NOTES ON PSEUDO-ANOSOVS WITH SMALL DILATATIONS COMING FROM THE MAGIC 3-MANIFOLD

EIKO KIN

1. Introduction

Let N be the exterior of the 3 chain link $\mathcal{C}_3$ (Figure 1) in the three sphere S^3. Gordon and Wu called N the *magic manifold*, because they found that N has many interesting non-hyperbolic fillings and this particular manifold plays a significant role for the study of non-hyperbolic fillings for cusped hyperbolic 3-manifolds. The magic manifold N is a hyperbolic surface bundles over the circle, and N has the smallest known volume among orientable 3-cusped hyperbolic 3-manifolds. Martelli and Petronio classified all the non-hyperbolic Dehn fillings of N in [18]. Let $N(r)$ be the manifold obtained from N by Dehn filling one cusp along the slope $r \in \mathbb{Q}$. The Whitehead link exterior and the Whitehead sister link (i.e, $(-2,3,8)$-pretzel link) exterior are homeomorphic to $N(1)$ and $N(\frac{3}{-2})$ respectively. It was proved by Agol [2] that the smallest volume among orientable 2-cusped hyperbolic 3-manifold is achieved by either $N(1)$ or $N(\frac{3}{-2})$. In the recent work of Gabai, Meyerhoff and Milley, the magic manifold N plays a central role for the minimizing problem on volumes of hyperbolic 3-manifolds. The main characters in this paper are manifolds N, $N(1)$, $N(\frac{3}{-2})$ and $N(\frac{1}{-2})$. The last 2-cusped 3-manifold $N(\frac{1}{-2})$ is homeomorphic to the exterior of the 6^2_2 link (Figure 1).

In [11, 12, 13, 14], we investigated the monodromies of fibrations of N extensively for the study of the minimal dilatations and their asymptotic behaviors. We found that N provides many interesting families of pseudo-Anosovs with small dilatations. In this paper, we give an expository account of results of [11, 12, 13, 14]. All the results in the paper are contained in those papers, and hence this paper has no new results. The purpose of this paper is to describe "places in N" where the pseudo-Anosovs with the smallest dilatations or with the smallest known dilatations "live". The main tool to do this is a fibered face of the Thurston norm ball for N.

Let $\Sigma_{g,n}$ be an orientable surface of genus g with n punctures, and let $\Sigma_g = \Sigma_{g,0}$ be a closed surface of genus g. We consider the mapping class group $\mathrm{Mod}(\Sigma)$ of $\Sigma = \Sigma_{g,n}$, that

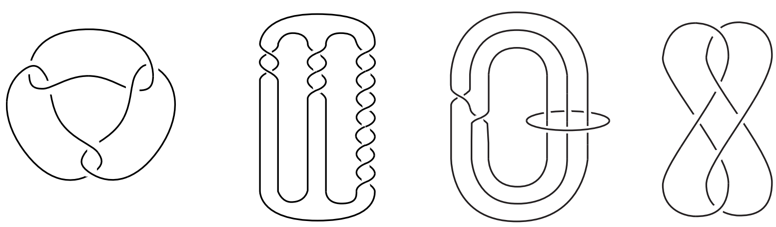

Figure 1. (from left to right) 3 chain link $\mathcal{C}_3$, $(-2,3,8)$-pretzel link, link 6^2_2, Whitehead link.

Received December 27, 2012.

is the group of isotopy classes of orientation preserving homeomorphisms on Σ. According to the work of Nielsen and Thurston, elements of $\mathrm{Mod}(\Sigma)$ are classified into three types: periodic, reducible, pseudo-Anosov. The last type, pseudo-Anosovs have many interesting and rich properties. The hyperbolization theorem by Thurston asserts that $\phi \in \mathrm{Mod}(\Sigma)$ is pseudo-Anosov if and only if the mapping torus $\mathbb{T}(\phi)$ of ϕ is a hyperbolic 3-manifold with finite volume.

Each pseudo-Anosov $\phi \in \mathrm{Mod}(\Sigma)$ has a representative $\Phi : \Sigma \to \Sigma$, called a *pseudo-Anosov homeomorphism*, which satisfies the following: there exists a constant $\lambda > 1$ and there exists a pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that

$$\Phi(\mathcal{F}^s) = \tfrac{1}{\lambda}\mathcal{F}^s \text{ and } \Phi(\mathcal{F}^u) = \lambda \mathcal{F}^u.$$

The constant $\lambda = \lambda(\Phi)$ is called the *dilatation* of Φ, and $\mathcal{F}^s$, $\mathcal{F}^u$ are called the *stable*, *unstable foliation* (or *invariant foliations*) of Φ. It is known that $\lambda(\Phi)$ does not depend on the choice of a pseudo-Anosov homeomorphism $\Phi \in \phi$, and hence the *dilatation* $\lambda(\phi)$ of ϕ is defined to be $\lambda(\Phi)$. We call the quantities

$$\mathrm{ent}(\phi) = \log \lambda(\phi) \text{ and } \mathrm{Ent}(\phi) = |\chi(\Sigma)| \log \lambda(\phi)$$

the *entropy* and *normalized entropy* of ϕ, where $\chi(\Sigma)$ is the Euler characteristic of Σ.

We fix Σ and consider the set of entropies defined on Σ:

$$\{\mathrm{ent}(\phi) \mid \phi \in \mathrm{Mod}(\Sigma) \text{ is pseudo-Anosov}\} \subset \mathbb{R}.$$

It is proved by Ivanov that this set is closed and discrete. In particular there exists a minimum. We denote by $\delta(\Sigma) > 1$, the minimal dilatation of pseudo-Anosov elements defined on Σ.

Problem 1.1 (Minimal dilatation problem)**.** *Determine the explicit value of $\delta(\Sigma)$. Identify a pseudo-Anosov element in* $\mathrm{Mod}(\Sigma)$ *which achieves $\delta(\Sigma)$.*

Let us set $\delta_{g,n} = \delta(\Sigma_{g,n})$ and $\delta_g = \delta_{g,0}$. The explicit values of δ_g's are known for the only cases $g = 1, 2$. It is known by [1]Penner [22] that $\log \delta_g \asymp \frac{1}{g}$. After the work of Penner, several authors examined the asymptotic behaviors of the minimal dilatations on surfaces varying topology, see [9, 1, 13, 20, 10, 24] and Table 1(1st column).

Problem 1.1 has several aspects, and there are many related questions.

Question 1.2 ([21] for (4))**.**

(1) *Is a pseudo-Anosov element $\phi \in$* $\mathrm{Mod}(\Sigma)$ *which achieves $\delta(\Sigma)$ unique up to conjugate?*
(2) *Identify the hyperbolic fibered 3-manifold $\mathbb{T}(\phi)$ of such a minimizer ϕ.*
(3) *What is the minimal polynomial of $\delta(\Sigma)$? (Note: The dilatation $\lambda(\phi)$ of a pseudo-Anosov ϕ is known to be an algebraic integer.)*
(4) *Do $\lim\limits_{g\to\infty} g \log \delta_g$, $\lim\limits_{g\to\infty} g \log \delta_g^+$, $\lim\limits_{n\to\infty} n \log \delta_{0,n}$ and $\lim\limits_{n\to\infty} n \log \delta_{1,n}$ exist? What are the values?*
(5) *Given $g \geq 2$, does $\lim\limits_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n}$ exist? What is its value?*

The smallest known upper bounds on Question 1.2(4)(5) are shown in Table 1(2nd column). We shall see that all families of pseudo-Anosovs ϕ's to give the upper bounds in Table 1(2nd column) 'come from' N. More precisely, these pseudo-Anosov mapping

[1]Let A_g and B_g be functions on g. We write $A_g \asymp B_g$ if there exists a constant c, independent of g, such that $\frac{A_g}{c} < B_g < cA_g$.

TABLE 1. asymptotic behaviors of minimal dilatations.

asymptotic behaviors	upper bounds
$\log \delta_g \asymp 1/g$ [22]	$\limsup\limits_{g\to\infty} g \log \delta_g \leq \log(\frac{3+\sqrt{5}}{2})$ [9, 1, 13]
$\log \delta_g^+ \asymp 1/g$ [20, 10]	$\limsup\limits_{\substack{g \not\equiv 0 \pmod 6 \\ g\to\infty}} g \log \delta_g^+ \leq \log(\frac{3+\sqrt{5}}{2})$ [9, 11] $\limsup\limits_{\substack{g \equiv 6 \pmod{12} \\ g\to\infty}} g \log \delta_g^+ \leq 2 \log \delta(D_5)$ [11]
$\log \delta_{0,n} \asymp 1/n$ [10]	$\limsup\limits_{n\to\infty} n \log \delta_{0,n} \leq 2\log(2+\sqrt{3})$ [10, 12]
$\log \delta_{1,n} \asymp 1/n$ [24]	$\limsup\limits_{n\to\infty} n \log \delta_{1,n} \leq 2\log \delta(D_4)$ [11]
Given $g \geq 2$, $\log \delta_{g,n} \asymp \frac{\log n}{n}$ [24]	$\limsup\limits_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2$ if g enjoys $(*)$ in Thm. 3.5 [14]

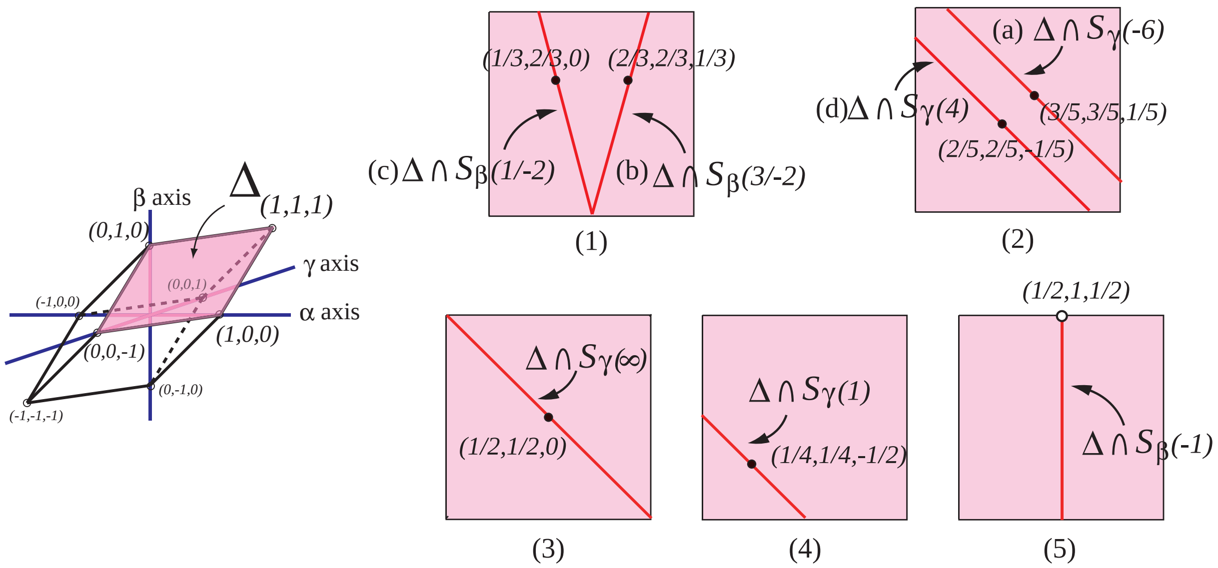

FIGURE 2. (left) Thurston norm ball U_N for N. (right) intersection of Δ and linear section $S_*(r)$. (1) $\Delta \cap S_\beta(\frac{1}{-2})$ (see (c) in the figure) and $\Delta \cap S_\beta(\frac{3}{-2})$ (see (b) in the figure); (2) $\Delta \cap S_\gamma(4)$ (see (d)) and $\Delta \cap S_\gamma(-6)$ (see (a)); (3) $\Delta \cap S_\gamma(\infty)$; (4) $\Delta \cap S_\alpha(1) = \Delta \cap S_\beta(1) = \Delta \cap S_\gamma(1)$; (5) $\Delta \cap S_\beta(-1)$.

classes ϕ's have the following property: The mapping torus $\mathbb{T}(\phi)$ is homeomorphic to N, or $\mathbb{T}(\phi)$ is obtained from N by Dehn filling cusps along the boundary slopes of a fiber of N. (i.e, N is a *parent manifold* of $\mathbb{T}(\phi)$.)

Let δ_g^+ be the minimal dilatation of pseudo-Anosovs with orientable invariant foliations defined on Σ_g. (Obviously $\delta_g \leq \delta_g^+$.) The explicit value of δ_g^+ is known for all $2 \leq g \leq 8$ except for $g = 6$ [1, 9, 13, 16, 26]. (See Table 5(3rd column).) The minimal dilatation $\delta(D_n)$ on an n-punctured disk D_n is determined for all $3 \leq n \leq 8$ [7, 8, 15, 17]. (See Table 10(3rd column).) These minimizers come from N in the same sense as above.

The paper is organized as follows. In Section 2, we first review the fibered face theory which is quite useful to find families of pseudo-Anosovs with small dilatations. Next, we describe the properties of fibrations on both N and manifolds $N(r)$'s. In Section 3, we examines the asymptotic behaviors of minimal dilatations given in Table 1. Especially we explain how the constants in the upper bounds of Table 1(2nd column) appear. These constants are related to an invariant "$\min \mathrm{Ent}$" of hyperbolic surface bundles over the circle. Figure 2(left) shows the Thurston norm ball of N. A particular fibered face Δ is shaded in the figure. By using Figure 2(right), we shall illustrate places in N where the pseudo-Anosovs with the smallest dilatations or with the smallest known dilatations live. (For the definition of the linear sections $S_\beta(r)$ etc, see Section 2.2. See also Figure 3.) We conclude the paper with conjectures and questions.

2. Preliminalies

2.1. **Basic facts on fibered face theory.** Let M be an oriented, hyperbolic 3-manifold possibly with boundary ∂M. We recall the Thurston norm $\|\cdot\| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$. See [23] fore more details. The Thurston norm $\|\cdot\|$ has the property such that for any integral class $a \in H_2(M, \partial M; \mathbb{R})$,

$$\|a\| = \min_F \{-\chi(F)\},$$

where the minimum is taken over all oriented surfaces F embedded in M, satisfying $a = [F]$, with no components of non-negative Euler characteristic. The surface F which realizes this minimum is called a *minimal representative* of a, and it is denoted by F_a. For a rational number r and an integral class $a \in H_2(M, \partial M; \mathbb{R})$, $\|ra\|$ is defined to be $\|ra\| = |r|\|a\|$. The norm $\|\cdot\|$ defined on rational classes admits a unique continuous extension to $H_2(M, \partial M; \mathbb{R})$ which is linear on the ray though the origin. The unit ball $U_M = \{a \in H_2(M, \partial M; \mathbb{R}) \mid \|a\| \leq 1\}$ is a compact, convex polyhedron.

Suppose that M is a surface bundles over the circle. We now recall Thurston's description of the relation between $\|\cdot\|$ and fibrations of M. Let Ω be a top dimensional face on ∂U_M. We denote the cone over Ω with the origin by C_Ω, and denote its interior by $int(C_\Omega)$. In [23], Thurston proved that if we let F be a fiber of a fibration of M, then there exists a top dimensional face Ω such that $[F]$ is an integral class of $int(C_\Omega)$. On the other hand, for *any* integral class $a \in int(C_\Omega)$, a minimal representative F_a becomes a fiber of the fibration associated to a. For this reason, such a face Ω is called a *fibered face* and an integral class $a \in int(C_\Omega)$ is called a *fibered class*. This property tells us that if M is a hyperbolic 3-manifold which is a surface bundles over the circle having the second Betti number more than 1, then it admits an infinite family of fibrations.

If a fibered class $a \in int(C_\Omega)$ is primitive, then the fibration associated to a has a connected fiber represented by F_a. Since M is hyperbolic, the mapping class $\phi_a = [\Phi_a]$ of the monodromy $\Phi_a : F_a \to F_a$ is pseudo-Anosov. The *dilatation* $\lambda(a)$ and *entropy* $\mathrm{ent}(a) = \log \lambda(a)$ are defined as the dilatation $\lambda(\phi_a)$ and entropy $\mathrm{ent}(\phi_a)$ of ϕ_a respectively.

We turn to the work of Fried, Matsumoto and McMullen. The entropy defined on primitive fibered classes is extended to rational classes as follows: For a rational number r and a primitive fibered class a, the entropy $\mathrm{ent}(ra)$ is defined by $\frac{1}{|r|}\mathrm{ent}(a)$. Let $int(C_\Omega(\mathbb{Q}))$ (resp. $int(C_\Omega(\mathbb{Z}))$) be the set of rational classes (resp. integral classes) in $int(C_\Omega)$. Fried proved that $\frac{1}{\mathrm{ent}} : int(C_\Omega(\mathbb{Q})) \to \mathbb{R}$ is concave [6], and in particular $\mathrm{ent} : int(C_\Omega(\mathbb{Q})) \to \mathbb{R}$ admits a unique continuous extension

$$\mathrm{ent} : int(C_\Omega) \to \mathbb{R}.$$

Moreover, Fried proved the following: The restriction of ent to the open fibered face $int(\Omega)$ has the property such that $\text{ent}(a)$ goes to ∞ as $a \in int(\Omega)$ goes to a point on $\partial\Omega$. Thus we have a continuous function

$$\text{Ent} = \|\cdot\| \, \text{ent}(\cdot) : int(C_\Omega) \to \mathbb{R}.$$

We call $\text{Ent}(a)$ the *normalized entropy* of $a \in int(C_\Omega)$. By definition of ent, we see that Ent is constant on each ray in $int(C_\Omega)$ through the origin. McMullen developed a theory of the *Teichmüller polynomial* P_Ω for a fibered face Ω of hyperbolic surface bundles over the circle, from which one can compute $\lambda(a)$ of each $a \in int(C_\Omega)$, see [21].

By Matsumoto [19] and by McMullen [21], it was proved that $\frac{1}{\text{ent}}$ on $int(\Omega)$ is strictly concave. This implies that ent is strictly convex on $int(\Omega)$ because ent is positive valued. Since $\|\cdot\|$ is constant ($= 1$) on a fibered face Ω, the normalized entropy Ent is strictly convex on $int(\Omega)$. Thus $\text{Ent}|_{int(\Omega)} : int(\Omega) \to \mathbb{R}$ has a minimum at a unique point in $int(\Omega)$. In other words, $\text{Ent} : int(C_\Omega) \to \mathbb{R}$ admits a minimum at a unique ray through the origin. We denote this minimum by $\min \text{Ent}(M, \Omega)$. We also denote by $\min \text{Ent}(M)$, $\min\limits_{\Omega}\{\min \text{Ent}(M, \Omega)\}$, where Ω is taken over all fibered faces for M.

2.2. **Properties of fibrations on the magic manifold.** In this section, we collect particular properties on N which are needed in the rest of the paper.

Let K_α, K_β and K_γ be the components of the 3 chain link $\mathcal{C}_3$. They bound the oriented disks F_α, F_β and F_γ with 2 holes. Let us set $\alpha = [F_\alpha]$, $\beta = [F_\beta]$, $\gamma = [F_\gamma] \in H_2(N, \partial N; \mathbb{Z})$. The Thurston (unit) ball U_N is the the parallelepiped with vertices $\pm\alpha$, $\pm\beta$, $\pm\gamma$, $\pm(\alpha + \beta + \gamma)$, see Figure 2(left). Every top dimensional face on ∂U_N is a fibered face by the symmetries of $H_2(N, \partial N)$. The set $\{\alpha, \beta, \gamma\}$ is a basis of $H_2(N, \partial N; \mathbb{Z})$, and $x\alpha + y\beta + z\gamma \in H_2(N, \partial N)$ is denoted by (x, y, z).

We denote by T_α, the torus which is the boundary of a regular neighborhood of K_α. We define the tori T_β and T_γ in the same manner. For a primitive integral class $a = (x, y, z) \in H_2(N, \partial N)$, let us set $\partial_\alpha F_a = \partial F_a \cap T_\alpha$ which consists of the parallel simple closed curves on T_α. We define $\partial_\beta F_a$ and $\partial_\gamma F_a$ in the same manner.

Pick a fibered face Δ on ∂U_N as in Figure 2(left) with vertices $(1, 0, 0)$, $(1, 1, 1)$, $(0, 1, 0)$ and $(0, 0, -1)$. The open face $int(\Delta)$ is written by

$$int(\Delta) = \{(x, y, z) \mid x + y - z = 1, \ x > 0, \ y > 0, \ x > z, \ y > z\}.$$

The Thurston norm of $(x, y, z) \in int(C_\Delta)$ is given by $x + y - z$.

Proposition 2.1 ([11]). *Let $a = (x, y, z)$ be a primitive fibered class in $int(C_\Delta)$.*

(1) *The number of the boundary components $\sharp(\partial F_a)$ of F_a is given by*

$$\sharp(\partial F_a) = \gcd(x, y + z) + \gcd(y, z + x) + \gcd(z, x + y),$$

where $\gcd(0, w)$ is defined by $|w|$. More precisely

$$\sharp(\partial_\alpha F_a) = \gcd(x, y + z), \ \sharp(\partial_\beta F_a) = \gcd(y, z + x), \ \sharp(\partial_\gamma F_a) = \gcd(z, x + y).$$

(2) *$\lambda(a) = \lambda_{(x,y,z)}$ equals the largest real root of*

$$f_{(x,y,z)}(t) = t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1,$$

where $f_{(x,y,z)}(t)$ is the specialization of the Teichmüler polynomial P_Δ at (x, y, z).

(3) *The inverse $\Phi^{-1}_{(x,y,z)}$ of $\Phi_{(x,y,z)} : F_{(x,y,z)} \to F_{(x,y,z)}$ is conjugate to the monodromy $\Phi_{(y,x,z)} : F_{(y,x,z)} \to F_{(y,x,z)}$ of the fibration on N associated to $(y, x, z) \in int(C_\Delta)$. In particular $\lambda_{(x,y,z)} = \lambda_{(y,x,z)}$.*

(4) $\min \mathrm{Ent}(N) = \min \mathrm{Ent}(N, \Delta) = \mathrm{Ent}((\frac{1}{2}, \frac{1}{2}, 0)) = 2\log(2+\sqrt{3}) \approx 2.6339$.

(5) *The stable foliation $\mathcal{F}_a$ of $\Phi_a : F_a \to F_a$ has the property such that each component of $\partial_\alpha F_a$, $\partial_\beta F_a$ and $\partial_\gamma F_a$ has $\frac{x}{\gcd(x,y+z)}$ prongs, $\frac{y}{\gcd(y,x+z)}$ prongs and $\frac{x+y-2z}{\gcd(z,x+y)}$ prongs respectively. Moreover $\mathcal{F}_a$ does not have singularities in the interior of F_a.*

(6) *$\mathcal{F}_a$ is orientable if and only if x and y are even and z is odd.*

We see that the slope of $\partial_\alpha F_a$ (resp. $\partial_\beta F_a$, $\partial_\gamma F_a$) is given by $b_\alpha(a) = \frac{y+z}{-x}$ (resp. $b_\beta(a) = \frac{z+x}{-y}$, $b_\gamma(a) = \frac{x+y}{-z}$). We call each of $b_\alpha(a)$, $b_\beta(a)$, $b_\gamma(a)$ the *boundary slope* of a.

By using the formula in Proposition 2.1, we recover the similar formula for *any* primitive fibered classes $a \in H_2(N, \partial N)$. This is because there is a homeomorphism $h : (S^3, \mathcal{C}_3) \to (S^3, \mathcal{C}_3)$ which sends K_α, K_β, K_γ to K_β, K_γ, K_α respectively, and $H_2(N, \partial N)$ has symmetries by the isomorphism $h_* : H_2(N, \partial N) \to H_2(N, \partial N)$ of order 3 induced from h.

It is known by [18] that $N(r)$ is hyperbolic if and only if $r \in \mathcal{H}yp = \mathbb{Q} \setminus \{-3, -2, -1, 0\}$. We now recall the description of fibered classes of the hyperbolic Dehn filling $N(r)$'s. Let $N(r)$ be the manifold obtained from N by Dehn filling the cusp specified by, say T_β, along the slope $r \in \mathbb{Q}$ or $r = \frac{1}{0} (= \infty)$. Then, there exists a natural injection

$$\iota_\beta : H_2(N(r), \partial N(r)) \to H_2(N, \partial N) \tag{1}$$

whose image equals the linear section $S_\beta(r)$, where

$$S_\beta(r) = \{(x, y, z) \in H_2(N, \partial N) \mid -ry = z + x\},$$

see [11, Proposition 2.11]. Choose $r \in \mathcal{H}yp$, and assume that $a \in S_\beta(r) = \mathrm{Im}\, \iota_\beta$ is a fibered class in $H_2(N, \partial N)$. Then, $\overline{a} = \iota_\beta^{-1}(a) \in H_2(N(r), \partial N(r))$ is also a fibered class of $N(r)$. We sometimes denote $N(r)$ by $N_\beta(r)$ when we need to specify the cusp which is filled.

Similarly, when $N(r)$ is the manifold obtained from N by Dehn filling the cusp specified by T_α or T_γ along the slope r, one has natural injections,

$$\iota_\alpha : H_2(N(r), \partial N(r)) \to H_2(N, \partial N),$$
$$\iota_\gamma : H_2(N(r), \partial N(r)) \to H_2(N, \partial N)$$

such that their images are

$$S_\alpha(r) = \{(x, y, z) \in H_2(N, \partial N) \mid -rx = y + z\},$$
$$S_\gamma(r) = \{(x, y, z) \in H_2(N, \partial N) \mid -rz = x + y\}.$$

We may denote by $N_\alpha(r)$ or $N_\gamma(r)$, the manifold $N(r)$ in this case. This description enables us to compute the Thurston norm of $N(r)$, especially the Thurston unit ball and fibered faces. For more detailed computation, see [11]. Figure 3 illustrates the intersection of the Thurston norm ball U_N and the linear section $S_*(r)$, $* \in \{\alpha, \beta, \gamma\}$.

Remark 2.2 (Lemmas 3.28 and 5.2 in [11])**.** Take $r \in \mathcal{H}yp$, and let $\overline{a} \in H_2(N(r), \partial N(r))$ be a primitive integral class. If $r \neq 1$, then $\sharp(\partial F_{\overline{a}})$ is bounded by a constant from above which depends on r. On the other hand, in the case $r = 1$, the genus of $F_{\overline{a}}$ is always equal to 1, and hence there exists no upper bound of $\sharp(\partial F_{\overline{a}})$.

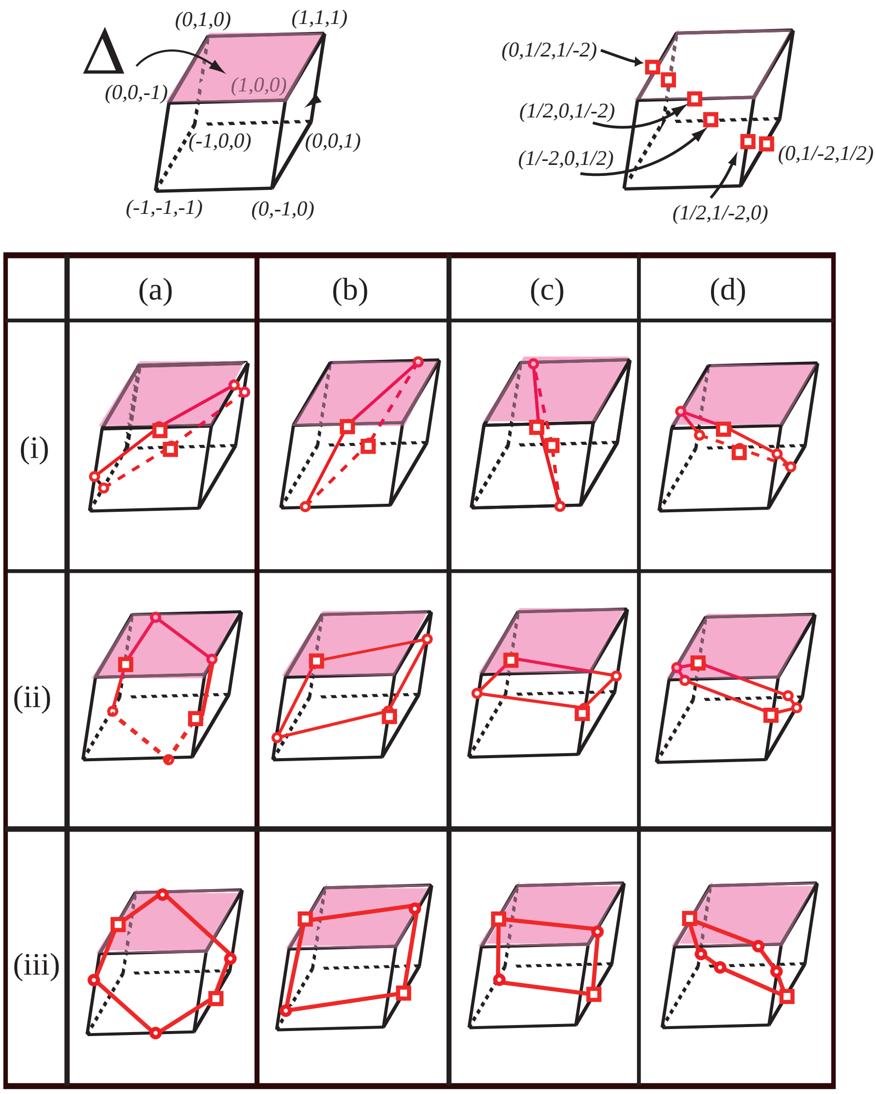

FIGURE 3. 1st row (i) $U_N \cap S_\beta(r)$, 2nd row (ii) $U_N \cap S_\gamma(r)$ and 3rd row (iii) $U_N \cap S_\alpha(r)$. [(a) $r \in (-\infty, -2)$, (b) $r \in (-2, -1)$, (c) $r \in (-1, 0)$, (d) $r \in (0, \infty)$.] [the fibered face Δ is shaded in the figure.]

2.3. **Entropy equivalence on the manifolds $N(r)$'s.** The notation "*entropy equivalence*" on fibered 3-manifolds was introduced in [11]. By using this equivalence relation, we will see in Theorem 2.3 that there are infinitely many entropy equivalent pairs among $N(r)$'s. The particular pair is $N(\frac{3}{-2})$ and $N(\frac{1}{-2})$. They are not homeomorphic to each other, but they have common properties on the normalized entropy.

We say that 3-manifolds M and M' are *Thurston norm equivalent*, denoted by $M \underset{\mathrm{T}}{\sim} M'$, if there exists an isomorphism $f : H_2(M, \partial M; \mathbb{Z}) \to H_2(M', \partial M'; \mathbb{Z})$ which preserves the Thurston norm, i.e, $\|a\| = \|f(a)\|$ for any $a \in H_2(M, \partial M; \mathbb{Z})$. We call such f the *Thurston norm preserving isomorphism*.

Let (M, Ω) and (M', Ω') be pairs of 3-manifolds M, M' and their fibered faces Ω, Ω' respectively. Possibly $M \simeq M'$. Then (M, Ω) and (M', Ω') are *entropy equivalent*, denoted by $(M,\Omega) \underset{\text{ent}}{\sim} (M',\Omega')$, if there exists a Thurston norm preserving isomorphism $f: H_2(M,\partial M;\mathbb{Z}) \to H_2(M',\partial M';\mathbb{Z})$ satisfying the following.

- $a \in int(C_\Omega(\mathbb{Z}))$ if and only if $f(a) \in int(C_{\Omega'}(\mathbb{Z}))$.
- $\mathrm{ent}(a) = \mathrm{ent}(f(a))$ for any $a \in int(C_\Omega(\mathbb{Z}))$.

The second bullet implies that $\mathrm{ent}(a) = \mathrm{ent}(f(a))$ for any $a \in int(C_\Omega)$ since $\mathrm{ent} : int(C_\Omega(\mathbb{Q})) \to \mathbb{R}$ admits a unique continuous extension. Thus if $(M,\Omega) \underset{\text{ent}}{\sim} (M',\Omega')$, then $\min \mathrm{Ent}(M,\Omega) = \min \mathrm{Ent}(M',\Omega')$.

Fibered 3-manifolds M and M' are *entropy equivalent*, denoted by $M \underset{\text{ent}}{\sim} M'$, if there exists a Thurston norm preserving isomorphism $f: H_2(M,\partial M;\mathbb{Z}) \to H_2(M',\partial M';\mathbb{Z})$ satisfying the following.

- $a \in H_2(M,\partial M;\mathbb{Z})$ is a fibered class if and only if $f(a) \in H_2(M',\partial M';\mathbb{Z})$ is a fibered class.
- Given a fibered face Ω of M, we have $\mathrm{ent}(a) = \mathrm{ent}(f(a))$ for any $a \in int(C_\Omega(\mathbb{Z}))$.

If $M \underset{\text{ent}}{\sim} M'$, then $\min \mathrm{Ent}(M) = \min \mathrm{Ent}(M')$.

We turn to the manifolds $N(r)$'s. Let $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ be coprime such that $r = \frac{p}{q} \in \mathcal{H}yp$. Then $N(r)$ has two kinds of fibered faces, *A-face* and *S-face*, see [11, Section 2.5]. When $r \in (-2,0)$, the Thurston norm ball of $N(r)$ is a parallelogram and every fibered face is an A-face. When $r \in (-\infty,-2) \cup (0,\infty)$ such that $|q| \neq 1$ (resp. $|q| = 1$), the Thurston norm ball for $N(r)$ is a hexagon (resp. rectangle) having two S-faces and four A-faces (resp. having two S-faces and two A-faces). cf. Figure 3. One can show that any two S-faces of $N(r)$ are entropy equivalent, and any two A-faces of $N(r)$ are entropy equivalent [11, Lemma 2.22]. In the case $r = 1$, by the symmetry of the Whitehead link exterior $N(1)$ itself, one can see that an S-face of $N(1)$ and an A-face of $N(1)$ are entropy equivalent [11, Proposition 3.26]. Moreover the fibered class $\overline{(1,1,-2)} \in H_2(N_\gamma(1),\partial N_\gamma(1))$ achieves $\min \mathrm{Ent}(N(1))$ [11, Corollary 3.27];

$$\min \mathrm{Ent}(N(1)) = \mathrm{Ent}(\overline{(1,1,-2)}) = 2\log \delta(D_4) \approx 1.6628.$$

An S-face of $N(r)$ may not be entropy equivalent to an A-face of $N(r)$ for other r.

Theorem 2.3 (Theorem 2.26 in [11]). *Let $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ be as above.*

1. *Suppose that $\frac{p}{q} \in (-\infty,-2)$ and $p+2q \neq 1$. Then $(N(\frac{p}{q}),\Omega_S) \underset{\text{ent}}{\sim} (N(\frac{2q+p}{-q}),\Omega_S)$.*
2. *Suppose that $\frac{p}{q} \in (-\infty,-1)$ and $|q| \neq 1$. Then $(N(\frac{p}{q}),\Omega_A) \underset{\text{ent}}{\sim} (N(\frac{-2q-p}{q}),\Omega_A)$.*
3. *Suppose that $\frac{p}{q} \in (-\infty,-1)$, $p+2q \neq 1$ and $|q| \neq 1$. Then $N(\frac{p}{q}) \underset{\text{ent}}{\sim} N(\frac{-2q-p}{q})$.*

In Proposition 2.4, we will see that the entropy function on N has symmetries. This property is a key for the proof of Theorem 2.3. By Theorem 2.3,

$$(N(-6)),\Omega_S) \underset{\text{ent}}{\sim} (N(4),\Omega_S) \text{ and } N(\tfrac{3}{-2}) \underset{\text{ent}}{\sim} N(\tfrac{1}{-2}).$$

Table 2 exhibits the computation of $\min \mathrm{Ent}$ for these manifolds. Readers may notice that we encountered these numbers $\min \mathrm{Ent}$ in the upper bounds of Table 1(2nd column). It turns out that the both $\min \mathrm{Ent}(N(r),\Omega_A)$ for $r = \frac{3}{-2}, \frac{1}{-2}$ and $\min \mathrm{Ent}(N(r),\Omega_S)$ for $r = -6, 4$ are achieved by fibered classes for $N(r)$, see Table 2. The topological types of the fibers are also shown in the table. (e.g. $\overline{a} = \overline{(3,3,1)} \in H_2(N_\gamma(-6),\partial N_\gamma(-6))$ achieves $\min \mathrm{Ent}(N(-6),\Omega_S)$, and $F_{\overline{a}} \simeq \Sigma_{2,2}$.)

TABLE 2. min Ent for some $N(r)$'s. [note: the technique in [11] does not work for the computation of $\min \mathrm{Ent}(N(r), \Omega_A)$ in the case $r = -6, 4$.]

$N(r)$	$\min \mathrm{Ent}(N(r), \Omega_S)$ fibered class, fiber	$\min \mathrm{Ent}(N(r), \Omega_A)$ fibered class, fiber	$\min \mathrm{Ent}(N(r))$
$N(\frac{3}{-2})$	none	$2\log(\frac{3+\sqrt{5}}{2})$ $(2,2,1),\ \Sigma_{1,2}$	$2\log(\frac{3+\sqrt{5}}{2}) \approx 1.9248$
$N(\frac{1}{-2})$	none	$2\log(\frac{3+\sqrt{5}}{2})$ $(1,2,0),\ \Sigma_{0,4}$	$2\log(\frac{3+\sqrt{5}}{2}) \approx 1.9248$
$N(-6)$	$4\log\delta(D_5)$ $(3,3,1),\ \Sigma_{2,2}$	?	$\leq 4\log\delta(D_5) \approx 2.1740$
$N(4)$	$4\log\delta(D_5)$ $(2,2,-1),\ \Sigma_{2,2}$	?	$\leq 4\log\delta(D_5) \approx 2.1740$

2.4. **Mysterious symmetries of entropy function on the magic manifold.** The entropy function on N has mysterious symmetries not coming from the symmetries of N itself, which we will recall below.

We take $(x,y,z) \in \Delta$. (Hence $x+y-z=1$.) Let us denote (x,y,z) by $[x,y]$. Then the open face $int(\Delta)$ is written by

$$int(\Delta) = \{[x,y] \mid 0 < x < 1,\ 0 < y < 1\}.$$

On the other hand if $(x,y,z) \in int(C_\Delta)$, then

$$(y-z, y, y-x), (y-z, x-z, -z), (x, x-z, x-y) \in int(C_\Delta).$$

These four classes have the same Thurston norm. Intriguingly, they have the same dilatation!

Proposition 2.4 (Lemma 2.5 in [11]). *The four classes*

$$(x,y,z), (y-z,y,y-x), (y-z,x-z,-z), (x,x-z,x-y) \in int(C_\Delta)$$

have the same dilatation. In particular,

$$[\tfrac{x}{x+y-z}, \tfrac{y}{x+y-z}], [\tfrac{y-z}{x+y-z}, \tfrac{y}{x+y-z}], [\tfrac{y-z}{x+y-z}, \tfrac{x-z}{x+y-z}], [\tfrac{x}{x+y-z}, \tfrac{x-z}{x+y-z}] \in int(\Delta)$$

have the same dilatation. (See Figure 4(left).)

We note that the topological types of $F_{(x,y,z)}$, $F_{(y-z,y,y-x)}$, $F_{(y-z,x-z,-z)}$, $F_{(x,x-z,x-y)}$ may be different. (e.g. $F_{(6,5,4)} \simeq \Sigma_{0,9}$, $F_{(1,5,-1)} \simeq \Sigma_{1,7}$, $F_{(1,2,-4)} \simeq \Sigma_{3,3}$ and $F_{(6,2,1)} \simeq \Sigma_{2,5}$.) On the other hand by Proposition 2.1(3), any two classes $a = [x,y], \widetilde{a} \in [y,x] \in int(\Delta)$ have the same dilatation. This together with Proposition 2.4 says that 8 classes $b_0, \widetilde{b_0}, \cdots, b_3, \widetilde{b_3} \in int(\Delta)$ as in Figure 4(right) have the same dilatation.

3. Asymptotic behaviors of minimal dilatations

3.1. **Sequence $\{\delta_g\}_{g\geq 2}$.** Let $\Phi : F \to F$ be the monodromy of a fibration on N, and let $\phi = [\Phi]$. Then the fibration extends naturally to a fibration on the closed manifold obtained from N by Dehn filling three cusps along boundary slopes of F. Also, Φ extends to the monodromy $\widehat{\Phi} : \widehat{F} \to \widehat{F}$ of the extended fibration, where the extended fiber $\widehat{F}$ is obtained from F by filing holes. Suppose that the stable foliation $\mathcal{F}$ of Φ has the property

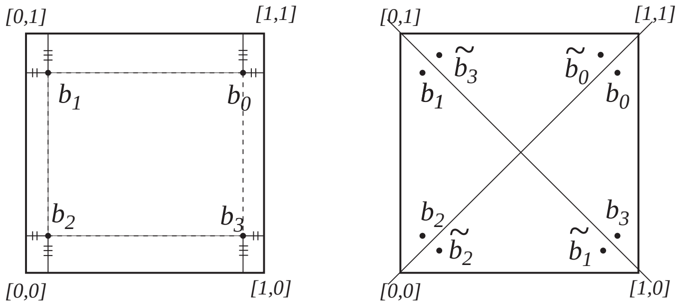

FIGURE 4. $b_0 = [\frac{x}{x+y-z}, \frac{y}{x+y-z}]$, $b_1 = [\frac{y-z}{x+y-z}, \frac{y}{x+y-z}]$, $b_2 = [\frac{y-z}{x+y-z}, \frac{x-z}{x+y-z}]$, $b_3 = [\frac{x}{x+y-z}, \frac{x-z}{x+y-z}] \in int(\Delta)$ and $\widetilde{b_i} \in int(\Delta)$.

such that any boundary component of F has no 1 prong. Then $\mathcal{F}$ extends canonically to the stable foliation $\widehat{\mathcal{F}}$ of $\widehat{\Phi}$, and $\widehat{\phi} = [\widehat{\Phi}]$ becomes pseudo-Anosov (including Anosov) with the same dilatation as that of ϕ. We consider the set $\mathcal{M}$ of (pseudo-Anosov) mapping classes coming from fibrations of N with this condition.

Now, let us denote by $\widehat{\mathcal{M}}$, the set of extensions $\widehat{\phi}$ of $\phi \in \mathcal{M}$ defined on the closed surfaces. Let $\widehat{\delta}_g$ be the minimum among dilatations of elements in $\widehat{\mathcal{M}} \cap \mathrm{Mod}(\Sigma_g)$. Clearly $\delta_g \leq \widehat{\delta}_g$. The equality holds when $g = 2$. (In fact δ_2 is achieved by $\widehat{\phi}_a \in \widehat{\mathcal{M}} \cap \mathrm{Mod}(\Sigma_2)$ when $a = (2, 2, -1)$ or $(2, 6, 1)$.)

The set $\mathcal{M}$ is large in the following sense. For any $r \in \mathcal{H}yp \setminus \{1\}$, there exist infinitely many primitive fibered classes $a_n = a_n(r) \in S_\beta(r)$ such that $\phi_{a_n} \in \mathcal{M}$ and the genus of F_{a_n} goes to ∞ as n goes to ∞. In [11], we addressed Question 1.2(4) (about the asymptotic behavior of $g \log \delta_g$) in $\widehat{\mathcal{M}}$.

Theorem 3.1 (Theorem 1.4 in [11]).

(1) *We have* $\lim_{g \to \infty} g \log \widehat{\delta}_g = \log(\frac{3+\sqrt{5}}{2})$.

(2) *For large g, $\widehat{\delta}_g$ is achieved by the monodromy of some Σ_g-bundle over the circle obtained from either $N(\frac{3}{-2})$ or $N(\frac{1}{-2})$ by Dehn filling both cusps.*

More precisely, one can show the following: For large g such that $g \equiv 0, 1, 5, 6, 7, 9 \pmod{10}$ (resp. such that $g \equiv 3, 8 \pmod{10}$), $\widehat{\delta}_g$ is achieved by the monodromy of some Σ_g-bundle over the circle obtained from $N(\frac{3}{-2})$ (resp. $N(\frac{1}{-2})$) by Dehn filling both cusps, see [11, Remark 3.18].

Table 3 shows the fibered class $(x, y, z) \in H_2(N, \partial N)$ which achieves $\widehat{\delta}_g$ for large g and the polynomial $f_{(x,y,z)}(t)$. Notice that such a fibered class (x, y, z) is in either $int(C_\Delta) \cap S_\beta(\frac{3}{-2})$ or $int(C_\Delta) \cap S_\beta(\frac{1}{-2})$, see (1) in Section 2.2. Its projective class $(x', y', z') \in int(\Delta)$ goes to the projective class of either $(2, 2, 1)$ or $(1, 2, 0)$ as the Thurston norm $\|(x, y, z)\|$ goes to ∞, see Figure 2(1).

For small g, our upper bound of δ_g is given by the brute computation, see Table 4. We note that in the case $g = 8, 13$, $\widehat{\delta}_g$ is not achieved by the monodromy of any Σ_g-bundle over the circle obtained from either $N(\frac{3}{-2})$ or $N(\frac{1}{-2})$ by Dehn filling [13, Proposition 4.37].

We describe the outline of the proof of Theorem 3.1(1). It is known that $N(-4) \simeq N(\frac{3}{-2})$, see [18]. We recall:

Claim 3.2 (Theorem 1.5 in [13]). *Let $r \in \{\frac{3}{-2}, \frac{1}{-2}, 2\}$. For each $g \geq 3$, there exist Σ_g-bundles over the circle obtained from $N(r)$ by Dehn filling both cusps along boundary slopes of fibers of $N(r)$. Among them, there exist monodromies $\Phi_g(r) : \Sigma_g \to \Sigma_g$ of the fibrations such that*

$$\lim_{g\to\infty} g \log \lambda(\Phi_g(r)) = \log(\tfrac{3+\sqrt{5}}{2}).$$

Let a_g be a primitive fibered class of $H_2(N, \partial N)$ such that $\phi_{a_g} \in \mathcal{M}$ and $\widehat{\delta}_g$ is achieved by $\widehat{\phi}_{a_g} \in \widehat{\mathcal{M}} \cap \mathrm{Mod}(\Sigma_g)$. Since $N(1)$ has no fiber of genus greater than 1, a_g does not have a boundary slope 1 for $g \geq 2$. By the analysis of $\mathrm{minEnt}(N(r), \Omega)$ (see [11, Theorem 1.11]), one can show that the set of normalized entropies of monodromies of the fibrations on the closed manifolds, obtained from N by Dehn filling all cusps along the slopes not in $\{-4, \frac{3}{-2}, \frac{1}{-2}, 2\}$, have no accumulation values $\leq 2\log(\frac{3+\sqrt{5}}{2})$. By using Claim 3.2, one can see that a_g has to have a boundary slope in $\{-4, \frac{3}{-2}, \frac{1}{-2}, 2\}$ eventually. Moreover the set of normalized entropies of the monodromies of the fibrations on the closed manifolds obtained from N by Dehn filling all cusps along the slopes, one of which is in $\{-4, \frac{3}{-2}, \frac{1}{-2}, 2\}$, have no accumulation values $< 2\log(\frac{3+\sqrt{5}}{2})$. Then Claim 3.2 leads to Theorem 3.1(1).

3.2. **Sequence $\{\delta_g^+\}_{g\geq 2}$.** Let $\widehat{\mathcal{M}}^+$ be the set of pseudo-Anosov elements of $\widehat{\mathcal{M}}$ with orientable invariant foliations. (One can use Proposition 2.1(6) to know whether $\widehat{\phi}_a \in \widehat{M}$ has orientable invariant foliations or not.) Let $\widehat{\delta}_g^+$ be the minimum among dilatations of elements in $\widehat{\mathcal{M}}^+ \cap \mathrm{Mod}(\Sigma_g)$. (Since $\widehat{\mathcal{M}}^+ \cap \mathrm{Mod}(\Sigma_g) \neq \emptyset$ for $g \geq 2$, $\widehat{\delta}_g^+$ is well-defined.) Clearly $\delta_g \leq \delta_g^+ \leq \widehat{\delta}_g^+$. The equality $\delta_g^+ = \widehat{\delta}_g^+$ holds for all $2 \leq g \leq 8$ except for $g = 6$, see Table 5.

Theorem 3.3 (Theorem 1.5 in [11]).

(1) *We have* $\displaystyle\lim_{\substack{g\not\equiv 0 \pmod 6\\ g\to\infty}} g \log \widehat{\delta}_g^+ = \log(\tfrac{3+\sqrt{5}}{2})$.

(2) *For large g such that $g \equiv 2, 4 \pmod 6$ or $g \equiv 3 \pmod{10}$ (resp. such that $g \equiv 1, 5, 7, 9 \pmod{10}$), $\widehat{\delta}_g^+$ is achieved by the monodromy of some Σ_g-bundle over the circle obtained from $N(\frac{1}{-2})$ (resp. $N(\frac{3}{-2})$) by Dehn filling both cusps.*

Table 6 shows the fibered class $(x, y, z) \in H_2(N, \partial N)$ which achieves $\widehat{\delta}_g^+$ for large $g \not\equiv 0 \pmod 6$ and the polynomial $f_{(x,y,z)}(t)$.

The proof of Theorem 3.3(1) is similar to that of Theorem 3.1(1). The difference is that in the case $g \equiv 0 \pmod 6$, there exist no examples of elements in $\widehat{\mathcal{M}}^+$ defined on Σ_g which occur as monodromies of fibrations on manifolds obtained from $N(\frac{1}{-2})$ or $N(\frac{3}{-2})$ by Dehn filling both cusps. This is the reason why we need the condition $g \not\equiv 0 \pmod 6$.

If we fix any $\epsilon > 0$ so that $1.97475 - \epsilon > 2\log(\frac{3+\sqrt{5}}{2})$, then for large g such that $g \equiv 0 \pmod 6$, we have

$$|\chi(\Sigma_g)| \log \widehat{\delta}_g^+ > 1.97475 - \epsilon > 2\log(\tfrac{3+\sqrt{5}}{2}),$$

see [11, Theorem 1.10].

The emphasis is that in the case $g \equiv 6 \pmod{12}$, elements of $\widehat{\mathcal{M}}^+$ provide a new family of pseudo-Anosovs defined on Σ_g with orientable invariant foliations obtained from $N(-6)$

or $N(4)$ by Dehn filling both cusps. By using the examples, we obtained the following bounds in [11, Theorem 1.7].

Theorem 3.4 (Upper bound on δ_g^+ for $g \equiv 6 \pmod{12}$)**.**

(1) $\delta_g^+ \leq \lambda_{(\frac{3g}{2}+1, \frac{3g}{2}-1, \frac{g}{2})}$ *if* $g \equiv 6, 30, 42, 54, 78 \pmod{84}$. *The specialization of the Teichmüler polynomial* P_Δ *at* $(\frac{3g}{2}+1, \frac{3g}{2}-1, \frac{g}{2}) \in S_\gamma(-6)$ *is*

$$f_{(\frac{3g}{2}+1, \frac{3g}{2}-1, \frac{g}{2})}(t) = (t^{\left(\frac{g}{2}\right)}+1)(t^{2g} - t^{\left(\frac{3g}{2}\right)} - t^{g+1} + t^g - t^{g-1} - t^{\left(\frac{g}{2}\right)} + 1).$$

(2) $\delta_g^+ \leq \lambda_{(g+2, g-2, -\frac{g}{2})}$ *if* $g \equiv 18, 66 \pmod{84}$. *The specialization of the Teichmüler polynomial* P_Δ *at* $(g+2, g-2, -\frac{g}{2}) \in S_\gamma(4)$ *is*

$$f_{(g+2, g-2, -\frac{g}{2})}(t) = (t^{\left(\frac{g}{2}\right)}+1)(t^{2g} - t^{\left(\frac{3g}{2}\right)} - t^{g+2} + t^g - t^{g-2} - t^{\left(\frac{g}{2}\right)} + 1).$$

The upper bound $\limsup_{\substack{g \equiv 6 \pmod{12} \\ g \to \infty}} g \log \delta_g^+ \leq 2 \log \delta(D_5)$ holds, since the ray of

$$\overline{(\tfrac{3g}{2}+1, \tfrac{3g}{2}-1, \tfrac{g}{2})} \in H_2(N_\gamma(-6), \partial N_\gamma(-6)) \text{ (resp. } \overline{(g+2, g-2, -\tfrac{g}{2})} \in H_2(N_\gamma(4), \partial N_\gamma(4)))$$

converges to the ray of $\overline{(3,3,1)}$ (resp. $\overline{(2,2,-1)}$) as g goes to ∞ which achieves

$$\min \mathrm{Ent}(N(-6), \Omega_S) \text{ (resp. } \min \mathrm{Ent}(N(4), \Omega_S)).$$

In particular the projective class of $(\frac{3g}{2}+1, \frac{3g}{2}-1, \frac{g}{2})$ (resp. $(g+2, g-2, -\frac{g}{2})$) lies on $int(\Delta) \cap S_\beta(-6)$ (resp. $int(\Delta) \cap S_\beta(4)$) and it converges to the projective class of $(3,3,1)$ (resp. $(2,2,1)$) as g goes to ∞, see Figure 2(2).

Table 1 in [11] exhibits upper bounds of δ_g^+ for small g such that $g \equiv 0 \pmod 6$ which improves the bound given in [20, 10].

3.3. **Sequences $\{\delta_{0,n}\}_{n \geq 4}$ and $\{\delta(D_n)\}_{n \geq 3}$.** The mapping class group $\mathrm{Mod}(D_n)$ on an n-punctured disk D_n is isomorphic to the subgroup of $\mathrm{Mod}(\Sigma_{0,n+1})$ consisting of the elements which fix a puncture of $\Sigma_{0,n+1}$. (Hence $\delta(D_n) \geq \delta_{0,n+1}$.) By using the usual isomorphism $\Gamma : B_n \to \mathrm{Mod}(D_n)$ from the n-braid group B_n to $\mathrm{Mod}(D_n)$, one represents each element of $\mathrm{Mod}(D_n)$ by an n-braid.

Let $\mathcal{N}_n$ be the set of primitive fibered classes $a \in H_2(N, \partial N)$ such that $F_a \simeq \Sigma_{0,n}$. In [12], we ask about which fibered class in $\mathcal{N}_n$ achieves the minimal dilatation. To give a statement more precisely, let us define an m-braid $T_{m,p}$ for $p \geq 1$ as follows.

$$T_{m,p} = (\sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-1})^p \sigma_{m-1}^{-2} = (\sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-1})^{p-1} \sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-2} \sigma_{m-2}^{-1}.$$

If one forgets the 1st strand of $T_{m,p}$, one obtains the $(m-1)$-braid, call it $T'_{m,p}$. Observe that $\lambda(T'_{m,p}) \leq \lambda(T_{m,p})$ if $T'_{m,p}$ is pseudo-Anosov. It was shown that the mapping torus $\mathbb{T}(\Gamma(T_{m,p}))$ is homeomorphic to N if $\gcd(m-1, p) = 1$ [12, Corollary 3.2]. Otherwise $\mathbb{T}(\Gamma(T_{m,p}))$ is toroidal, i.e, $\Gamma(T_{m,p})$ is reducible [12, Lemma 3.11]. Table 7 describes our result in [12, Theorem 1.1] which answers the above question. For $n \geq 9$, the fibered class $s_n = (x, y, z)$ which achieves the minimal dilatation in $\mathcal{N}_n$ and its mapping class ϕ_{s_n} are given in the table. (The statement in the case $4 \leq n \leq 8$ can be found in [12, Theorem 1.1].) Here, we have a remark on the same table(4th column). By Proposition 2.1(1), $\sharp(\partial_\alpha F_{s_n}) = 1$ holds. (Also $\sharp(\partial_\beta F_{s_n}) = 1$.) Hence the monodromy $\Phi_{s_n} : F_{s_n}(\simeq \Sigma_{0,n}) \to F_{s_n}$ of the fibration associated to s_n on N is described by an element in $\mathrm{Mod}(D_{n-1})$, and hence by an $(n-1)$-braid. (In this case it turns out that the braid is given by $T_{n-1,p}$ for some p.)

We denote by $T_{(n-1)}$, the braid $T_{n-1,*}$ in Table 7(4th column) which represents ϕ_{s_n} for the fibered class s_n. For example, when $n = 2k+1$, $T_{(2k)} = T_{2k,2}$. The stable foliation $\mathcal{F}_{s_n}$ has the property such that the boundary component of F_{s_n} which lies on the torus T_α has $x(\neq 1)$ prong, see Proposition 2.1(5). This implies that $T'_{(n-1)} \in B_{n-2}$ is pseudo-Anosov and $\lambda(T'_{(n-1)}) = \lambda(T_{(n-1)})$. One can use both $(n-2)$-braids $T_{(n-2)}$ and $T'_{(n-1)}$ for upper bounds of $\delta(D_{n-2})$, see Table 8(5th column). We would like to point out that $T'_{(2k)} = T'_{2k,2} \in B_{2k-1}$ is conjugate to the braid called $\sigma_{k-2,k}$ in [10]. For small n, our upper bound of $\delta(D_{n-2})$ is given in Table 9.

The minimal dilatation $\delta(D_n)$ is determined for all $3 \leq n \leq 8$ [7, 8, 15, 17]. In these cases, the minimizers "come from" N. More precisely, the minimal representative $F_{(x,y,z)}$ of the fibered class $(x,y,z) \in H_2(N, \partial N)$ in Table 10 is homeomorphic to $\Sigma_{0,n+2}$. It turns out that the mapping class $\phi_{(x,y,z)}$ is of the form $T_{n+1,p}$ for some p. Except for $n = 6$, the braid $T'_{n+1,p} \in B_n$ in Table 10(6th column) achieves the minimal dilatation $\delta(D_n)$. In the case $n = 6$, the braid $T_{6,2}$ achieves the minimal dilatation $\delta(D_6)$.

Observe that $s_n \in int(C_\Delta) \cap S_\gamma(\infty)$ and the ray of s_n converges to the ray of $[\frac{1}{2}, \frac{1}{2}] = (\frac{1}{2}, \frac{1}{2}, 0) \in int(\Delta)$ as n goes to ∞, see Figure 2(3). By Proposition 2.1(4), we obtain

$$\limsup_{n\to\infty} n \log \delta(D_n),\ \limsup_{n\to\infty} n \log \delta_{0,n} \leq \min \mathrm{Ent}(N) = 2\log(2+\sqrt{3}).$$

3.4. **Sequence** $\{\delta_{1,n}\}_{n\geq 1}$**.** Let $\mathcal{W}_n \subset H_2(N(1), \partial N(1))$ be the set of primitive fibered classes whose minimal representatives are homeomorphic to $\Sigma_{1,n}$, see Remark 2.2. In Table 11, one can find the fibered class $\overline{w_n} = \overline{(x,y,z)} \in H_2(N_\gamma(1), \partial N_\gamma(1))$ which achieves the minimal dilatation in $\mathcal{W}_n$, see [11, Proposition 3.30]. The dilatation of $w_n \in H_2(N, \partial N)$ is equal to the dilatation of $\overline{w_n}$, since $\mathcal{F}_{w_n}$ has the property such that the boundary components of F_{w_n} which lie on T_γ has 3 prong, see Proposition 2.1(5). Thus we have

$$\delta_{1,n} \leq \lambda(\overline{w_n}) = \lambda(w_n) = \lambda_{(x,y,z)}.$$

For the polynomial $f_{(x,y,z)}(t)$ in this case, see Table 11(3rd column).

The ray of $\overline{w_n} \in H_2(N_\gamma(1), \partial N_\gamma(1))$ converges to the ray of $\overline{(1,1,-2)}$ as n goes to ∞ which achieves $\min \mathrm{Ent}(N(1))$, see Figure 2(4). Thus

$$\limsup_{n\to\infty}\ n \log \delta_{1,n} \leq \min \mathrm{Ent}(N(1)) = 2\log \delta(D_4).$$

Table 12 shows our upper bound of $\delta_{1,n}$ for small n due to the brute computation. It turns out that this coincides with the upper bound given by Table 11.

3.5. $g > 1$**, Sequence** $\{\delta_{g,n}\}_{n\geq 1}$**.** So far, for the upper bounds of normalized entropies of pseudo-Anosovs, we used the following property of hyperbolic surface bundles over the circle M: Let Ω be a fibered face of M and let $\mathcal{D} \subset int(\Omega)$ be any compact set. Then there exists a constant $c = c_{\mathcal{D}} > 0$ such that for any fibered class $a \in int(C_\Omega)$, we have $\mathrm{Ent}(a) = \mathrm{Ent}(\Phi_a) \leq c$ whenever the projective class a' of a is in the compact set $\mathcal{D}$. However for any fixed $g \geq 2$, the same technique doesn't work in order to give an upper bound of $\delta_{g,n}$ varying n because of Tsai's result $\log \delta_{g,n} \asymp \frac{\log n}{n}$. Her result implies that if there exists a sequence of primitive fibered class classes $\{a_i\}$ with $a_i = a_{g,i} \in int(C_\Omega)$ such that the fiber of the fibration associated to a_i is a surface of genus g and n_i boundary components with $n_i \to \infty$, then accumulation points of the sequence of projective classes $\{a'_i\}$ must lie on the boundary of Ω.

In [14], we found such a sequence $\{a_i\} = \{a_{g,i}\}$ of the primitive fibered class $a_i \in int(C_\Delta) \cap S_\beta(-1)$ of N for each $g \geq 2$ with the best possible asymptotic behavior, i.e, $\log \lambda(a_i) = \log \lambda(\Phi_{a_i}) \asymp \frac{\log \|a_i\|}{\|a_i\|}$. These examples have the property such that the projective class a'_i goes to a particular point $(\frac{1}{2}, 1, \frac{1}{2}) \in \partial\Delta$ as i goes to ∞, see Figure 2(5). By using the sequence $\{a_i\}$, we proved the following.

Theorem 3.5 ([14]). *Given $g \geq 2$, there exists a sequence $\{n_i\}_{i=0}^{\infty}$ with $n_i \to \infty$ such that $\limsup\limits_{i\to\infty} \frac{n_i \log \delta_{g,n_i}}{\log n_i} \leq 2$. Furthermore, if $g \geq 2$ enjoys*

$(*)\quad \gcd(2g+1, s) = 1$ *or* $\gcd(2g+1, s+1) = 1$ *for each* $0 \leq s \leq g$,

then

$$\limsup_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2. \tag{2}$$

For example, $(*)$ holds for $g = 4$ since 9 is relatively prime to $1, 2, 4$ and 5, but $(*)$ does not hold for $g = 7$ because $\gcd(15, 5) = 5$ and $\gcd(15, 6) = 3$. Observe that g enjoys $(*)$ if $2g + 1$ is prime. (Hence infinitely many g's satisfy $(*)$.)

The inequality (2) in Theorem 3.5 improves the upper bound $\limsup\limits_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2(2g+1)$ (see [14]) obtained from Tsai's examples. Note that this upper bound holds for any $g \geq 2$.

4. Questions and conjectures

We close with some questions and conjectures about pseudo-Anosovs with the minimal dilatations and their mapping tori.

Conjecture 4.1 ([11]).

1. *We have $\lim\limits_{g\to\infty} g \log \delta_g = \log(\frac{3+\sqrt{5}}{2})$. For large g, δ_g is achieved by the monodromy of some Σ_g-bundle over the circle obtained from either $N(\frac{3}{-2})$ or $N(\frac{1}{-2})$ by Dehn filling both cusps.*
2. *We have $\lim\limits_{\substack{g\not\equiv 0 \pmod 6\\ g\to\infty}} g \log \delta_g^+ = \log(\frac{3+\sqrt{5}}{2})$. For large g such that $g \not\equiv 0 \pmod 6$, δ_g^+ is achieved by the monodromy of some Σ_g-bundle over the circle obtained from $N(\frac{3}{-2})$ or $N(\frac{1}{-2})$ by Dehn filling both cusps.*

Conjecture 4.2 ([12]).

1. *$\delta(D_{2k-1}) = \lambda(T'_{2k,2})$ for $k \geq 5$.*
2. *$\delta(D_{4k}) = \lambda(T'_{4k+1,2k-1})$ for $k \geq 3$.*
3. *$\delta(D_{10}) = \lambda(T_{10,2})$, and $\delta(D_{8k+2}) = \lambda(T'_{8k+3,2k+1})$ for $k \geq 2$.*
4. *$\delta(D_{8k+6}) = \lambda(T'_{8k+7,2k+1})$ for $k \geq 1$.*

Conjecture 4.3 ([11]). *We have $\lim\limits_{n\to\infty} n \log \delta_{1,n} = 2 \log \delta(D_4)$. For large n, $\delta_{1,n}$ is achieved by the monodromy of a fibration on $N(1)$.*

Question 4.4 ([14]). *Can one eliminate the condition $(*)$ in Theorem 3.5? i.e, given $g \geq 2$, does $\limsup\limits_{n\to\infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2$ hold?*

Finally, we ask about questions related to the finiteness theorem for small dilatation pseudo-Anosov homeomorphisms [5, 3]. Given a pseudo-Anosov $\Phi : \Sigma \to \Sigma$, let $\Sigma^\circ \subset \Sigma$ be the surface obtained by removing all the singularities of the stable foliation for Φ, and $\Phi|_{\Sigma^\circ} : \Sigma^\circ \to \Sigma^\circ$ denotes the restriction of Φ to Σ°. Observe that $\lambda(\Phi) = \lambda(\Phi|_{\Sigma^\circ})$. The finiteness theorem implies that the following sets are finite.

$$\mathcal{U} = \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid \Phi \text{ is pseudo-Anosov on } \Sigma = \Sigma_g \text{ such that } \lambda(\Phi) = \delta_g,\ g \geq 2\},$$
$$\mathcal{U}_{\text{braid}} = \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid \Phi \text{ is pseudo-Anosov on } \Sigma = D_n \text{ such that } \lambda(\Phi) = \delta(D_n),\ n \geq 3\},$$
$$\mathcal{U}_{g=1} = \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid \Phi \text{ is pseudo-Anosov on } \Sigma = \Sigma_{1,n} \text{ such that } \lambda(\Phi) = \delta_{1,n},\ n \geq 1\}.$$

We know that $N \in \mathcal{U} \cap \mathcal{U}_{\text{braid}} \cap \mathcal{U}_{g=1}$. Since pseudo-Anosov mapping classes with the smallest known dilatations defined on either Σ_g, D_n or $\Sigma_{1,n}$ come from N, we ask:

Question 4.5. *It is true that $\mathcal{U} = \mathcal{U}_{\text{braid}} = \mathcal{U}_{g=1} = \{N\}$?*

On the other hand, by the fact that given $g \geq 2$, $\log \delta_{g,n} \asymp \frac{\log n}{n}$, one can not appeal to the finiteness theorem for the following set $\mathcal{U}_g$ for $g \geq 2$.

$$\mathcal{U}_g = \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid \Phi \text{ is pseudo-Anosov on } \Sigma = \Sigma_{g,n} \text{ such that } \lambda(\Phi) = \delta_{g,n},\ n \geq 1\}.$$

The examples which provide the upper bound in Theorem 3.5 are monodromies of fibrations on manifolds obtained from the single manifold N by Dehn fillings. For this reason, we would like to ask:

Question 4.6. *Is there any $g \geq 2$ such that $\mathcal{U}_g$ is a finite set?*

5. Tables

Table 3. fibered class $(x, y, z) \in H_2(N, \partial N)$ which achieves $\widehat{\delta}_g$ for large g, see [11, Theorem 1.4, Remark 3.18]. [notice that (x, y, z) is in either $S_\beta(\frac{3}{-2})$ or $S_\beta(\frac{1}{-2})$.]

g	$(x,y,z) \in H_2(N, \partial N)$	$f_{(x,y,z)}(t)$
$0, 1, 5, 6 \pmod{10}$	$(2g+5, 2g+6, g+4) \in S_\beta(\frac{3}{-2})$	$(t^{g+3}+1)(t^{2g+4} - t^{g+3} - t^{g+2} - t^{g+1} + 1)$
$7, 9 \pmod{10}$	$(2g+6, 2g+8, g+6) \in S_\beta(\frac{3}{-2})$	$(t^{g+4}+1)(t^{2g+4} - t^{g+4} - t^{g+2} - t^{g} + 1)$
$[3 \pmod{10}]$		
$3, 13 \pmod{30}$	$(g+1, 2g+8, 3) \in S_\beta(\frac{1}{-2})$	$(t^{g+4}+1)(t^{2g+2} - t^{g+4} - t^{g+1} - t^{g-2} + 1)$
$23 \pmod{30}$	$(g+1, 2g+4, 1) \in S_\beta(\frac{1}{-2})$	$(t^{g+2}+1)(t^{2g+2} - t^{g+2} - t^{g+1} - t^{g} + 1)$
$[8 \pmod{10}]$		
$8 \pmod{30}$	$(g+1, 2g+4, 1) \in S_\beta(\frac{1}{-2})$	$(t^{g+2}+1)(t^{2g+2} - t^{g+2} - t^{g+1} - t^{g} + 1)$
$18, 28 \pmod{30}$	$(g+1, 2g+8, 3) \in S_\beta(\frac{1}{-2})$	$(t^{g+4}+1)(t^{2g+2} - t^{g+4} - t^{g+1} - t^{g-2} + 1)$

TABLE 4. upper bounds of δ_g for small g. [see also [9, 1, 13].]

g	$(x,y,z) \in H_2(N,\partial N)$	$(\delta_g \leq)\lambda_{(x,y,z)} \approx$
3	$(4,14,3) \in S_\beta(\frac{1}{-2})$	1.4012
4	$(5,16,3) \in S_\beta(\frac{1}{-2})$	1.2612
5	$(13,12,5) \in S_\beta(\frac{3}{-2})$	1.1487
6	$(15,14,6) \in S_\beta(\frac{3}{-2})$	1.1287
7	$(16,14,5) \in S_\beta(\frac{3}{-2})$	1.1154
8	$(17,18,7) \in S_\beta(\frac{4}{-3})$	1.1040
9	$(20,18,7) \in S_\beta(\frac{3}{-2})$	1.0928
10	$(23,22,10) \in S_\beta(\frac{3}{-2})$	1.0837
11	$(25,24,11) \in S_\beta(\frac{3}{-2})$	1.0770
12	$(25,22,8) \in S_\beta(\frac{3}{-2})$	1.0726
13	$(27,21,8) \in S_\beta(\frac{5}{-3})$	1.0716
14	$(29,26,10) \in S_\beta(\frac{3}{-2})$	1.0629
15	$(33,32,15) \in S_\beta(\frac{3}{-2})$	1.0583
16	$(35,34,16) \in S_\beta(\frac{3}{-2})$	1.0549
17	$(36,34,15) \in S_\beta(\frac{3}{-2})$	1.0522
18	$(19,44,3) \in S_\beta(\frac{1}{-2})$	1.0525
19	$(40,38,17) \in S_\beta(\frac{3}{-2})$	1.0470
20	$(43,42,20) \in S_\beta(\frac{3}{-2})$	1.0447

TABLE 5. fibered class $(x,y,z) \in H_2(N,\partial N)$ which achieves δ_g^+ for small g.

g	$(x,y,z) \in H_2(N,\partial N)$	$\delta_g^+ = \lambda_{(x,y,z)} \approx$	minimal polynomial (a factor of $f_{(x,y,z)}(t)$)
2	$(2,6,1) \in S_\beta(\frac{1}{-2})$	1.7220 [26]	$t^4 - t^3 - t^2 - t + 1$
3	$(4,14,3) \in S_\beta(\frac{1}{-2})$	1.4012 [16]	$t^6 - t^4 - t^3 - t^2 + 1$
4	$(4,10,1) \in S_\beta(\frac{1}{-2})$	1.2806 [16]	$t^8 - t^5 - t^4 - t^3 + 1$
5	$(18,22,15) \in S_\beta(\frac{3}{-2})$	1.1762 [16]	$t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$
7	$(20,22,13) \in S_\beta(\frac{3}{-2})$	1.1154 [16, 1, 13]	$t^{14} + t^{13} - t^9 - t^8 - t^7 - t^6 - t^5 + t + 1$
8	$(8,18,1) \in S_\beta(\frac{1}{-2})$	1.1287 [16, 9]	$t^{16} - t^9 - t^8 - t^7 + 1$

TABLE 6. fibered class $(x,y,z) \in H_2(N,\partial N)$ which achieves $\widehat{\delta}_g^+$ for large $g \not\equiv 0 \pmod 6$, see [11, Theorem 1.5]. [notice that (x,y,z) is in either $S_\beta(\frac{3}{-2})$ or $S_\beta(\frac{1}{-2})$.]

g	$(x,y,z) \in H_2(N,\partial N)$	$f_{(x,y,z)}(t)$
$7,9 \pmod{10}$	$(2g+6,2g+8,g+6) \in S_\beta(\frac{3}{-2})$	$(t^{g+4}+1)(t^{2g+4} - t^{g+4} - t^{g+2} - t^g + 1)$
$1,5 \pmod{10}$	$(2g+8,2g+12,g+10) \in S_\beta(\frac{3}{-2})$	$(t^{g+6}+1)(t^{2g+4} - t^{g+6} - t^{g+2} - t^{g-2} + 1)$
$[3 \pmod{10})]$ $3,13 \pmod{30}$ $23 \pmod{30}$	 $(g+1,2g+8,3) \in S_\beta(\frac{1}{-2})$ $(g+1,2g+4,1) \in S_\beta(\frac{1}{-2})$	 $(t^{g+4}+1)(t^{2g+2} - t^{g+4} - t^{g+1} - t^{g-2} + 1)$ $(t^{g+2}+1)(t^{2g+2} - t^{g+2} - t^{g+1} - t^g + 1)$
$2,4 \pmod 6$	$(g,2g+2,1) \in S_\beta(\frac{1}{-2})$	$(t^{g+1}+1)(t^{2g} - t^{g+1} - t^g - t^{g-1} + 1)$

TABLE 7. for $n \geq 9$, fibered class s_n which achieves the minimal dilatation in $\mathcal{N}_n$ and its mapping class ϕ_{s_n}, see [12, Theorem 1.1]. [notice that $s_n \in S_\gamma(\infty)$.]

n	$s_n = (x,y,z) \in H_2(N, \partial N)$	$f_{(x,y,z)}(t)$	ϕ_{s_n}
$2k+1$	$(k-1, k, 0) \in S_\gamma(\infty)$	$t^{2k-1} - 2(t^{k-1} + t^k) + 1$	$T_{2k,2}$
$4k+2$	$(2k+1, 2k-1, 0) \in S_\gamma(\infty)$	$t^{4k} - 2(t^{2k-1} + t^{2k+1}) + 1$	$T_{4k+1,2k-1}$
$8k+4$	$(4k-1, 4k+3, 0) \in S_\gamma(\infty)$	$t^{8k+2} - 2(t^{4k-1} + t^{4k+3}) + 1$	$T_{8k+3,2k+1}$
$8(k+1)$	$(4k+5, 4k+1, 0) \in S_\gamma(\infty)$	$t^{8k+6} - 2(t^{4k+1} + t^{4k+5}) + 1$	$T_{8k+7,2k+1}$

TABLE 8. upper bounds of $\delta(D_{n-2})$, see [12, Corollary 4.1]. [see also [10, 25].]

n	$s_n = (x,y,z) \in H_2(N, \partial N)$	$f_{(x,y,z)}(t)$	filling	braid $\in B_{n-2}$
$2k+1$	$(k-1, k, 0) \in S_\gamma(\infty)$	$t^{2k-1} - 2(t^{k-1} + t^k) + 1$	α	$T'_{2k,2}$
$4k+2$	$(2k+1, 2k-1, 0) \in S_\gamma(\infty)$	$t^{4k} - 2(t^{2k-1} + t^{2k+1}) + 1$	α	$T'_{4k+1,2k-1}$
$8k+4$	$(4k-1, 4k+3, 0) \in S_\gamma(\infty)$	$t^{8k+2} - 2(t^{4k-1} + t^{4k+3}) + 1$	α	$T'_{8k+3,2k+1}$
$8(k+1)$	$(4k+5, 4k+1, 0) \in S_\gamma(\infty)$	$t^{8k+6} - 2(t^{4k+1} + t^{4k+5}) + 1$	α	$T'_{8k+7,2k+1}$

TABLE 9. upper bounds of $\delta(D_{n-2})$ for small n. [see also [10, 25].]

n	$(x,y,z) \in H_2(N, \partial N)$	$(\delta(D_{n-2}) \leq)\lambda_{(x,y,z)} \approx$	filling	braid $\in B_{n-2}$
11	$(4,5,0) \in S_\gamma(\infty)$	1.3437	α	$T'_{10,2}$
12	$(4,5,0) \in S_\gamma(\infty)$	1.3437	none	$T_{10,2}$
13	$(5,6,0) \in S_\gamma(\infty)$	1.2724	α	$T'_{12,2}$
14	$(7,5,0) \in S_\gamma(\infty)$	1.2514	α	$T'_{13,5}$
15	$(6,7,0) \in S_\gamma(\infty)$	1.2257	α	$T'_{14,2}$
16	$(9,5,0) \in S_\gamma(\infty)$	1.2225	α	$T'_{15,3}$
17	$(7,8,0) \in S_\gamma(\infty)$	1.1926	α	$T'_{16,2}$
18	$(9,7,0) \in S_\gamma(\infty)$	1.1812	α	$T'_{17,7}$
19	$(8,9,0) \in S_\gamma(\infty)$	1.1680	α	$T'_{18,2}$
20	$(7,11,0) \in S_\gamma(\infty)$	1.1643	α	$T'_{19,5}$
21	$(9,10,0) \in S_\gamma(\infty)$	1.1490	α	$T'_{20,2}$
22	$(11,9,0) \in S_\gamma(\infty)$	1.1419	α	$T'_{21,9}$
23	$(10,11,0) \in S_\gamma(\infty)$	1.1338	α	$T'_{22,2}$
24	$(13,9,0) \in S_\gamma(\infty)$	1.1307	α	$T'_{23,5}$
25	$(11,12,0) \in S_\gamma(\infty)$	1.1215	α	$T'_{24,2}$
26	$(13,11,0) \in S_\gamma(\infty)$	1.1166	α	$T'_{25,11}$
27	$(12,13,0) \in S_\gamma(\infty)$	1.1112	α	$T'_{26,2}$
28	$(11,15,0) \in S_\gamma(\infty)$	1.1086	α	$T'_{27,7}$
29	$(13,14,0) \in S_\gamma(\infty)$	1.1025	α	$T'_{28,2}$
30	$(15,13,0) \in S_\gamma(\infty)$	1.0990	α	$T'_{29,13}$
31	$(14,15,0) \in S_\gamma(\infty)$	1.0951	α	$T'_{30,2}$
32	$(17,13,0) \in S_\gamma(\infty)$	1.0930	α	$T'_{31,7}$

TABLE 10. fibered class $(x,y,z) \in H_2(N, \partial N)$ which achieves $\delta(D_n)$ for small n, see [12, Section 4.1]. [for the minimal polynomial of $\delta(D_n)$, see the 4th column.]

n	$(x,y,z) \in H_2(N, \partial N)$	$\delta(D_n) = \lambda_{(x,y,z)} \approx$	minimal polynomial	filling	braid $\in B_n$
3	$(2,1,0) \in S_\gamma(\infty)$	$\frac{3+\sqrt{5}}{2}$ [7]	$t^2 - 3t + 1$	α	$T'_{4,1}$
4	$(3,1,0) \in S_\gamma(\infty)$	2.2966 [15]	$t^4 - 2t^3 - 2t + 1$	α	$T'_{5,1}$
5	$(2,3,0) \in S_\gamma(\infty)$	1.7220 [8]	$t^4 - t^3 - t^2 - t + 1$	α	$T'_{6,2}$
6	$(2,3,0) \in S_\gamma(\infty)$	1.7220 [17]	$t^4 - t^3 - t^2 - t + 1$	none	$T_{6,2}$
7	$(3,4,0) \in S_\gamma(\infty)$	1.4655 [17]	$t^3 - t^2 - 1$	α	$T'_{8,2}$
8	$(5,3,0) \in S_\gamma(\infty)$	1.4134 [17]	$t^8 - 2t^5 - 2t^3 + 1$	α	$T'_{9,5}$

REFERENCES

[1] J. W. Aaber and N. M. Dunfield, *Closed surface bundles of least volume*, Algebr. Geom. Topol. **10** (2010), 2315–2342.

[2] I. Agol, *The minimal volume orientable hyperbolic* 2*-cusped* 3*-manifolds*, Proc. Amer. Math. Soc. **138** (2010), 3723–3732.

[3] I. Agol, *Ideal triangulations of pseudo-Anosov mapping tori*, Topology and Geometry in Dimension Three: Triangulations, Invariants, and Geometric Structures, edited by W. Li, L. Bartolini, J. Johnson, F. Luo, R. Myers and J. H. Rubinstein, Contemp. Math. **560** (2011), 1–17.

Table 11. fibered class $\overline{w_n} \in H_2(N_\gamma(1), \partial N_\gamma(1))$ which achieves the minimal dilatation in $\mathcal{W}_n$, see [11, Proposition 3.30]. [notice that $w_n \in S_\gamma(1)$.]

n	$w_n = (x, y, z) \in H_2(N, \partial N)$	$f_{(x,y,z)}(t)$	filling
2	$(1, 1, -2) \in S_\gamma(1)$	$t^4 - 2t^3 - 2t + 1$	γ
$2k-1$	$(k, k-1, -2k+1) \in S_\gamma(1)$	$t^{4k-2} - t^{3k-1} - t^{3k-2} - t^k - t^{k-1} + 1$	γ
$4k$	$(2k+1, 2k-1, -4k) \in S_\gamma(1)$	$t^{8k} - t^{6k+1} - t^{6k-1} - t^{2k+1} - t^{2k-1} + 1$	γ
$4k+2$	$(2k+3, 2k-1, -4k-2) \in S_\gamma(1)$	$t^{8k+4} - t^{6k+5} - t^{6k+1} - t^{2k+3} - t^{2k-1} + 1$	γ

Table 12. upper bounds of $\delta_{1,n}$ for small n.

n	$(x, y, z) \in H_2(N, \partial N)$	$(\delta_{1,n} \leq)\lambda_{(x,y,z)} \approx$	filling
2	$(1, 1, -2) \in S_\gamma(1)$	2.2966	γ
3	$(2, 1, -3) \in S_\gamma(1)$	1.7816	γ
4	$(3, 1, -4) \in S_\gamma(1)$	1.5823	γ
5	$(3, 2, -5) \in S_\gamma(1)$	1.4012	γ
6	$(5, 1, -6) \in S_\gamma(1)$	1.4012	γ
7	$(4, 3, -7) \in S_\gamma(1)$	1.2703	γ
8	$(5, 3, -8) \in S_\gamma(1)$	1.2369	γ
9	$(5, 4, -9) \in S_\gamma(1)$	1.2039	γ
10	$(7, 3, -10) \in S_\gamma(1)$	1.1932	γ
11	$(6, 5, -11) \in S_\gamma(1)$	1.1637	γ
12	$(7, 5, -12) \in S_\gamma(1)$	1.1502	γ
13	$(7, 6, -13) \in S_\gamma(1)$	1.1367	γ
14	$(9, 5, -14) \in S_\gamma(1)$	1.1301	γ
15	$(8, 7, -15) \in S_\gamma(1)$	1.1174	γ
16	$(9, 7, -16) \in S_\gamma(1)$	1.1101	γ
17	$(9, 8, -17) \in S_\gamma(1)$	1.1028	γ
18	$(11, 7, -18) \in S_\gamma(1)$	1.0986	γ
19	$(10, 9, -19) \in S_\gamma(1)$	1.0915	γ
20	$(11, 9, -20) \in S_\gamma(1)$	1.0870	γ

[4] J. H. Cho and J. Y. Ham, *The minimal dilatation of a genus-two surface*, Exp. Math. **17** (2008), 257–267.

[5] B. Farb, C. J. Leininger and D. Margalit, *Small dilatation pseudo-Anosov homeomorphisms and 3-manifolds*, Adv. Math. **228** (2011), 1466–1502.

[6] D. Fried, *Flow equivalence, hyperbolic systems and a new zeta function for flows*, Comment. Math. Helv. **57** (1982), 237–259.

[7] M. Handel, *The forcing partial order on the three times punctured disk*, Ergodic Theory Dynam. Systems **17** (1997), 593–610.

[8] J. Y. Ham and W. T. Song, *The minimum dilatation of pseudo-Anosov 5-braids*, Exp. Math. **16** (2007), 167–179.

[9] E. Hironaka, *Small dilatation mapping classes coming from the simplest hyperbolic braid*, Algebr. Geom. Topol. **10** (2010), 2041–2060.

[10] E. Hironaka and E. Kin, *A family of pseudo-Anosov braids with small dilatation*, Algebr. Geom. Topol. **6** (2006), 699–738.

[11] E. Kin, S. Kojima and M. Takasawa, *Minimal dilatations of pseudo-Anosovs generated by the magic 3-manifold and their asymptotic behavior*, preprint (2011), arXiv:1104.3939v3

[12] E. Kin and M. Takasawa, *Pseudo-Anosov braids with small entropy and the magic* 3*-manifold*, Comm. Anal. Geom. **19** (2011), 1–54.
[13] E. Kin and M. Takasawa, *Pseudo-Anosovs on closed surfaces having small entropy and the Whitehead sister link exterior*, preprint (2010), arXiv:1003.0545, to appear in "J. Math. Soc. Japan".
[14] E. Kin and M. Takasawa, *The boundary of a fibered face of the magic* 3*-manifold and the asymptotic behavior of the minimal pseudo-Anosovs dilatations*, preprint (2012), arXiv:1205.2956
[15] K. H. Ko, J. Los and W. T. Song, *Entropies of braids*, J. Knot Theory Ramifications **11** (2002), 647–666.
[16] E. Lanneau and J. L. Thiffeault, *On the minimum dilatation of pseudo-Anosov homeomorphisms on surfaces of small genus*, Ann. Inst. Fourier **61** (2011), 105–144.
[17] E. Lanneau and J. L. Thiffeault, *On the minimum dilatation of braids on the punctured disc*, Geom. Dedicata **152** (2011), 165–182.
[18] B. Martelli and C. Petronio, *Dehn filling of the "magic"* 3*-manifold*, Comm. Anal. Geom. **14** (2006), 969–1026.
[19] S. Matsumoto, *Topological entropy and Thurston's norm of atoroidal surface bundles over the circle*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 763–778.
[20] H. Minakawa, *Examples of pseudo-Anosov homeomorphisms with small dilatations*, J. Math. Sci. Univ. Tokyo **13** (2006), 95–111.
[21] C. McMullen, *Polynomial invariants for fibered* 3*-manifolds and Teichmüler geodesic for foliations*, Ann. Sci. École Norm. Sup. **33** (2000), 519–560.
[22] R. C. Penner, *Bounds on least dilatations*, Proc. Amer. Math. Soc. **113** (1991), 443–450.
[23] W. Thurston, *A norm of the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. **59** (1986), 99–130.
[24] C. Y. Tsai, *The asymptotic behavior of least pseudo-Anosov dilatations*, Geom. Topol. **13** (2009), 2253–2278.
[25] R. Venzke, *Braid forcing, hyperbolic geometry, and pseudo-Anosov sequences of low entropy*, PhD thesis, California Institute of Technology (2008).
[26] A. Y. Zhirov, *On the minimum dilation of pseudo-Anosov diffeomorphisms on a double torus*, Russian Mathematical Surveys **50** (1995), 223–224.

Department of Mathematics, Osaka University
E-mail address: kin@math.sci.osaka-u.ac.jp