Dynamics of the monodromies of the fibrations on the magic 3-manifold

Eiko Kin, Osaka University

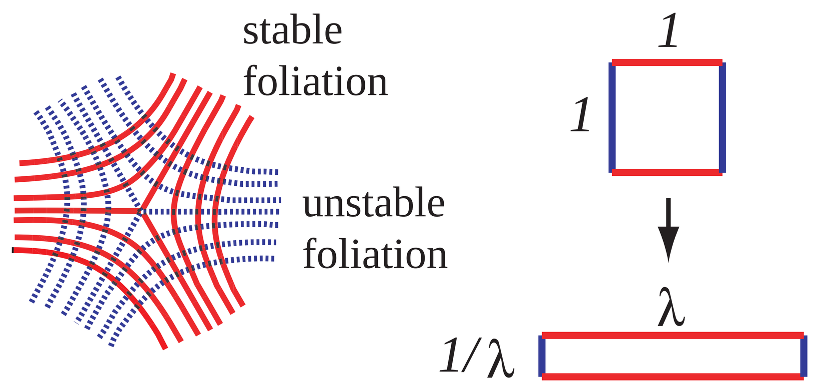
The Magic manifold N

is the hyperbolic and fibered 3-manifold defined by $N := S^3 \setminus$ (the 3-chaink link). It plays a significant rule to study

- exceptional Dehn surgeries of hyperbolic 3-manifolds (Gordon-Wu)
- hyperbolic 3-manifolds with small volumes (Martelli-Petronio, Gabai-Meyerhoff-Milley)
- pseudo-Anosovs with small dilatations (Takasawa-K)

Pseudo-Anosovs and dilatations

Consider the mapping class group $\mathrm{Mod}(\Sigma)$ on $\Sigma = \Sigma_{g,n}$; the closed orientable surface of genus g by removing $n \geq 0$ punctures. Each pseudo-Anosov mapping class $\phi \in \mathrm{Mod}(\Sigma)$ is equipped with some algebraic integer $\lambda(\phi) > 1$ called the dilatation.



Minimal dilatation problem.
Let $\delta_{g,n} = \min\{\lambda(\phi) \mid$ pseudo-Anosov $\phi \in \mathrm{Mod}(\Sigma_{g,n})\}$, and let $\delta_g = \delta_{g,0}$. Determine $\delta_{g,n}$. Find pseudo-Anosov elements which achieve $\delta_{g,n}$.

asymptotic behaviors	smallest known upper bounds
$\log \delta_g \asymp 1/g$	**(U1)** $\limsup_{g\to\infty} g \log \delta_g \leq \log(\frac{3+\sqrt{5}}{2})$
$\log \delta_{0,n} \asymp 1/n$	**(U2)** $\limsup_{n\to\infty} n \log \delta_{0,n} \leq 2\log(2+\sqrt{3})$
$\log \delta_{1,n} \asymp 1/n$	**(U3)** $\limsup_{n\to\infty} n \log \delta_{1,n} \leq 2 \log \delta(D_4) \approx 1.6628$
Given $g \geq 2$, $\log \delta_{g,n} \asymp \frac{\log n}{n}$	**(U4)** $\limsup_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2$ if $2g+1$ is prime

ref. [Penner], [Hironaka-K], [Tsai], [Hironaka], [Aaber-Dunfield], [Takasawa-K]. Here D_n is an n-punctured disk.

Why is the magic manifold mysterious?

The mapping tori of all potential candidates with the smallest dilatations are homeomorphic to N, or they are obtained from N by Dehn fillings along the boundary slops of fibers. Said differently, all examples with the smallest known dilatations are coming from a single 3-manifold N.

Question and Theorem

Question.
Choose any fibered class a of N. Construct the fiber F_a and the monodromy $\Phi_a : F_a \to F_a$ associated to a. What do these pseudo-Anosovs Φ_a look like?

Theorem. There exists an algorithm to build the followings for each fibered class a of N.
(1) The fiber F_a and the monodromy $\Phi_a : F_a \to F_a$.
(2) The invariant train track τ_a and the train track map $g_a : \tau_a \to \tau_a$ associated to $\phi_a = [\Phi_a]$.

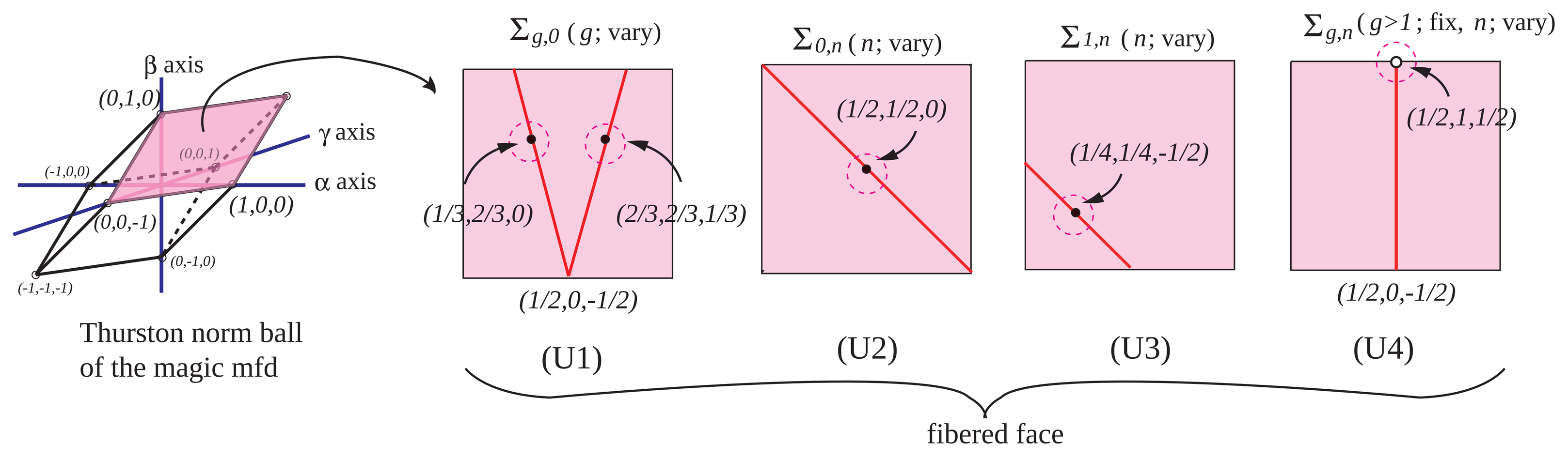
★ We also construct the metrized, directed graph Γ_a from $g_a : \tau_a \to \tau_a$. Then we build the curved graph G_a induced by Γ_a which reflects the 'shape' of Φ_a. We compute the clique polynomial $Q_a(t)$ of G_a to capture the dilatation λ_a.

Hint: First construct (the 3-braid $\sigma_1^2\sigma_2^{-1}$ monodromy) $\Phi_{\alpha+\beta}$ of the fibration associated to the fibered class $\alpha+\beta$ by using the 'pillow model'. Then the mapping torus $\mathbb{T}_{\Phi_{\alpha+\beta}} \simeq N$. Next construct the branched surface $\mathcal{B}$ which carries the fiber F_a of each fibered class $a \in int(C_\Delta)$. The first return map $\Phi_a : F_a \to F_a$ of the suspension flow $\Phi^t_{\alpha+\beta}$ gives us the the desired monodromy. Third, consider the stable foliation $\mathcal{F}_{\alpha+\beta}$ of $\Phi_{\alpha+\beta}$, and take its suspension $\widehat{\mathcal{F}}_{\alpha+\beta}$. Construct another branched surface $\mathcal{B}_\Delta$ which carries $\widehat{\mathcal{F}}_{\alpha+\beta}$. To obtain $g_a : \tau_a \to \tau_a$, we need to view the intersection $\mathcal{B} \cap \mathcal{B}_\Delta$. □

Reference

[1] C. McMullen, *Entropy and the clique polynomial*. Preprint (2013).
[2] E. Kin, S. Kojima and M. Takasawa, *Minimal dilatations of pseudo-Anosovs generated by the magic 3-manifold and their asymptotic behavior.* Algebraic and Geometric Topology 13 (2013), 3537-3602.

Pseudo-Anosovs with small dilatation coming from N

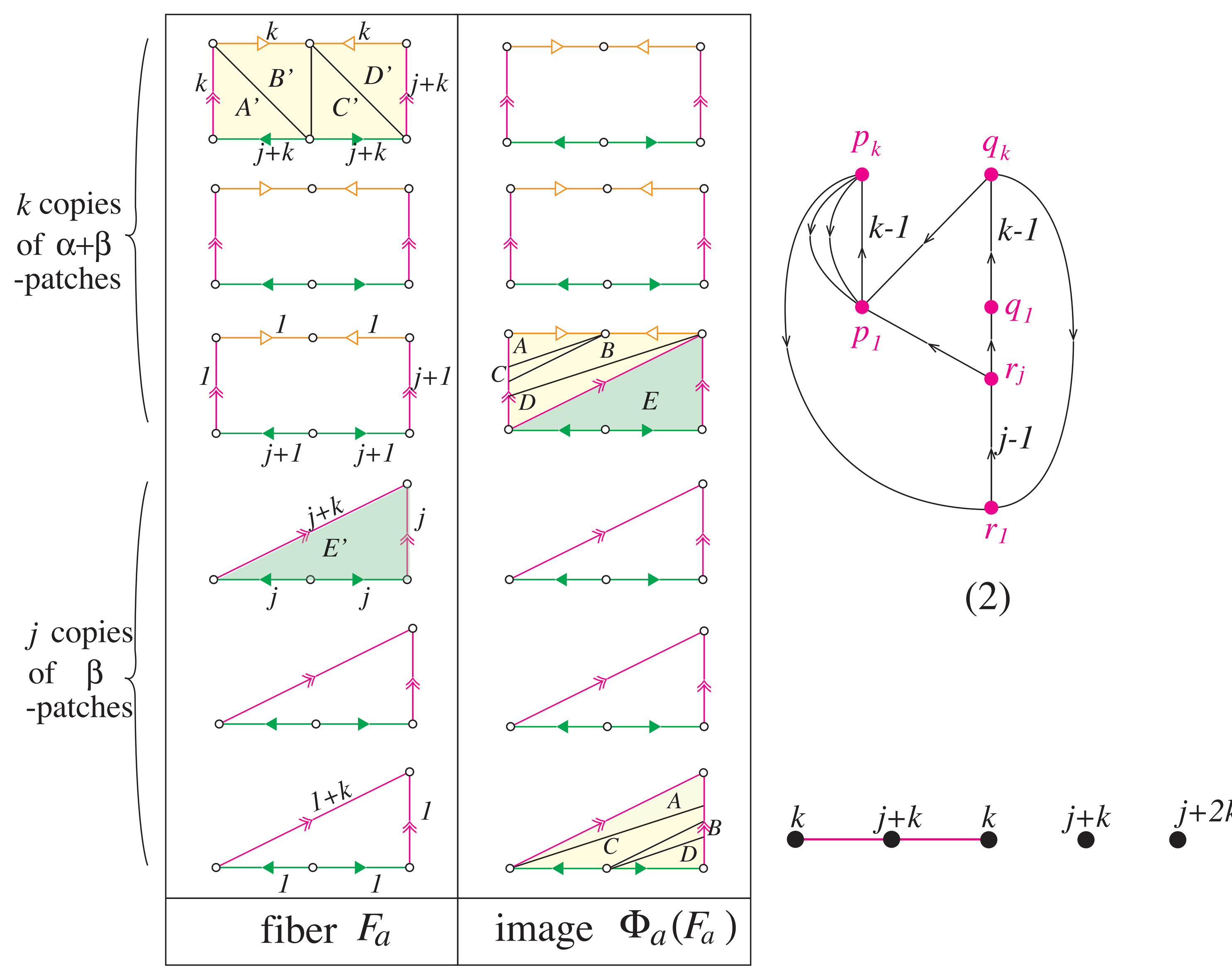


Interesting families of pseudo-Anosovs coming from N

Example. [minimizers of $\delta(D_n)$ for $n = 3,4,5,6,7,8$] Consider the fibered classes of the form $a = (j,k)_0 = (k, j+k, 0)$. Then the clique polynomial $Q_a(t)$ is given by

$$Q_a(t) = 1 - (2t^k + 2t^{j+k} + t^{j+2k})$$

whose largest root equals λ_a. In the figure below, (1) $\Phi_a : F_a \to F_a$, (2) Γ_a and (3) G_a.

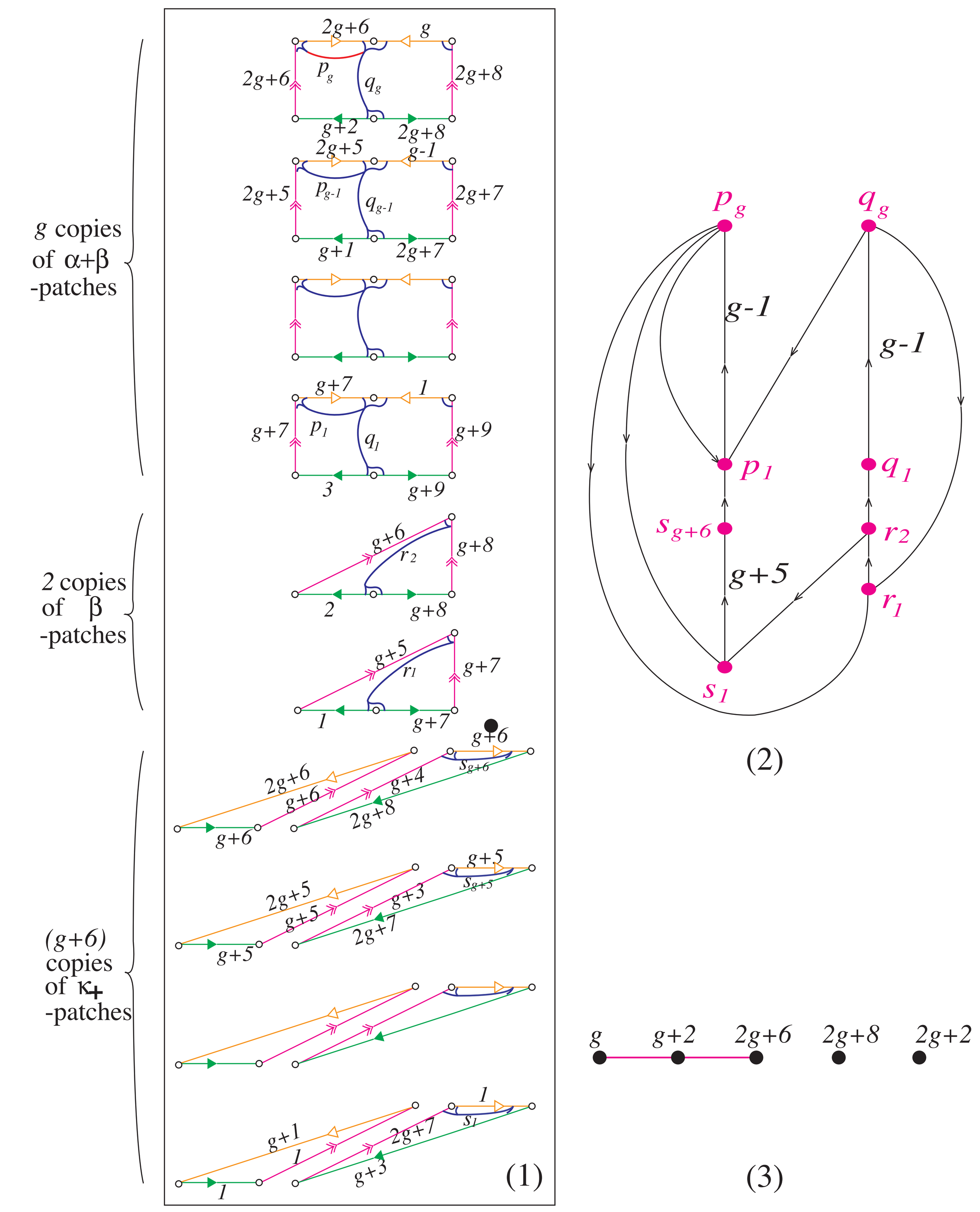


(1) (2) (3)

Example. [minimizer of δ_7^+] Suppose that $g \equiv 7, 9 \pmod{10}$, and let $a = (g+6, 2, g)_+ = (2g+6, 2g+8, g+6)$. Then F_a has genus g. We have the clique polynomial

$$Q_a(t) = f_a(t) = (t^{g+4}+1)(t^{2g+4} - t^{g+4} - t^{g+2} - t^g + 1),$$

and its largest root gives us the dilatation λ_a. In the figure below, (1) $\tau_a \subset F_a$, (2) Γ_a and (3) W_a.

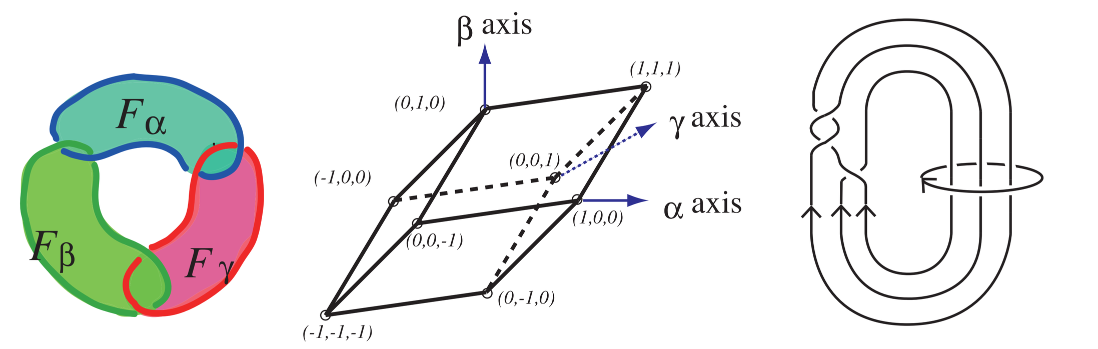


(1) (2) (3)

Background

Fact. (Thurston)
Let M be an oriented hyperbolic 3-manifold. The unit ball U_M with respect to the Thurston norm $\|\cdot\| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ is a compact, convex polyhedron.

Example. (Thurston norm ball of N.) $\alpha = [F_\alpha]$, $\beta = [F_\beta]$, $\gamma = [F_\gamma] \in H_2(N, \partial N; \mathbb{Z})$.



Theorem and Definition. (Thurston)
Suppose that M is a hyperbolic surface bundle over the circle. Then there exists a top dimensional face Ω on ∂U_M such that each integral class $\in int(\mathrm{Cone}(\Omega))$ corresponds to a fiber of some fibration of M. Such an integral class is called the fibered class, and the face Ω is called the fibered face.

Fact. (Fried, S. Matsumoto, McMullen)
Let Ω be a fibered face of M.

- Let $\mathrm{ent}(a) := \log \lambda(\Phi_a)$, where Φ_a is the monodromy of the fibration associated to a fibered class a. This defines a map $\mathrm{ent} : int(C_\Omega(\mathbb{Z})) \to \mathbb{R}$. It admits a continuous extension $\mathrm{ent} : int(C_\Omega) \to \mathbb{R}$.
- $1/\mathrm{ent} : int(\Omega) \to \mathbb{R}$ is strictly concave. If $a \in int(\Omega)$ goes to $\partial\Omega$, then $\mathrm{ent}(a)$ goes to ∞.
- Teichmüler polynomial polynomial P_Ω captures the dilatations λ_a of all fibered classes a in $int(C_\Omega)$.

Theorem. (McMullen [1])
(1) Let Γ be a directed graph with a metric $m : E(\Gamma) \to \mathbb{R}_+$. Let $\lambda(\Gamma, m)$ be the growth rate. Then the smallest positive root of the *Perron polynomial* $P(t)$ of (Γ, m) is given by $\frac{1}{\lambda(\Gamma,m)}$. The function $h(m) = \log \lambda(\Gamma, m)$ is convex of m.

(2) Let G be an undirected graph with a weight $\omega : V(G) \to \mathbb{R}_+$. The *clique polynomial* $Q(t)$ of (G, ω) captures the growth rate $\lambda(G, \omega)$.

(3) Given (Γ, m), one can define the *curved complex* (G, ω) of (Γ, m). In this case, the Perron polynomial $P(t)$ of (Γ, m) coincide with the clique polynomial $Q(t)$ of (G, ω)