ell+1} \cup \cdots \cup \Sigma_{i+\ell}.$$  

Notice that if we fix $n \geq 2$, then $(h^n \tilde{\psi})^\ell(\Sigma_i) \cap \Sigma_i = \emptyset$ for each $\ell > 0$, and hence $R_n = \tilde{S}/(h^n \tilde{\psi})$ is a surface. In fact $R_n$ is a disk with $2n$ punctures, and hence we can think of $R_n$ as a sphere with $2n + 1$ punctures (see Figures 9 and 10). Note that one of the punctures of $R_n$, say $p_\infty$, comes from the preimage of the boundary of the disk under the projection $p : \tilde{S} \to S = D_3$. By Theorem 1.1, we know $h^{-1}$ descends to the monodromy $\psi_n$, and we see that $\psi_n$ maps $p_\infty$ to itself. Thus $\psi_n \in \text{Mod}(D_{2n})$. By Theorem A, we have $\ell_C(\psi_n) \leq C/n^2$ for some constant $C$, and hence $L_C(\text{Mod}(D_{2n})) \leq C/n^2$.

For the case where the number of the punctures of $D_n$ is odd, we turn to the pseudo-Anosov braid $\phi = \beta^2 \in B_3$. The hyperbolic fibered 3-manifold $M_\phi$ has a
fiber $S = D_3$ with monodromy $\phi$. The dual to $c = c_1$ is the primitive cohomology class fixed by $\phi$. Consider the $\mathbb{Z}$-cover $\tilde{S}$ corresponding to this cohomology class. Let $\tilde{\phi} = (\tilde{\psi})^2 : \tilde{S} \to \tilde{S}$ be the canonical lift of $\phi$ as before. By using the proper arc $\phi(c) = \beta^2(c)$ (see Figure 8(3)), we see where each copy $\Sigma_i$ maps on $\tilde{S}$ under $\tilde{\phi}$. We use the same argument as above replacing $\tilde{\psi}$ with $\tilde{\phi}$, and construct a surface $\tilde{S}/\langle h^n \tilde{\phi} \rangle$ concretely. Then we find that this surface is a sphere with $2n + 2$ punctures which is a fiber of $M_\phi$ for $n$ large. Also we see that $\phi_n$ fixes one of the punctures of the fiber (which comes from the preimage of the boundary of the disk). Thus $\phi_n \in \text{Mod}(D_{2n+1})$. By Theorem A, we have $\ell_C(\text{Mod}(D_{2n+1})) \leq C'/n^2$. This tells us that $L_C(\text{Mod}(D_{2n+1})) \leq C'/n^2$. This completes the proof.

**Proof of Theorem B(2).** We first consider the case where the number of punctures is odd. Let $L_W$ be the Whitehead link in $S^3$. The complement $S^3 \setminus L_W$ is a fibered hyperbolic 3-manifold with a fiber $S_{1,2}$. Consider its pseudo-Anosov monodromy $\psi$ defined on the fiber $S_{1,2}$ (see [KR, Appendix B] for more details), and we use a basis $\alpha, \beta, \gamma \in H_1(S_{1,2}; \mathbb{Z})$ as in Figure 11(1). Let $m$ be a simple closed curve in $S_{1,2}$, and let $\ell$ be an oriented loop based at one of the punctures of $S_{1,2}$ as in Figure 11(2). Let $c$ be a representative of the generator $\beta \in H_1(S_{1,2}; \mathbb{Z})$ as in Figure 11(3). We set $\psi = T_m^{-1} f_\ell \in \text{Mod}(S_{1,2})$ where $f_\ell$ is the mapping class which represents the point-pushing map along $\ell$ (see Figure 11(3)). Then $\psi$ is the monodromy of a fibration on $S^3 \setminus L_W$, i.e., $M_\psi$ is homeomorphic to $S^3 \setminus L_W$. In particular $\psi$ is pseudo-Anosov since $S^3 \setminus L_W$ is hyperbolic. Observe that the induced map $\psi_* : H_1(S_{1,2}; \mathbb{Z}) \to H_1(S_{1,2}; \mathbb{Z})$ sends $\alpha, \beta$ and $\gamma$ to $\alpha - \beta - \gamma, \beta + \gamma$ and $\gamma$ respectively. Then the cohomology class $\xi_0 \in H^1(S_{1,2}; \mathbb{Z})$ which is dual to $c$, is primitive and fixed by $\psi$. We consider the $\mathbb{Z}$-cover $\tilde{S}$ over $S = S_{1,2}$ corresponding to $\xi_0$, and we take a lift $\tilde{\psi} : \tilde{S} \to \tilde{S}$ such that $\tilde{\psi}(\Sigma_i) \subset \Sigma_{i-1} \cup \Sigma_i$ (see Figures 12 and 13).
By the same argument as in the proof of Theorem B(1), we verify that $R_n$ is a torus with $2n + 1$ punctures if $n \geq 2$. By Theorem A, we conclude that $L_\mathcal{C}(\text{Mod}(S_{1,2n+1})) \leq C/n^2$ for some constant $C > 0$.

We turn to the case where the number of punctures is even. Let $a$ and $b$ be simple closed curves in $S_{1,2}$ as in Figure 13(1), and let $c$ be as before, i.e., $\beta = [c]$. Consider $\psi = T_b^{-1}T_a \in \text{Mod}(S_{1,2})$ which is pseudo-Anosov by Penner’s construction. The induced map $\psi_*$ maps a basis $a, \beta$ and $\gamma$ of $H_1(S_{1,2};\mathbb{Z})$ to $\alpha + \beta + \gamma$, $\beta$, and $\gamma$, respectively. Thus $\psi$ fixes a primitive cohomology class $\xi_0 \in H^1(S_{1,2};\mathbb{Z})$ which is dual to $c$. Consider the $\mathbb{Z}$-cover $\tilde{S}$ over $S$ corresponding to $\xi_0$ (Figure 13(2)) and pick a lift of $\tilde{\psi} : \tilde{S} \to \tilde{S}$ of $\psi$. We can apply Theorem A.
for the fiber \((S_{1,2}, \psi)\) of the mapping torus \(M_\psi\) together with \(\xi_0 \in H^1(S_{1,2}; \mathbb{Z})\) fixed by \(\psi\). Theorem 1.1 says that for all \(n\) sufficiently large, \(R_n\) is a fiber of \(M_\psi\). In this case \(R_n\) is a torus with \(2n + n_0\) punctures, where \(n_0\) is an even number which depends on the choice of the lift \(\tilde{\psi}\). By Theorem A we conclude that \(L_C(\text{Mod}(S_{1,2n})) < C'/n^2\) for some constant \(C' > 0\). This completes the proof. \(\square\)

5 Proof of Theorem C

This section includes the proof of Theorem C.

In the proof of Theorem B(1), we used the hyperbolic fibered 3-manifold \(M_\beta = M_{\sigma_1^{-2}\sigma_2}\), the so-called \textit{magic manifold} and its double cover \(M_\beta\), depending on the parity of the number of punctures in the disk. Here we only use \(M_\beta\) and a sequence \((R_n, \psi_n)\) of the fibers \(R_n = D_{2n}\) of \(M_\beta\) with the monodromy \(\psi_n\) for \(n \geq 2\) as in the proof of Theorem B(1). \textit{Train tracks} play an important role in the proof. Terminology related to train tracks can be found in [BH95] or [FM12] for example.

We think of \(R_n\) as a sphere with \(2n + 1\) punctures. An invariant train track \(\tau_n\) and a train track representative \(p_n : \tau_n \to \tau_n\) of \(\psi_n : S_{0,2n+1} \to S_{0,2n+1}\) are studied in [Kin15, Example 4.6]. Figure 14 shows the train track \(\tau_n \subset S_{0,2n+1}\) and its image \(\psi_n(\tau_n)\). Each of the monogon components of \(S_{0,2n+1} \setminus \tau_n\) (bounded by loop edges of \(\tau_n\)) contains a puncture of \(S_{0,2n+1}\), the \((n-1)\)-gon of \(S_{0,2n+1} \setminus \tau_n\) contains another puncture, and the other connected component of \(S_{0,2n+1} \setminus \tau_n\)
contains the other puncture $p_\infty$ in the proof of Theorem B(1). Recall that $\psi_n$ maps $p_\infty$ to itself. Figure 15 gives the directed graph $\Gamma_n$ of $\tau_n$ for $n \geq 3$. The set of vertices of $\Gamma_n$ equals the set of non-loop edges $r, p_1, q_1, \cdots, p_{n-1}, q_{n-1}$ of $\tau_n$ as shown in Figure 14. The edges of $\Gamma_n$ tell the locations of $p_n(e), p_{n+1}(e), \cdots$ in $S_{0,2n+1}$ for each non-loop edge $e$ of $\tau_n$. More precisely, $j$ edges of $\Gamma_n$ running from the vertex $e$ to the vertex $e'$ mean that $p_n(e)$ passes through the edge $e'$ of $\tau_n$ $j$ times. One can construct $\Gamma_n$ viewing $\psi_n(\tau_n)$ and $\tau_n$. The “vertical” consecutive edges of $\Gamma_n$ in Figure 15 reveal the dynamics of $\psi_n$:

We first prove the following.

**Proposition 5.1.** For $n \geq 4$, we have

$$L_C(\text{Mod}(D_{2n-1})) \leq \frac{1}{n^2 - 4n + 2} \quad \text{and} \quad L_C(\text{Mod}(D_{2n})) \leq \frac{1}{n^2 - 4n + 2}.$$

**Proof.** We first prove the latter upper bound. We assume $n \geq 4$. Let $N(\tau_n) \subset S_{0,2n+1}$ be a fibered neighborhood of $\tau_n$ (see [PP87, page 360] for the definition) equipped with a retraction $N(\tau_n) \smallsetminus \gamma \tau$. For a connected subset $\tau' \subset \tau_n$, we define a fibered neighborhood $N(\tau')$ of $\tau'$ as follows.

$$N(\tau') = N(\tau_n) \cap U(\tau'),$$

where $U(\tau')$ is a small neighborhood of $\tau'$ in the 2-sphere $S^2$. We take $n$ points $v_0, v_1, v_2, \cdots, v_{n-1} \subset \tau_n$, each of which lies on an edge of the $(n-1)$-gon, see Figure 16(1). For $1 \leq i < j \leq n-1$, let $\tau(i,j)$ be the connected component of $\tau_n \setminus \{v_{i-1}, v_j\}$ containing $p_i, q_i, p_{i+1}, q_{i+1}, \cdots, p_j, q_j$ (see Figure 16(2)). We consider its fibered neighborhood $N(\tau(i,j))$, and we set

$$N(p_i q_i p_{i+1} q_{i+1} \cdots p_j q_j) = N(\tau(i,j)).$$
Figure 16: (1) Points $v_0, v_1, \ldots, v_{n-1}$. (2) $\tau(2, n-1) \subset N(p_2q_2 \cdots p_{n-1}q_{n-1})$. (3) $\tau(1) \subset N(rp_1q_1)$. 

For $1 \leq j \leq n-2$, let $\tau(j)$ be the connected component of $\tau_n \setminus \{v_j, v_{n-1}\}$ containing $r, p_1, q_1, \ldots, p_j, q_j$ (see Figure 16(3)). Let

$$N(rp_1q_1 \cdots p_jq_j) = N(\tau(j)).$$

The notation $N(rp_1q_1 \cdots p_jq_j)$ tells a property that it contains $r, p_1, q_1, \ldots, p_j, q_j$. The same thing holds for $N(p_iq_i p_{i+1}q_{i+1} \cdots p_{n-1}q_{n-1})$.

We take an essential arc $c$ connecting the two punctures as in Figure 17(1). Then $c$ is carried by $\tau_n$. Notice that if $i \geq 2$, then $N(p_iq_ip_{i+1}q_{i+1} \cdots p_{n-1}q_{n-1})$ is disjoint from $c$. Since $c \subset N(rp_1q_1)$, we have

$$\psi_n(c) \subset N(p_1q_1p_2q_2),$$
$$\psi_n^2(c) \subset N(p_2q_2p_3q_3),$$
$$\vdots$$
$$\psi_n^{1+(n-3)}(c) = \psi_n^{n-2}(c) \subset N(p_{n-2}q_{n-2}p_{n-1}q_{n-1})$$

(see Figures 15 and 17). Observe that $\psi_n^n(\psi_n^{n-2}(c)) = \psi_n^n(c) \subset N(rp_1q_1p_2q_2)$. We have

$$\psi_n^{n+1}(c) \subset N(p_1q_1p_2q_2p_3q_3),$$
$$\vdots$$
$$\psi_n^{(n+1)+(n-4)}(c) = \psi_n^{2n-3}(c) \subset N(p_{n-3}q_{n-3}p_{n-2}q_{n-2}p_{n-1}q_{n-1}).$$

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In the same manner, for $2 \leq k \leq n - 2$, we have
\[
\psi_{n}^{(k-1)n-k} \subset \mathcal{N}(p_{n-k}q_{n-k} \cdots p_{n-1}q_{n-1}).
\]
When $k = n - 2$,
\[
\psi_{n}^{(n-3)n-(n-2)} = \psi_{n}^{n^2-4n+2} \subset \mathcal{N}(p_{n-1}q_{n-1}).
\]
Hence clearly we have
\[
d_{\text{AC}}(c, \psi_{n}^{n^2-4n+2}(c)) = 1.
\]
If we consider a regular neighborhood of $c$ in $S^2$, then we obtain an essential simple closed curve $\alpha$ in $S^0_{2n+1}$ as the boundary of the neighborhood in question. Notice that $\alpha$ is also carried by $\tau_n$ and $\alpha \subset \mathcal{N}(rp_{1}q_{1})$. The above argument shows that $\psi_{n}^{n^2-4n+2}(\alpha) \subset \mathcal{N}(p_{2}q_{2} \cdots p_{n-1}q_{n-1})$ and $\alpha$ is disjoint from $\psi_{n}^{n^2-4n+2}(\alpha)$. Recall that $\psi_{n}$ is defined on $R_{n} = D_{2n}$. This together with Lemma 2.1 implies that
\[
L_{C}(\text{Mod}(D_{2n})) \leq \frac{1}{n^2 - 4n + 2}.
\]
To show the former upper bound of $L_{C}(\text{Mod}(D_{2n-1}))$ in the claim, we fill the puncture in the $(n - 1)$-gon of $S_{0,2n+1} \setminus \tau_{n}$. The assumption $n - 1 \geq 3$ ensures that $\tau_{n}$ extends to a train track $\tau_{n}$ in $S_{0,2n}$ and $\psi_{n} : S_{0,2n+1} \to S_{0,2n+1}$ extends to $\overline{\psi}_{n} : S_{0,2n} \to S_{0,2n}$ which is still pseudo-Anosov. In particular $\overline{\psi}_{n}$ maps the puncture $p_{\infty}$ to itself. We can think of $\overline{\psi}_{n} : S_{0,2n} \to S_{0,2n}$ as an element of
Figure 18: (1) $x_{2k} \in SW_{2k}$. (2) $y_{2k} \in SW_{2k}$.

Mod($D_{2n-1}$). The train track representative $p_n : \tau_n \to \tau_n$ also extends to a train track representative $\overline{p}_n : \tau_n \to \tau_n$ of $S_{0,2n} \to S_{0,2n}$. All non-loop edges of $\tau_n$ are coming from those of $\tau_n$, and hence the directed graph $\Gamma_n$ for $\overline{p}_n : \tau_n \to \tau_n$ is the same as $\Gamma_n$ for $p_n : \tau_n \to \tau_n$. For the arc $\overline{c}$ and the simple closed curve $\overline{\alpha}$ in $S_{0,2n}$ coming from $c$ and $\alpha$ in $S_{0,2n+1}$, respectively, the above argument tells us that

$$d_{AC}(\overline{c}, \overline{\psi}_n^{n^2-4n+2}(\overline{\tau})) = 1 \quad \text{and} \quad d_C(\overline{\alpha}, \overline{\psi}_n^{n^2-4n+2}(\overline{\pi})) = 1.$$  \hfill (5.1)

The latter equality in (5.1) with Lemma 2.1 gives the desired upper bound. \hfill \Box

We are now ready to prove Theorem C.

Proof of Theorem C. By Lemma 3.1 together with either of the equalities for $\overline{\psi}_n : S_{0,2n} \to S_{0,2n}$ in (5.1), we have $L_C(H(S_{n-1})) \leq \frac{1}{(g+1)^2 - 4(g+1) + 2}$ for $g \geq 3$. Thus for $g \geq 3$,

$$L_C(H(S_g)) \leq \frac{1}{(g+1)^2 - 4(g+1) + 2} = \frac{1}{g^2 - 2g - 1}.$$  \hfill \Box

6 Proof of Theorem D

In this section, we finally prove Theorem D.

Proof of Theorem D. The proof is separated into two cases, depending on the parity of the genera. First of all we introduce spherical braids $x_{2k}, y_{2k} \in SB_{2k}$ for $k \geq 5$ as shown in Figure 18. It is straightforward to see that they are elements of $SW_{2k}$. We define $w_{2k} \in SW_{2k}$ for each $k \geq 5$ as follows.

$$w_{4n+8} = x_{4n+8}(y_{4n+8})^n \quad \text{if} \quad 2k = 4n + 8 \quad \text{for some} \ n \geq 1,$$

$$w_{4n+10} = (x_{4n+10})^2(y_{4n+10})^n \quad \text{if} \quad 2k = 4n + 10 \quad \text{for some} \ n \geq 0.$$  

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Consider an element in the Hilden group $SH_{2k}$ corresponding to $w_{2k}$ (see Section 3.2) and its mapping torus $M_{w_{2k}}$. In [HK17] it is shown that when $2k = 4n + 8$ for $n \geq 1$, $M_{w_{2k}}$ is homeomorphic to the mapping torus $M_{\psi}$ of the element in $SH_4$ corresponding to the pseudo-Anosov braid $w \in SW_6$ (see Figure 6(3)). In other words, $M_{\psi}$ is hyperbolic and it has a fiber $S_{0, 2k}$ with pseudo-Anosov monodromy $w_{2k}$ when $2k = 4n + 8$. We claim that a sequence of fibers $(S_{0, 4n+10}, w_{4n+10})$ of $M_{\psi}$ comes from a fibered 3-manifold as in Theorem A. More precisely, if we remove the 6th strand of $w$, then we obtain a spherical braid with 5 strands. Regarding such a braid as the one on the disk, we have a 5-braid, say $\psi \in B_5$. Clearly $M_{\psi}$ is homeomorphic to $M_{w_{2k}}$. We consider a fiber $S = D_5$ with monodromy $\psi$ of the mapping torus $M_{\psi} \cong M_{w_{2k}}$. Since $\psi_n$ maps the generator $t_5$ to itself (see the 5th strand of the braid $w$ in Figure 6(3)), the cohomology class $\xi_0 \in H^1(S; \mathbb{Z})$ which is dual to the proper arc $c = c_5$ is fixed by $\psi$. Let $\tilde{S}$ be the $\mathbb{Z}$-cover of $S$ corresponding to $\xi_0$. We consider the canonical lift $\tilde{\psi} : \tilde{S} \to \tilde{S}$ of $\psi$. Then $R_n = \tilde{S}/(h^n\tilde{\psi})$ is a fiber of $M_{\psi}$ with monodromy $\psi_n$ for $n$ large. In this case, $R_n$ is a sphere with $4n + 8$ punctures, and we find that the monodromy $\psi_n$ is given by the braid $w_{4n+8} \in SW_{4n+8}$ from the argument in [HK17, Section 3]. By the proof of Theorem A, there exist $\alpha \in AC(R_n)^0$ and $m \geq n^2$ such that $d_{AC}(\alpha, (\psi_n)^m(\alpha)) = 1$.

Notice that a lift $\tilde{\psi} = \tilde{\psi}_n$ of $\psi_n$ under the map $q$ is an element of $\mathcal{H}(\mathbb{H}_{2n+3})$ (see Section 3.2). By Lemma 3.1, $\ell_C(\tilde{\psi}) \leq 1/m$, which implies $\ell_C(\tilde{\psi}) \leq C/n^2$ for some constant $C > 0$. Thus we have $L_C(\mathcal{H}(\mathbb{H}_{2n+3})) \leq C/n^2$ in the case of the odd genus.

To obtain the upper bound $L_C(\mathcal{H}(\mathbb{H}_{2n+4})) \leq C'/n^2$ for some $C' > 0$ in the case of the even genus, we take the second power $\psi^2 \in B_5$ of the above $\psi$ and we set $\phi = \psi^2$. We consider a fiber $S = D_5$ with monodromy $\phi$ in the mapping torus $M_{\phi}$. Note that $\phi$ fixes the same $\xi_0 \in H^1(S; \mathbb{Z})$. Let $\tilde{S}$ be the $\mathbb{Z}$-cover over $S$ as before and let $\tilde{\phi} = (\psi)^2 : \tilde{S} \to \tilde{S}$ which is the canonical lift of $\phi$. Now we apply Theorem A for the fiber $(S, \phi)$ of $M_{\phi}$ together with $\xi_0$. One sees that for $n$ large, $\tilde{S}/(h^n\tilde{\phi})$ is a fiber of $M_{\phi}$ which is the sphere with $4n + 10$ punctures. The same argument as in [HK17, Section 3] tells us that the monodromy of the fiber $\tilde{S}/(h^n\tilde{\phi})$ is described by the braid $w_{4n+10} \in SW_{4n+10}$. As in the case of the odd genus, we obtain the desired upper bound of $L_C(\mathcal{H}(\mathbb{H}_{2n+4}))$. This completes the proof. 

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\textbf{References}


