ON HYPERBOLIC SURFACE BUNDLES OVER THE CIRCLE AS BRANCHED DOUBLE COVERS OF THE 3-SPHERE

SUSUMU HIROSE AND EIKO KIN

Abstract. The branched virtual fibering theorem by Sakuma states that every closed orientable 3-manifold with a Heegaard surface of genus g has a branched double cover which is a genus g surface bundle over the circle. It is proved by Brooks that such a surface bundle can be chosen to be hyperbolic. We prove that the minimal entropy over all hyperbolic, genus g surface bundles as branched double covers of the 3-sphere behaves like $1/g$. We also give an alternative construction of surface bundles over the circle in Sakuma's theorem when closed 3-manifolds are branched double covers of the 3-sphere branched over links. A feature of surface bundles coming from our construction is that the monodromies can be read off the braids obtained from the links as the branched set.

1. Introduction

This paper concerns the *branched* virtual fibering theorem by Makoto Sakuma. To state his theorem let $\Sigma = \Sigma_{g,p}$ be an orientable, connected surface of genus g with p punctures possibly $p = 0$, and let us set $\Sigma_g = \Sigma_{g,0}$. The mapping class group $\mathrm{Mod}(\Sigma)$ is the group of isotopy classes of orientation-preserving self-homeomorphisms on Σ which preserve the punctures setwise. By Nielsen-Thurston classification, elements in $\mathrm{Mod}(\Sigma)$ fall into three types: periodic, reducible, pseudo-Anosov [9]. To each pseudo-Anosov element ϕ, there is an associated dilatation (stretch factor) $\lambda(\phi) > 1$ (see [4] for example). We call the logarithm of the dilatation $\log(\lambda(\phi))$ the *entropy* of ϕ.

Choosing a representative $f : \Sigma \to \Sigma$ of ϕ we define the *mapping torus* T_ϕ by

$$T_\phi = \Sigma \times \mathbb{R}/\sim,$$

where $(x,t) \sim (f(x), t+1)$ for $x \in \Sigma$, $t \in \mathbb{R}$. We call Σ the *fiber surface* of T_ϕ. The 3-manifold T_ϕ is a Σ-bundle over the circle with the monodromy ϕ. By Thurston [10] T_ϕ admits a hyperbolic structure of finite volume if and only if ϕ is pseudo-Anosov.

The following theorem is due to Sakuma [8, Addendum 1]. See also [3, Section 3].

Theorem 1 (Branched virtual fibering theorem). *Let M be a closed orientable 3-manifold. Suppose that M admits a genus g Heegaard splitting. Then there is a 2-fold branched cover $\widetilde{M}$ of M which is a Σ_g-bundle over the circle.*

It is proved by Brooks [3] that $\widetilde{M}$ in Theorem 1 can be chosen to be hyperbolic if $g \geq \max(2, g(M))$, where $g(M)$ is the Heegaard genus of M. See also [6] by Montesinos.

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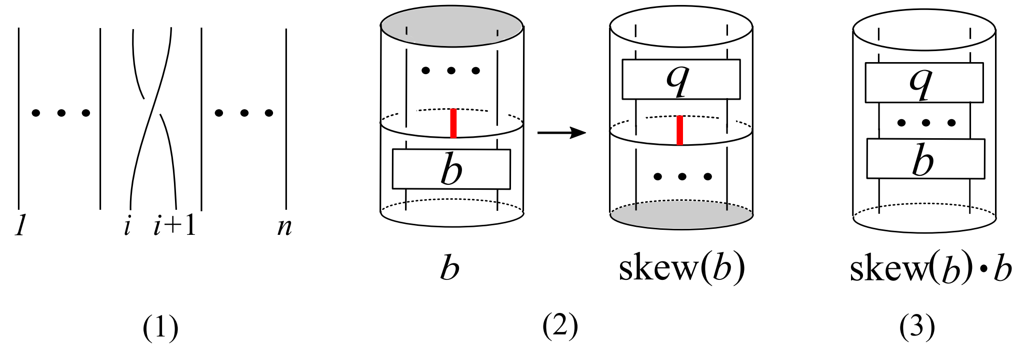

FIGURE 1. (1) $\sigma_i \in B_n$. (2) Involution on the cylinder. Thick segment in the middle is the fixed point set of the involution. (3) $\widetilde{b} = \mathrm{skew}(b) \cdot b$ is invariant under such an involution.

Let $D_g(M)$ be the subset of $\mathrm{Mod}(\Sigma_g)$ consisting of elements ϕ such that T_ϕ is homeomorphic to a 2-fold branched cover of M branched over some link. By Theorem 1 we have $D_g(M) \neq \emptyset$. By Brooks together with the stabilization of Heegaard splittings, there is a pseudo-Anosov element in $D_g(M)$ for each $g \geq \max(2, g(M))$. The set of fibered 3-manifolds T_ϕ over all $\phi \in D_g(M)$ possesses various properties inherited under branched covers of M. It is natural to ask about the dynamics of pseudo-Anosov elements in $D_g(M)$. We are interested in the set of entropies of pseudo-Anosov mapping classes.

We fix a surface Σ and consider the set of entropies

$$\{\log \lambda(\phi) \mid \phi \in \mathrm{Mod}(\Sigma) \text{ is pseudo-Anosov}\}$$

which is a closed, discrete subset of $\mathbb{R}$ ([1]). For any subset $G \subset \mathrm{Mod}(\Sigma)$ let $\delta(G)$ denote the minimum of dilatations $\lambda(\phi)$ over all pseudo-Anosov elements $\phi \in G$. Then $\delta(G) \geq \delta(\mathrm{Mod}(\Sigma))$. For real valued functions f and h, we write $f \asymp h$ if there is a universal constant c such that $h/c \leq f \leq ch$. It is proved by Penner [7] that

$$\log \delta(\mathrm{Mod}(\Sigma_g)) \asymp \frac{1}{g}.$$

A question arises: what can we say about the asymptotic behavior of the minimal entropies $\log \delta(D_g(M))$'s for each closed 3-manifold M? In this paper we consider this question when M is the 3-sphere S^3. Our main theorem is the following.

Theorem A. *We have* $\log \delta(D_g(S^3)) \asymp \dfrac{1}{g}$.

Let $q_L : M_L \to S^3$ denote the 2-fold branched covering map of S^3 branched over a link L in S^3. Along the way in the proof of Theorem A we give an alternative proof of Theorem 1 when $M = M_L$ in Theorem B. A feature of surface bundles $\widetilde{M_L}$ coming from our construction is that their monodromies can be read off the braids obtained from the links as the branched set. To state Theorem B, we need 3 ingredients.

1. Involution skew : $B_n \to B_n$**.** Let B_n be the (planar) braid group with n strands and let σ_i denote the Artin generator of B_n as in Figure 1(1). We define an

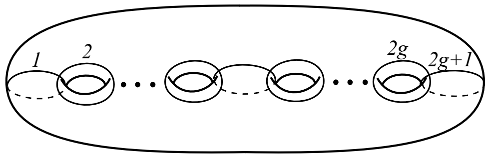

FIGURE 2. Simple closed curves on Σ_g.

involution

$$\begin{array}{rcl} \mathrm{skew}: B_n & \to & B_n \\ \sigma_{n_1}^{\epsilon_1}\sigma_{n_2}^{\epsilon_2}\cdots\sigma_{n_k}^{\epsilon_k} & \mapsto & \sigma_{n-n_k}^{\epsilon_k}\cdots\sigma_{n-n_2}^{\epsilon_2}\sigma_{n-n_1}^{\epsilon_1}, \qquad \epsilon_i = \pm 1. \end{array}$$

We say that $b \in B_n$ is *skew-palindromic* if $\mathrm{skew}(b) = b$. The braid $\mathrm{skew}(b)\cdot b$ is skew-palindromic for any $b \in B_n$. (There is a skew-palindromic braid which can not be written by $\mathrm{skew}(b)\cdot b$ for any b, for example $\sigma_1\sigma_2\sigma_3 \in B_4$.) We write

$$\widetilde{b} = \mathrm{skew}(b)\cdot b.$$

Note that $\mathrm{skew}: B_n \to B_n$ is induced by the involution on the cylinder as shown in Figure 1(2) and skew-palindromic braids are invariant under such an involution.

In the later section, the map skew is also regarded as a map from the braid group on the sphere or the annulus to itself. The above assertion for the braid $\widetilde{b} = \mathrm{skew}(b)\cdot b$ also holds in this setting.

2. Homomorphism $\mathfrak{t}: B_{2g+2} \to \mathrm{Mod}(\Sigma_g)$. Let t_i denote the right-handed Dehn twist about the simple closed curve with the number i in Figure 2. Then there is a homomorphism

$$\mathfrak{t}: B_{2g+2} \to \mathrm{Mod}(\Sigma_g)$$

which sends σ_i to t_i for $i = 1, \ldots, 2g+1$, since $\mathrm{Mod}(\Sigma_g)$ has the braid relation. (We apply elements of mapping class groups from right to left.) The *hyperelliptic mapping class group* $\mathcal{H}(\Sigma_g)$ is the subgroup of $\mathrm{Mod}(\Sigma_g)$ consisting of elements with representative homeomorphisms that commute with some fixed hyperelliptic involution. By Birman-Hilden [2], $\mathcal{H}(\Sigma_g)$ is generated by t_i's. Thus

$$\mathcal{H}(\Sigma_g) = \mathfrak{t}(B_{2g+2}).$$

3. Circular plat closure $C(b)$. We use two types of links in S^3 obtained from braids. One is the *closure* $\mathrm{cl}(\beta)$ of $\beta \in B_{g+1}$ as in Figure 3(1). The other is the *circular plat closure* $C(b)$ of $b \in B_{2g+2}$ with even strands as in Figure 3(2)(3). We also use the link $C(b) \cup W$, the union of $C(b)$ and the trivial link $W = O \cup O'$ with 2 components as shown in Figure 3(4).

Any link in S^3 can be represented by $C(\beta')$ for some braid β'. To see this, it is well-known that L is the closure $\mathrm{cl}(\beta)$ for some $\beta \in B_{g+1}$ ($g \geq 1$). The desired braid $\beta' \in B_{2g+2}$ can be obtained from β by adding $g+1$ straight strands as in Figure 3(1).

For a braid $b \in B_{2g+2}$ let $q = q_{C(b)}: M_{C(b)} \to S^3$ be the 2-fold branched covering map of S^3 branched over $C(b)$. There is a $(g+1)$-bridge sphere S for the link

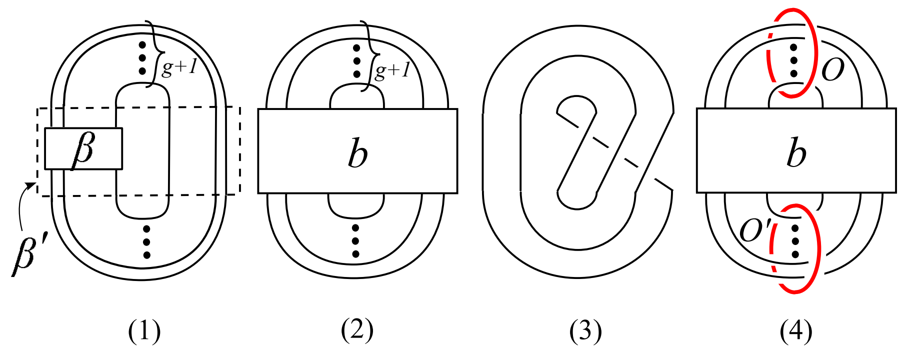

FIGURE 3. (1) $\mathrm{cl}(\beta)$ for $\beta \in B_{g+1}$ and $\beta' \in B_{2g+2}$. (2) $C(b)$ for $b \in B_{2g+2}$. (3) $C(\sigma_3\sigma_4\sigma_5)$. (4) $C(b) \cup W$, where $W = O \cup O'$.

$C(b) \subset S^3$. Hence $M_{C(b)}$ admits a genus g Heegaard splitting with the Heegaard surface $q^{-1}(S)$. Then we have the following result.

Theorem B. *Let $\widetilde{M_{C(b)}}$ be the 2-fold branched cover of $M_{C(b)}$ branched over the link $q^{-1}(W)$. Then $\widetilde{M_{C(b)}}$ is homeomorphic to the mapping torus $T_{\mathfrak{t}(\widetilde{b})}$.*

Remark 2. *To be precise, $\widetilde{M_{C(b)}}$ is the 2-fold branched cover of $M_{C(b)}$ branched over $q^{-1}(W)$ obtained as the $\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$-cover of S^3 branched over the link $C(b) \cup W$ associated with the epimorphism $H_1(S^3 \setminus (C(b) \cup W)) \to \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$ which maps the meridians of $C(b)$ to the generator of the first factor and the meridians of W to the generator of the second factor.*

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2. PROOF OF THEOREM B

Proof of Theorem B. We construct the 3-sphere S^3 from two copies of the 3-ball B^3 by gluing their boundaries together. Consider the link $C(b) \cup W$ so that $C(b)$ is contained in one of the 3-balls, and W is given by the union of the four thick segments in the two 3-balls, see Figure 4(2). Let S be the sphere in S^3 which is the union of the two shaded disks in the same figure. The 2-sphere S is a $(g+1)$-bridge sphere for $C(b)$, and the preimage $q^{-1}(S)$ is a genus g Heegaard surface of $M_{C(b)}$.

Let $q_W : M_W \to S^3$ be the 2-fold branched covering map of S^3 branched over W (Figure 4(1)). The preimage $q_W^{-1}(B^3)$ is homeomorphic to the solid torus $D^2 \times S^1$. Then M_W is obtained from two copies of $D^2 \times S^1$ by gluing their boundaries together, and hence M_W is homeomorphic to $S^2 \times S^1$. Observe that the link $q_W^{-1}(C(b))$ is the closure of the *spherical* braid $\widetilde{b} = \mathrm{skew}(b) \cdot b$, i.e.,

$$q_W^{-1}(C(b)) = \mathrm{cl}(\widetilde{b}) \subset S^2 \times S^1.$$

$$\begin{tikzcd}
T_{t(\widetilde{b})} \cong \widetilde{M_{C(b)}} \arrow[rr] \arrow[d, "p"'] & & M_{C(b)} \arrow[d, "q=q_{C(b)}"] \\
M_W \cong S^2 \times S^1 \arrow[rr, "q_W"] & & S^3
\end{tikzcd}$$

(3)

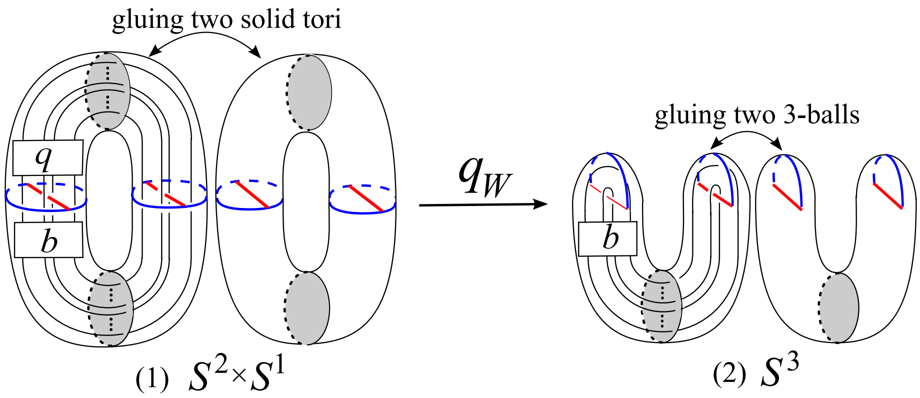

FIGURE 4. (1) $\mathrm{cl}(\widetilde{b}) \subset S^2 \times S^1$. (2) $C(b) \cup W \subset S^3$. (3) Diagram: $\widetilde{M_{C(b)}} \to M_{C(b)}$ is the 2-fold branched cover of $M_{C(b)}$ branched over $q^{-1}(W)$.

Let $p : N_{\mathrm{cl}(\widetilde{b})} \to S^2 \times S^1$ be the 2-fold branched covering map of $S^2 \times S^1$ branched over $\mathrm{cl}(\widetilde{b})$. The 2-fold branched cover of the level surface $S^2 \times \{u\}$ for $u \in S^1$ branched at the $2g+2$ points in $\Big((S^2 \times \{u\}) \cap \mathrm{cl}(\widetilde{b})\Big)$ is a closed surface of genus g. Thus $N_{\mathrm{cl}(\widetilde{b})}$ is a Σ_g-bundle over S^1 with the monodromy $\mathfrak{t}(\widetilde{b})$, i.e., $N_{\mathrm{cl}(\widetilde{b})}$ is homeomorphic to $T_{\mathfrak{t}(\widetilde{b})}$.

We can observe that the composition

$$q_W \circ p : T_{\mathfrak{t}(\widetilde{b})} \cong N_{\mathrm{cl}(\widetilde{b})} \to S^3$$

is the $\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/2\mathbb{Z}$-cover of S^3 branched over the link $C(b) \cup W$ described in Remark 2. Hence $T_{\mathfrak{t}(\widetilde{b})} \cong N_{\mathrm{cl}(\widetilde{b})}$ is identified with 2-fold branched cover $\widetilde{M_{C(b)}}$. This completes the proof. □

3. PROOF OF THEOREM A

Given a braid b we first give a construction of a braid b' (with more strands than b) such that $C(b)$ is ambient isotopic to $C(b')$.

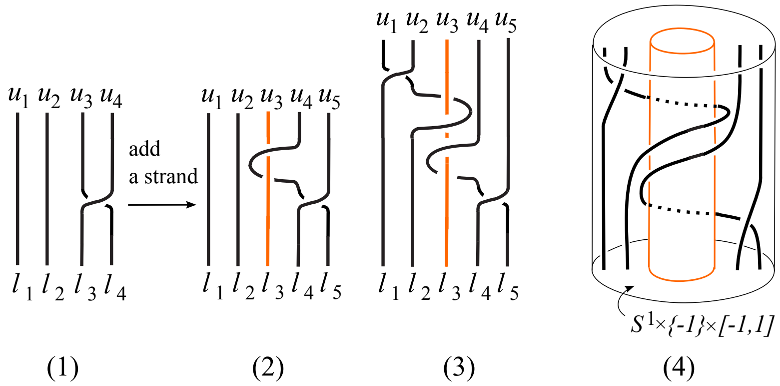

FIGURE 5. (1) $b = \sigma_3$. (2) $b^\circ = \sigma_3^2\sigma_4$. (3) $\widetilde{b^\circ}$. ($\widetilde{b} = \sigma_1\sigma_3$ is obtained from $\widetilde{b^\circ}$ by removing the strand with endpoints ℓ_3 and u_3.) (4) Braid $\widetilde{b}$ on $\mathcal{A}$.

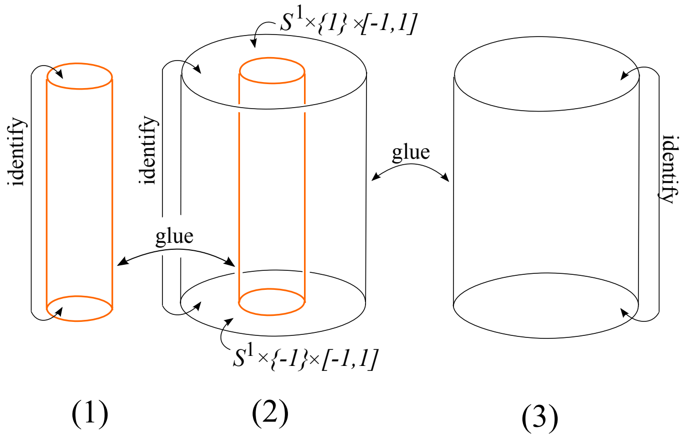

FIGURE 6. (1) Solid torus $(D^2 \times S^1)_{-1}$. (2) $S^1 \times S^1 \times [-1,1]$ obtained from $S^1 \times [-1,1] \times [-1,1]$ by identifying the two annuli $S^1 \times \{1\} \times [-1,1]$ and $S^1 \times \{-1\} \times [-1,1]$. (3) Solid torus $(D^2 \times S^1)_{+1}$.

The bottom and top endpoints of a planar braid with n strands are denoted by $l_1, \ldots, l_n$ and $u_1, \ldots, u_n$ from left to right. For a braid $b \in B_{2g+2}$ with even strands, we choose a braid $b^\circ \in B_{2g+3}$ obtained from b by adding a strand, say $b^\circ(g+2)$ connecting the middle of the two points l_{g+1} and l_{g+2} with the middle of the two points u_{g+1} and u_{g+2}. Of course b° is not unique. For example when $b = b_1 = \sigma_3 \in B_4$, one can choose $b^\circ(= b_1^\circ) = \sigma_3^2\sigma_4 \in B_5$. See Figure 5.

We consider $\widetilde{b^\circ} = \mathrm{skew}(b^\circ) \cdot b^\circ \in B_{2g+3}$ with bottom endpoints $l_1, \ldots, l_{2g+3}$ and top endpoints $u_1, \ldots, u_{2g+3}$. The braid $\widetilde{b^\circ}$ has the strand $\widetilde{b^\circ}(g+2)$ with endpoints l_{g+2} and u_{g+2}. If we remove this strand from $\widetilde{b^\circ}$, then we obtain $\widetilde{b} = \mathrm{skew}(b) \cdot b$.

Now we construct $S^2 \times S^1$ from three pieces, two solid tori $(D^2 \times S^1)_{\pm 1}$ and the product $S^1 \times S^1 \times [-1,1]$ of a torus $S^1 \times S^1$ and the interval $[-1,1]$ by gluing $(\partial D^2 \times S^1)_i$ to $S^1 \times S^1 \times \{i\}$ together for $i = \pm 1$. See Figure 6. We think of S^1 as the quotient space $[-1,1]/(-1 \sim 1)$, and consider the product

$$S^1 \times S^1 \times [-1,1] = S^1 \times [-1,1]/(-1 \sim 1) \times [-1,1].$$

For the braided link $\mathrm{br}(\widetilde{b^\circ}) = \mathrm{cl}(\widetilde{b^\circ}) \cup A$ in S^3, we perform the 0-surgery along the braid axis A. Then the image of $\mathrm{cl}(\widetilde{b^\circ})$ forms a link in $S^2 \times S^1$, which continues to denote by the same symbol $\mathrm{cl}(\widetilde{b^\circ})$. We deform this link $\mathrm{cl}(\widetilde{b^\circ})$ in $S^2 \times S^1$ so that the knot $\mathrm{cl}(\widetilde{b^\circ}(g+2))$ becomes the core of $(D^2 \times S^1)_{-1}$ and $\mathrm{cl}(\widetilde{b}) = \mathrm{cl}(\widetilde{b^\circ}) \setminus \mathrm{cl}(\widetilde{b^\circ}(g+2))$ is contained in $S^1 \times S^1 \times [-1,1]$. One can regard $\widetilde{b}$ as a braid on the annulus $\mathcal{A} := S^1 \times \{-1\} \times [-1,1]$ which is embedded in $S^1 \times [-1,1] \times [-1,1]$, and one can think of the link $\mathrm{cl}(\widetilde{b})$ as the closure of the braid $\widetilde{b}$ on $\mathcal{A}$. See Figure 5(4). Let

$$R : S^2 \times S^1 \to S^2 \times S^1$$

be the deck transformation of $q_W : S^2 \times S^1 \to S^3$ in the proof of Theorem B. Then q_W sends the fixed point set of the involution R to the trivial link W (Figure 4(1)(2)). Let

$$f : S^1 \times S^1 \to S^1 \times S^1$$

be any orientation-preserving homeomorphism. We may assume that f commutes with the involution

$$\begin{array}{rcl} \iota : S^1 \times S^1 & \to & S^1 \times S^1 \\ (x,y) & \mapsto & (-x,-y). \end{array}$$

We consider the homeomorphism

$$\Phi_f = f \times \mathrm{id}_{[-1,1]} : S^1 \times S^1 \times [-1,1] \to S^1 \times S^1 \times [-1,1].$$

The image of $\mathrm{cl}(\widetilde{b})$ under Φ_f may or may not be the closure of some braid on $\mathcal{A}$. We assume the former case $(*)$:

$(*)$ $\Phi_f(\mathrm{cl}(\widetilde{b})) = \mathrm{cl}(\gamma)$ for some braid γ on $\mathcal{A}$.

Then the involution $R|_{S^1 \times S^1 \times [-1,1]} = \iota \times \mathrm{id}_{[-1,1]}$ has the following property.

$$R(\mathrm{cl}(\beta)) = \mathrm{cl}(\mathrm{skew}(\beta)) \text{ for any braid } \beta \text{ on } \mathcal{A}.$$

cf. Figure 1(2). Since f commutes with ι, it follows that Φ_f commutes with $R|_{S^1 \times S^1 \times [-1,1]}$. Hence we have

$$\mathrm{cl}(\gamma) = \Phi_f(\mathrm{cl}(\widetilde{b})) = \Phi_f(\mathrm{cl}(\mathrm{skew}(\widetilde{b}))) = \Phi_f R(\mathrm{cl}(\widetilde{b})) = R\Phi_f(\mathrm{cl}(\widetilde{b})) = R(\mathrm{cl}(\gamma)).$$

(The first and last equality come from the assumption $(*)$, and the second equality holds since $\widetilde{b}$ is skew-palindromic.) Thus $\mathrm{cl}(\gamma) = R(\mathrm{cl}(\gamma))$. We further assume that

$(**)$ $\big(\Phi_f(\mathrm{cl}(\widetilde{b})) =\big)\mathrm{cl}(\gamma) = \mathrm{cl}(\widetilde{b_f})$ for some braid b_f on $\mathcal{A}$.

Remark 3. *Clearly $(**)$ implies $\mathrm{cl}(\gamma) = R(\mathrm{cl}(\gamma))$. It is likely the converse holds.*

Now, we think of the braid b_f on $\mathcal{A}$ as a planar braid as in Figure 5, and consider the link $C(b_f)$ in S^3. We have the following lemma.

Lemma 4. *Under the assumptions $(*)$ and $(**)$, $C(b)$ and $C(b_f)$ are ambient isotopic.*

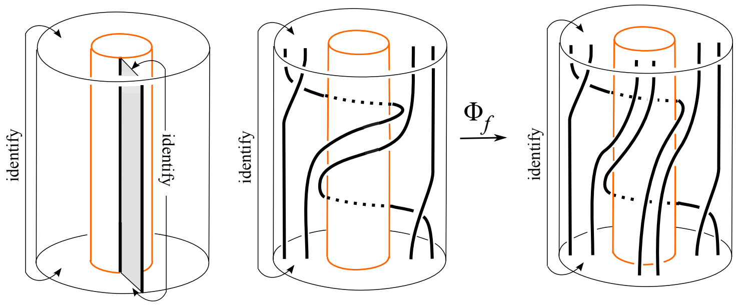

FIGURE 7. (1) Shaded region indicates annulus $\mathbb{A}$ with boundary $\{v\} \times S^1 \times \{\pm 1\}$. (3) $\mathrm{cl}(\widetilde{b_1})$ and (4) $\Phi_f(\mathrm{cl}(\widetilde{b_1})) = \mathrm{cl}(\widetilde{b_2})$ contained in $S^1 \times S^1 \times [-1,1]$.

Proof. Note that the quotient $\big(S^1 \times S^1 \times [-1,1]\big)/R$ is homeomorphic to $S^2 \times [-1,1]$. Since Φ_f commutes with $R|_{S^1 \times S^1 \times [-1,1]}$, Φ_f induces a self-homeomorphism

$$\underline{\Phi}_f : S^2 \times [-1,1] \to S^2 \times [-1,1].$$

Since $\Phi_f(\mathrm{cl}(\widetilde{b})) = \mathrm{cl}(\widetilde{b_f})$ we have $\underline{\Phi}_f(C(b)) = C(b_f)$. Any orientation-preserving self-homeomorphism on S^2 is isotopic to the identity, and S^3 is a union of $S^2 \times [-1,1]$ and two 3-balls by gluing the boundaries together. Thus $\underline{\Phi}_f$ extends to a self-homeomorphism on S^3 which sends $C(b)$ to $C(b_f)$. This completes the proof. $\square$

Let us consider the mapping class group $\mathrm{Mod}(D_n)$ of the n-punctured disk D_n preserving the boundary ∂D of the disk setwise. We have a surjective homomorphism

$$\Gamma : B_n \to \mathrm{Mod}(D_n)$$

which sends each generator σ_i to the right-handed half twist between the ith and $(i+1)$st punctures. We say that $\beta \in B_n$ is *pseudo-Anosov* if $\Gamma(\beta) \in \mathrm{Mod}(D_n)$ is of the pseudo-Anosov type. When β is a pseudo-Anosov braid, the dilatation $\lambda(\beta)$ is defined by the dilatation of $\lambda(\Gamma(\beta))$.

We consider the above homomorphism $\Gamma : B_{2g+2} \to \mathrm{Mod}(D_{2g+2})$ when $n = 2g+2$. Recall the homomorphism $\mathfrak{t} : B_{2g+2} \to \mathrm{Mod}(\Sigma_g)$. The following lemma relates dilatations of β and $\mathfrak{t}(\beta)$.

Lemma 5. *Let $\beta \in B_{2g+2}$ be pseudo-Anosov and let $\Phi_\beta : D_{2g+2} \to D_{2g+2}$ be a pseudo-Anosov homeomorphism which represents $\Gamma(\beta) \in \mathrm{Mod}(D_{2g+2})$. Suppose that the stable foliation $\mathcal{F}_\beta$ for Φ_β defined on D_{2g+2} is not 1-pronged at the boundary ∂D of the disk. Then $\mathfrak{t}(\beta) \in \mathrm{Mod}(\Sigma_g)$ is pseudo-Anosov, and $\lambda(\mathfrak{t}(\beta)) = \lambda(\beta)$ holds.*

Proof. Since $\mathcal{F}_\beta$ is not 1-pronged at ∂D, $\Phi_\beta : D_{2g+2} \to D_{2g+2}$ induces a pseudo-Anosov homeomorphism $\Phi'_\beta : \Sigma_{0,2g+2} \to \Sigma_{0,2g+2}$ by filling ∂D with a disk. By Birman-Hilden [2], we have a surjective homomorphism

$$\mathfrak{q} : \mathcal{H}(\Sigma_g) \to \mathrm{Mod}(\Sigma_{0,2g+2})$$

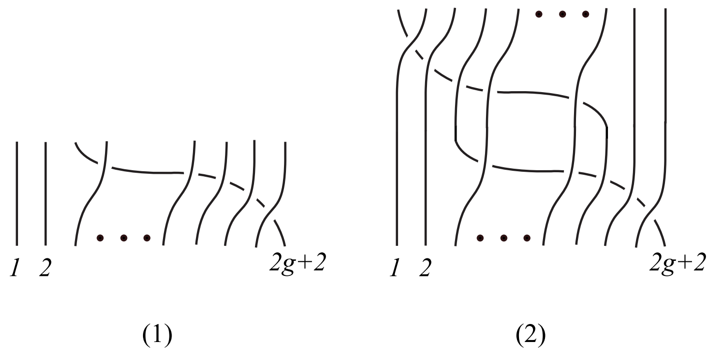

FIGURE 8. (1) $b_g \in B_{2g+2}$. (2) $\widetilde{b_g} \in B_{2g+2}$.

sending the Dehn twist t_i to the right-handed half twist h_i between the ith and $(i+1)$st punctures for $i = 1, \cdots, 2g+1$. Consider the 2-fold branched cover $\Sigma_g \to \Sigma_{0,2g+2}$ branched at the $2g+2$ marked points (corresponding to the punctures of $\Sigma_{0,2g+2}$). Then there is a lift $f_\beta : \Sigma_g \to \Sigma_g$ of $\Phi'_\beta : \Sigma_{0,2g+2} \to \Sigma_{0,2g+2}$ which satisfies $\mathfrak{t}(\beta) = [f_\beta] \in \mathcal{H}(\Sigma_g)$. Note that the stable foliation $\mathcal{F}_\beta$ for Φ_β extends to the stable foliation $\mathcal{F}'_\beta$ for Φ'_β by the assumption that $\mathcal{F}_\beta$ is not 1-pronged at ∂D. The stable foliation $\mathcal{F}'_\beta$ defined on $\Sigma_{0,2g+2}$ is lifted to the stable foliation for f_β defined on Σ_g. Thus f_β is a pseudo-Anosov homeomorphism which represents $\mathfrak{t}(\beta) = [f_\beta]$, and we have

$$\lambda([f_\beta]) = \lambda([\Phi'_\beta]) = \lambda([\Phi_\beta]) = \lambda(\beta).$$

This completes the proof. $\square$

Proof of Theorem A. For $g \geq 1$, we consider $b_g = \sigma_3\sigma_4 \cdots \sigma_{2g+1} \in B_{2g+2}$ and

$$\widetilde{b_g} = \sigma_1\sigma_2 \cdots \sigma_{2g-1} \cdot \sigma_3\sigma_4 \cdots \sigma_{2g+1} \in B_{2g+2},$$

see Figure 8. By Penner's result it is enough to prove that $\mathfrak{t}(\widetilde{b_g})$ is a pseudo-Anosov element in $D_g(S^3)$ for large g and $\log \lambda(\mathfrak{t}(\widetilde{b_g})) \asymp 1/g$ holds.

Applying Theorem B for the braid b_g we have the 2-fold branched cover

$$T_{\mathfrak{t}(\widetilde{b_g})} \to M_{C(b_g)}$$

branched over $q^{-1}(W)$. We first prove that $M_{C(b_g)} \cong S^3$ for $g \geq 1$. Clearly $C(b_1) = C(\sigma_3)$ is a trivial knot, and hence $M_{C(b_1)} \cong S^3$. We add a strand to $b_1 = \sigma_3 \in B_4$ so that $b_1^\circ = \sigma_3^2\sigma_4 \in B_5$, and think of $\widetilde{b_1^\circ}$ as a braid on $\mathcal{A}$, see Figure 5. Choose any $v \in S^1$ and consider the annulus $\mathbb{A}$ in $S^1 \times S^1 \times [-1,1]$ with boundary $\{v\} \times S^1 \times \{\pm 1\}$, see Figure 7(1). Let $f : S^1 \times S^1 \to S^1 \times S^1$ be the Dehn twist on $S^1 \times S^1$ about $\{v\} \times S^1$. Then the self-homeomorphism Φ_f on $S^1 \times S^1 \times [-1,1]$ is an annulus twist about $\mathbb{A}$, see Figure 7(2)(3). Observe that $\Phi_f(\mathrm{cl}(\widetilde{b_1})) = \mathrm{cl}(\widetilde{b_2})$ satisfies the assumptions $(*)$ and $(**)$. By repeating this process it is not hard to see that

$$\Phi_f(\mathrm{cl}(\widetilde{b_{j-1}})) = \mathrm{cl}(\widetilde{b_j}) \text{ for each } j \geq 2.$$

Thus $\Phi_f(\mathrm{cl}(\widetilde{b_{j-1}}))$ satisfies $(*)$ and $(**)$ for each j. Lemma 4 tells us that $C(b_g)$ is a trivial knot for $g \geq 1$ since so is $C(b_1)$. Thus $M_{C(b_g)} \cong S^3$, and $\mathfrak{t}(\widetilde{b_g}) \in D_g(S^3)$ for $g \geq 1$ by Theorem B.

The proof of Theorem D in [5] says that $\Gamma(\widetilde{b_g}) \in \mathrm{Mod}(D_{2g+2})$ is pseudo-Anosov for $g \geq 2$ and $\log \lambda(\widetilde{b_g}) \asymp \dfrac{1}{g}$ holds. Moreover the stable foliation of the pseudo-Anosov reprerentative for $\Gamma(\widetilde{b_g})$ satisfies the assumption of Lemma 5, see the proof of Step 2 in [5, Theorem D]. Thus $\mathfrak{t}(\widetilde{b_g})$ is pseudo-Anosov with $\lambda(\mathfrak{t}(\widetilde{b_g})) = \lambda(\widetilde{b_g})$ by Lemma 5, and we obtain the desired claim $\log \lambda(\mathfrak{t}(\widetilde{b_g})) = \log \lambda(\widetilde{b_g}) \asymp \dfrac{1}{g}$. This completes the proof. □

We end this paper with a question.

Question 6. *Let M be a closed 3-manifold M which is the 2-fold branched cover of S^3 branched over some link. Then does it hold $\log \delta(D_g(M)) \asymp \dfrac{1}{g}$?*

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