Minimal dilatations of pseudo-Anosovs and a mystery of the magic 3-manifold

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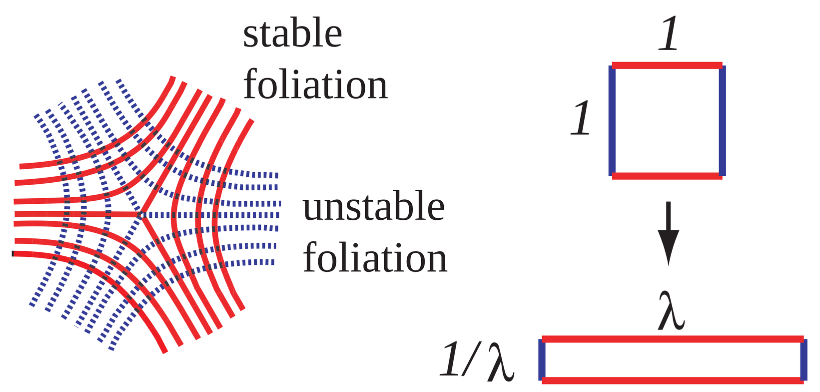
The Magic 3-manifold?

It is the hyperbolic manifold defined by $N := S^3 \setminus$ (the 3-chain link). It plays a significant rule to study the exceptional Dehn surgeries of hyperbolic 3-manifolds (Gordon-Wu). Also, Interesting hyperbolic 3-manifolds with small volumes can be obtained from N by Dehn fillings (Martelli-Petronio, Gabai-Meyerhoff-Milley). Let $N(r)$ be the manifold obtained from N by Dehn filling a cusp along the slope r. Our main examples of this presentation are $N(\frac{3}{-2})$, $N(\frac{1}{-2})$ and $N(1)$.

Pseudo-Anosovs and dilatations

Consider the mapping class group $\mathrm{Mod}(\Sigma)$ on $\Sigma = \Sigma_{g,n}$; the closed orientable surface of genus g by removing $n \geq 0$ punctures. Each pseudo-Anosov mapping class $\phi \in \mathrm{Mod}(\Sigma)$ is equipped with some algebraic integer $\lambda(\phi) > 1$ called the dilatation.



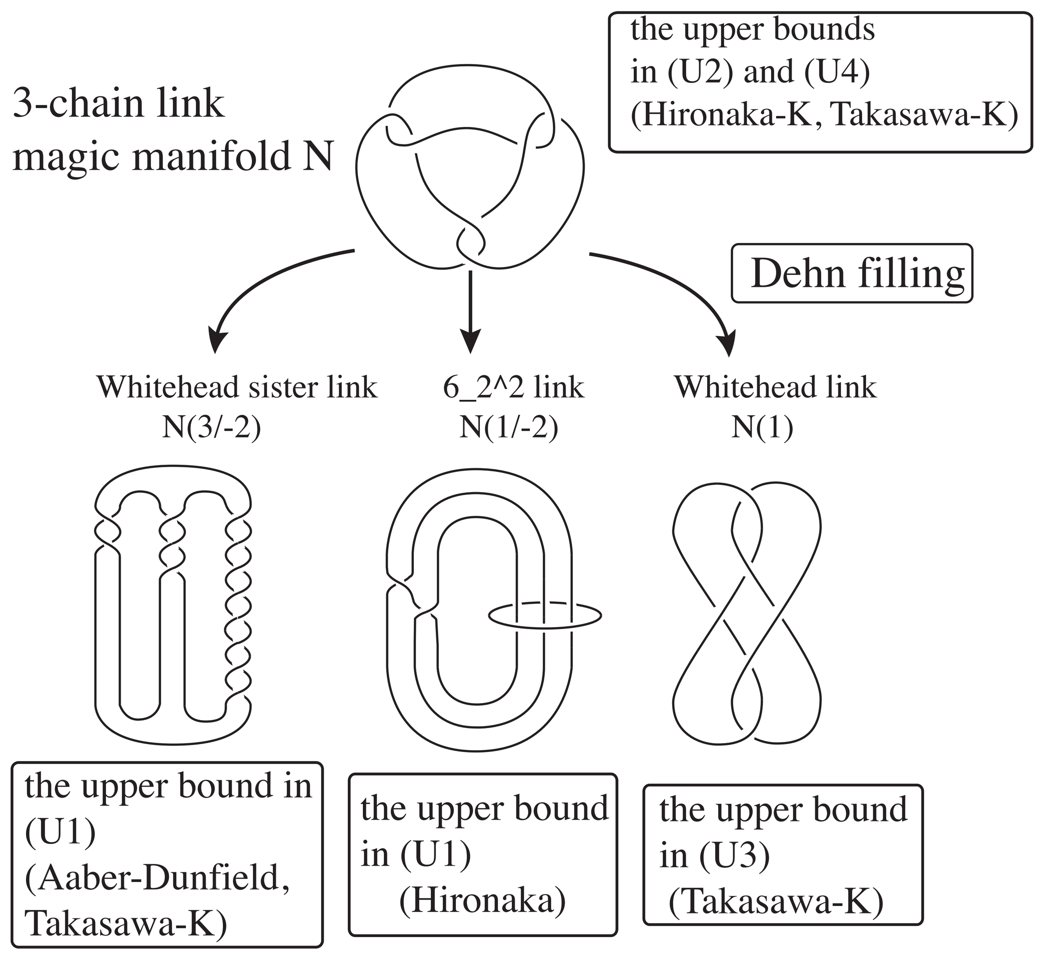
Problem.
Let $\delta_{g,n} = \min\{\lambda(\phi) \mid \text{pseudo-Anosov } \phi \in \mathrm{Mod}(\Sigma_{g,n})\}$, and let $\delta_g = \delta_{g,0}$. Determine $\delta_{g,n}$. Find pseudo-Anosov elements which achieve $\delta_{g,n}$.

asymptotic behaviors	smallest known upper bounds
$\log \delta_g \asymp 1/g$	**(U1)** $\limsup_{g\to\infty} g \log \delta_g \leq \log(\frac{3+\sqrt{5}}{2})$
$\log \delta_{0,n} \asymp 1/n$	**(U2)** $\limsup_{n\to\infty} n \log \delta_{0,n} \leq 2\log(2+\sqrt{3})$
$\log \delta_{1,n} \asymp 1/n$	**(U3)** $\limsup_{n\to\infty} n \log \delta_{1,n} \leq 2\log \delta(D_4) \approx 1.6628$
Given $g \geq 2$, $\log \delta_{g,n} \asymp \frac{\log n}{n}$	**(U4)** Theorem B in this poster

ref. [Penner], [Hironaka-K], [Tsai], [Hironaka], [Aaber-Dunfield], [Takasawa-K]. Here D_n is an n-punctured disk.

Why is the magic manifold mysterious?

The mapping tori of all potential candidates with the smallest dilatations are homeomorphic to N, or they are obtained from N by Dehn fillings. Interestingly, all examples with the smallest known dilatations are coming from the single 3-manifold N.



Reference

[1] E. Kin, S. Kojima and M. Takasawa, *Minimal dilatations of pseudo-Anosovs generated by the magic 3-manifold and their asymptotic behavior.* to appear in "Algebraic and geometric topology"
[2] E. Kin and M. Takasawa, *The boundary of a fibered face of the magic 3-manifold and the asymptotic behavior of the minimal pseudo-Anosovs dilatations.* preprint.

Main results

Consider the set

$$\mathcal{M} = \{\text{the monodromy of a fibration on the magic manifold } N \text{ with the condition } (*)\}.$$

$(*)$ The stable foliation of the monodromy associated to a fibration on N has no 1 prong.

Let $\widehat{\mathcal{M}}$ be the set of extensions $\widehat{\Phi}$ of $\Phi \in \mathcal{M}$ defined on the closed surfaces. (Note: $\widehat{\Phi}$ is pseudo-Anosov with the same dilatation as Φ.) Let $\widehat{\delta}_g$ be the minimum among dilatations of elements in $\widehat{\mathcal{M}} \cap \mathrm{Mod}(\Sigma_{g,0})$. (Clearly $\delta_g \leq \widehat{\delta}_g$. $\delta_2 = \widehat{\delta}_2$ holds.)

Theorem A. (Kojima-Takasawa-K) The upper bound (U1) of the table is "best possible" in N. i.e,

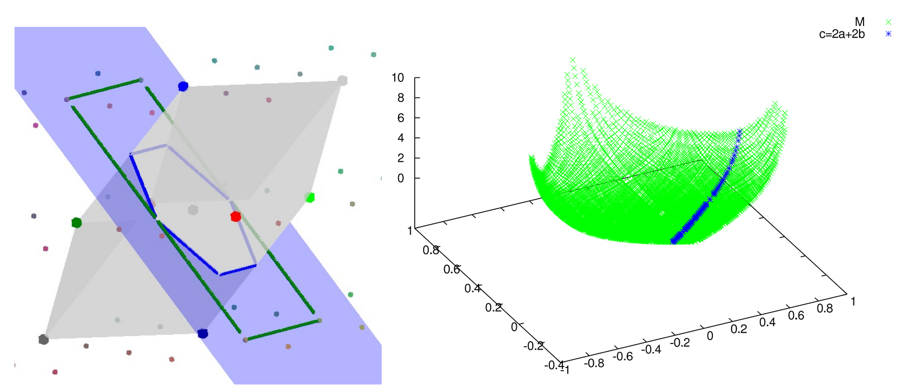
(1) $\lim_{g\to\infty} g \log \widehat{\delta}_g = \log(\frac{3+\sqrt{5}}{2})$.

(2) For large g, $\widehat{\delta}_g$ is achieved by the monodromy of some Σ_g-bundle over the circle obtained from either $N(\frac{3}{-2})$ or $N(\frac{1}{-2})$ by Dehn filling both cusps.

Hint: Study the invariants $\min \mathrm{Ent}(N(r), \Omega_r)$ for each $r \in \mathbb{Q} \setminus \{-3,-2,-1,0\}$. Show that the smallest one and the second smallest one are

$$\min \mathrm{Ent}(N(1)) = 2\log \delta(D_4) \approx 1.6628,$$
$$\min \mathrm{Ent}(N(r)) = 2\log(\tfrac{3+\sqrt{5}}{2}) \approx 1.9248 \text{ for } r = \tfrac{3}{-2}, \tfrac{1}{-2}.$$

If we let a_g be the fibered class of N which realizes $\widehat{\delta}_g$, then show that one of the boundary slopes of the fiber associated to a_g is either $\frac{1}{-2}$ or $\frac{3}{-2}$ for g large. □



Conjecture.

$$\delta_g = \widehat{\delta}_g \quad \text{for } g \text{ large.}$$

Question.
Tsai proved that for any fixed $g \geq 2$, $\log \delta_{g,n} \asymp \frac{\log n}{n}$. Given $g \geq 2$, does $\lim_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n}$ exist? What is its value?

Theorem B. (Takasawa-K) If $2g+1$ is relatively prime to s or $s+1$ for each $0 \leq s \leq g$, then

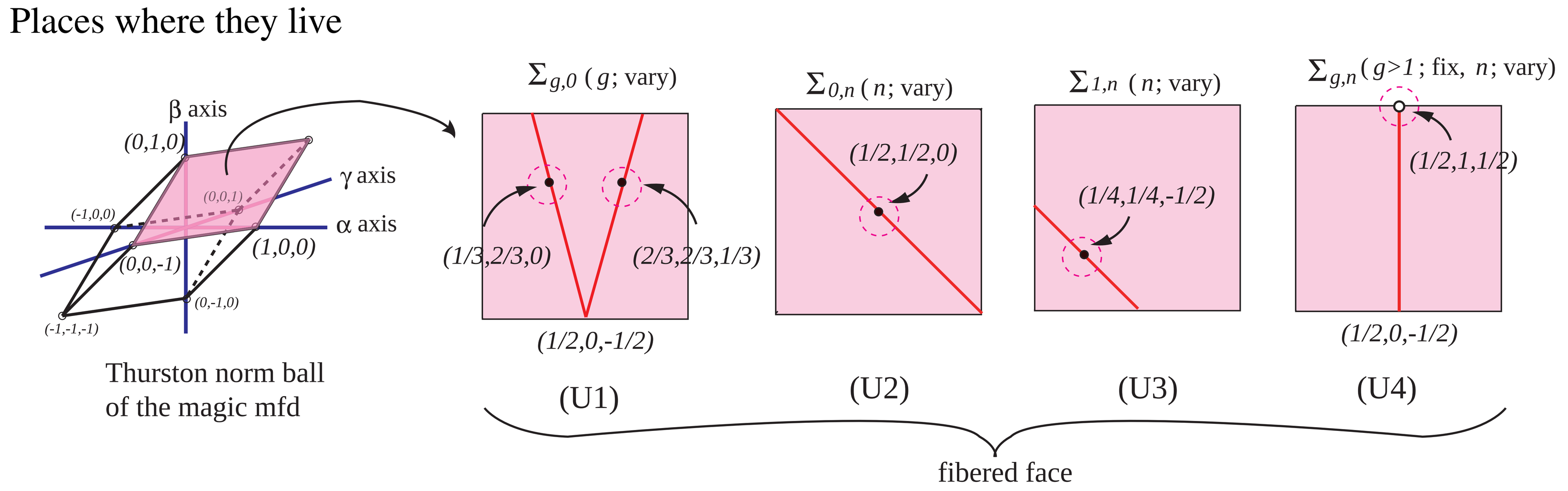
$$\limsup_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2.$$

In particular if $2g+1$ is prime for $g \geq 2$, then the above inequality holds.

Hint: Find suitable fibered classes of N whose projective classes go to the boundary of the fibered face. □

Pseudo-Anosovs with small dilatation coming from N

Places where they live



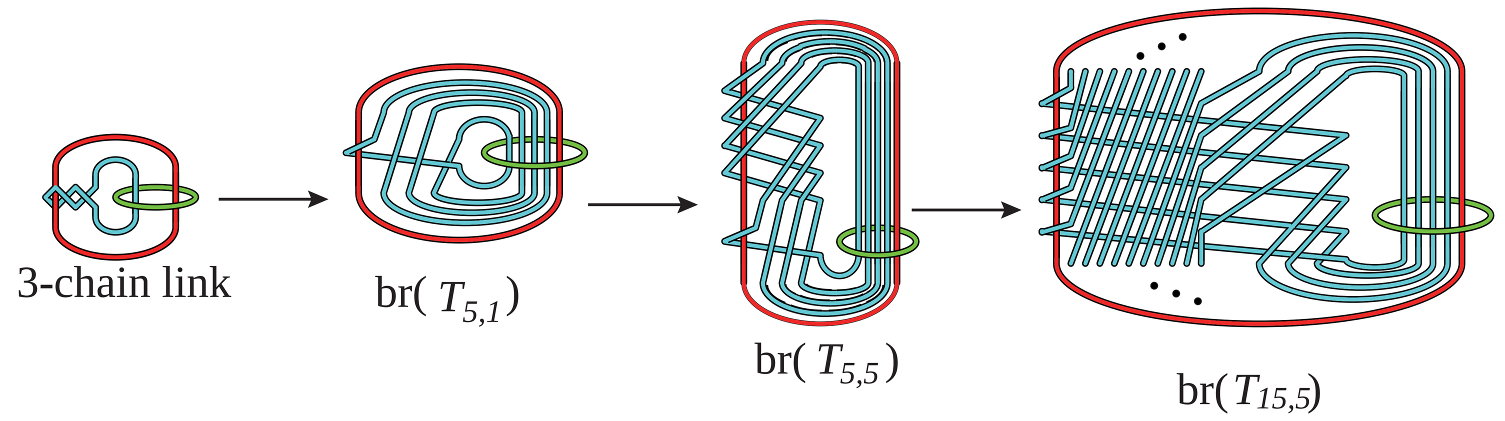
An interesting family of pseudo-Anosov braids

For $m \geq 3$, $p \geq 1$, let

$$T_{m,p} := (\sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-1})^{p-1} \sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-2} \sigma_{m-1}^{-1} \in B_m.$$

By forgetting the 1st strand of $T_{m,p}$, one obtains $T'_{m,p} \in B_{m-1}$.

Theorem. (Takasawa-K)
$S^3 \setminus \mathrm{br}(T_{m,p}) \simeq N \iff m-1$ and p are relatively prime.
(Here $\mathrm{br}(b)$ is the closure of a braid b with its axis.)



Why interesting?

- $\delta(D_n)$'s have been computed for $3 \leq n \leq 8$ (Ko-Los-Song, Ham-Song, Lanneau-Thiffeault). The following braids reach the minimal dilatations;

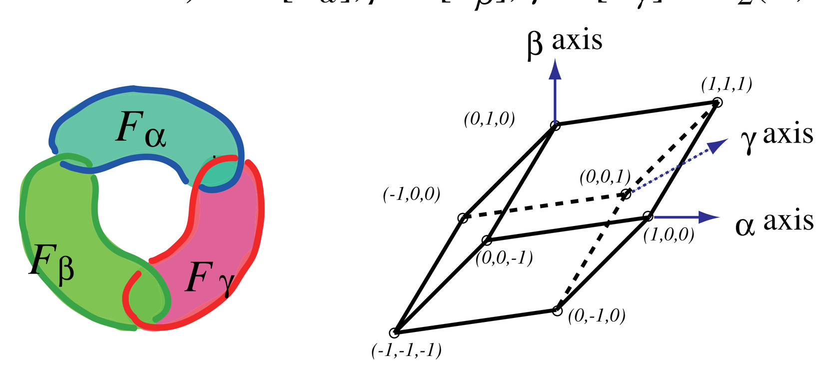
$$T'_{4,1} \in B_3,\ T'_{5,1} \in B_4,\ T'_{6,2} \in B_5,\ T_{6,2} \in B_6,\ T'_{8,2} \in B_7,\ T'_{9,5} \in B_8.$$

- For $g \geq 2$, $T'_{2g+2} \in B_{2g+1}$ is conjugate to the Hironaka-Kin's braid $\sigma_{g-1,g+1}$ with the smallest known dilatation.

Background

Fact. (Thurston)
Let M be an oriented hyperbolic 3-manifold. The unit ball U_M with respect to the Thurston norm $\|\cdot\| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ is a compact, convex polyhedron.

Example. (Thurston norm ball of N.) $\alpha = [F_\alpha]$, $\beta = [F_\beta]$, $\gamma = [F_\gamma] \in H_2(N, \partial N; \mathbb{Z})$.



Theorem and Definition. (Thurston)
Suppose that M is a hyperbolic surface bundle over the circle. Then there exists a top dimensional face Ω on ∂U_M such that each integral class $\in int(\mathrm{Cone}(\Omega))$ corresponds to a fiber of some fibration of M. Such an integral class is called the fibered class, and the face Ω is called the fibered face.

Fact. (Fried, S. Matsumoto, McMullen)
Let Ω be a fibered face of M.

- Let $\mathrm{ent}(a) := \log \lambda(\Phi_a)$, where Φ_a is the monodromy of the fibration associated to a fibered class a. This defines a map $\mathrm{ent} : int(C_\Omega(\mathbb{Z})) \to \mathbb{R}$. Then it admits a continuous extension

$$\mathrm{ent} : int(C_\Omega) \to \mathbb{R}.$$

- $1/\mathrm{ent} : int(\Omega) \to \mathbb{R}$ is strictly concave. If $a \in int(\Omega)$ goes to $\partial\Omega$, then $\mathrm{ent}(a)$ goes to ∞.
- McMullen polynomial P_Ω captures the dilatations of all fibered classes in $int(C_\Omega)$.

Invariant of hyperbolic fibered 3-manifolds

$$\min \mathrm{Ent}(M, \Omega) := \min\{\mathrm{ent}(a) \mid a \in int(\Omega)\}$$
$$\min \mathrm{Ent}(M) := \min_{\Omega}\{\min \mathrm{Ent}(M, \Omega)\},$$

where Ω is taken over all fibered faces of M.