Pseudo-Anosov braids with small entropy and the magic 3-manifold

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Abstract

We consider a surface bundle over the circle, the so called magic manifold M. We determine homology classes whose minimal representatives are genus 0 fiber surfaces for M, and describe their monodromies by braids. Among those classes whose representatives have n punctures for each n, we decide which one realizes the minimal entropy. We show that for each $n \geq 9$ (resp. $n = 3,4,5,7,8$), there exists a pseudo-Anosov homeomorphism $\Phi_n : D_n \to D_n$ with the smallest known entropy (resp. the smallest entropy) which occurs as the monodromy on an n-punctured disk fiber for the Dehn filling of M. A pseudo-Anosov homeomorphism $\Phi_6 : D_6 \to D_6$ with the smallest entropy occurs as the monodromy on a 6-punctured disk fiber for M.

Keywords: mapping class group, pseudo-Anosov, entropy, hyperbolic volume, magic manifold

Mathematics Subject Classification : Primary 37E30, 57M27, Secondary 57M50

1 Introduction

Let $\mathcal{M}(\Sigma)$ be the mapping class group of an orientable surface $\Sigma = \Sigma_{g,p}$ of genus g with p punctures. Assuming that $3g-3+p \geq 1$, elements of $\mathcal{M}(\Sigma)$ are classified into three types: periodic, pseudo-Anosov and reducible [29]. There exist two numerical invariants of pseudo-Anosov mapping classes ϕ. One is the entropy $\mathrm{ent}(\phi)$ which is the logarithm of the dilatation $\lambda(\phi)$. The

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other is the volume $\mathrm{vol}(\phi)$ which comes from the hyperbolization theorem by Thurston [30]. His theorem asserts that ϕ is pseudo-Anosov if and only if its mapping torus

$$\mathbb{T}(\phi) = \Sigma \times [0,1]/\sim,$$

where $\sim$ identifies $(x,1)$ with $(f(x),0)$ for any representative $f \in \phi$, is hyperbolic. We denote the volume of $\mathbb{T}(\phi)$ by $\mathrm{vol}(\phi)$.

Let $\mathcal{M}^{\mathrm{pA}}(\Sigma)$ be the set of pseudo-Anosov elements of $\mathcal{M}(\Sigma)$. Fixing Σ, the dilatation $\lambda(\phi)$ for $\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)$ is known to be an algebraic integer with a bounded degree depending only on Σ. The set of dilatations $\lambda(\phi)$ for $\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)$ bounded by each constant from above is finite, see [13]. In particular the set

$$\mathrm{Dil}(\Sigma) = \{\lambda(\phi) > 1 \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)\}$$

achieves its infimum $\lambda(\Sigma)$.

We turn to volume. The set

$$\{v \mid v \text{ is the volume of a hyperbolic 3-manifold}\}$$

is a well-ordered closed subset of $\mathbb{R}$ of order type ω^ω [27]. In particular any subset achieves its infimum. Let $\mathrm{vol}(\Sigma) = \min\{\mathrm{vol}(\phi) \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)\}$. It is of interest to compute $\lambda(\Sigma)$ (resp. $\mathrm{vol}(\Sigma)$) and to determine the mapping class realizing the minimum. Another problem related to the minimal dilatation (resp. minimal volume) are as follows. For a non-negative integer c, we set

$$\begin{aligned} \lambda(\Sigma; c) &= \min\{\lambda(\phi) \mid \phi \in \mathcal{M}^{\mathrm{p}A}(\Sigma),\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}, \\ \mathrm{vol}(\Sigma; c) &= \min\{\mathrm{vol}(\phi) \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma),\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}. \end{aligned}$$

A problem is to compute $\lambda(\Sigma; c)$ (resp. $\mathrm{vol}(\Sigma; c)$) and to find a mapping class realizing the minimum.

In [15], the authors and S. Kojima obtain experimental results concerning the entropy and volume. In the case the mapping class group $\mathcal{M}(D_n)$ of an n-punctured disk D_n, they observe that for many pairs (n, c), there exists a mapping class simultaneously reaching both $\lambda(D_n; c)$ and $\mathrm{vol}(D_n; c)$. Experiments tell us that in case $c = 3$, the mapping tori reaching both minima are homeomorphic to the magic manifold M_{magic} which is the exterior of the 3 chain link $\mathcal{C}_3$ illustrated in Figure 1. Moreover when $c = 2$, it is observed that there exists a mapping class ϕ realizing both $\lambda(D_n; 2)$ and $\mathrm{vol}(D_n; 2)$ and its mapping torus $\mathbb{T}(\phi)$ is homeomorphic to a Dehn filling of M_{magic} along one cusp. This study motivates the present paper which concerns the fibrations in M_{magic}. The magic manifold has the smallest known volume among orientable hyperbolic 3-manifolds having 3 cusps. Many manifolds having at most 2 cusps with small volume are obtained from M_{magic} by Dehn fillings, see [21]. Also, some important examples for the study of the exceptional Dehn fillings can be obtained from the Dehn fillings of M_{magic}, see [9].

Let M be a hyperbolic 3-manifold with boundary which fibers over the circle. We assume that M admits infinitely many different fibrations. Thurston introduced the norm function $X_T : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$, and showed that the unit ball U with respect to X_T is a compact, convex polyhedron [28]. He described the relation between the function X_T and fibrations of M as follows. For each fiber F of M, the homology class $[F] \in H_2(M, \partial M; \mathbb{R})$ lies in the open cone $int(C_\Delta)$ with the origin over a top dimensional face Δ of ∂U. Conversely for any integral class $a \in int(C_\Delta)$, there exists a fiber F of M representing a. Using this description of the fibers, the entropy function $\mathrm{ent}(\cdot) : int(C_\Delta) \to \mathbb{R}$ can be defined as follows. For each primitive integral class $a \in int(C_\Delta)$, the monodromy $\Phi_a : F_a \to F_a$ on a connected surface F_a representing a is pseudo-Anosov, and one defines the entropy of a by $\mathrm{ent}(a) = \log(\lambda(\Phi_a))$. Fried proves that this function defined on primitive integral classes admits a unique continuous extension to a homogeneous function on $int(C_\Delta)$ [6].

One sees that $\mathcal{M}(D_n)$ is isomorphic to the subgroup of $\mathcal{M}(\Sigma_{0,n+1})$ consisting of the elements which fix a puncture of $\Sigma_{0,n+1}$. By using the natural surjective homomorphism $\Gamma : B_n \to \mathcal{M}(D_n)$ from the n-braid group B_n to $\mathcal{M}(D_n)$, one represents each element of $\mathcal{M}(D_n)$ by an n-braid. A braid b is called *pseudo-Anosov* if $\Gamma(b)$ is a pseudo-Anosov mapping class. If this is the case, the dilatation $\lambda(b)$ of b is defined by the dilatation of $\Gamma(b)$. Let $T_{m,p}$ be the following m-braid for $m \geq 3$ and $p \geq 1$ which is a main example in this paper.

$$T_{m,p} = (\sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-1})^p \sigma_{m-1}^{-2}.$$

For example, $T_{6,2} = (\sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_5)^2 \sigma_5^{-2} = \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_5^{-1}$ (Figure 8(left)). The braid $T_{m,p}$ is a horseshoe braid if $\gcd(p, m-1) = 1$ and $1 < p \leq \frac{m-1}{2}$ (Proposition 4.14). If $\gcd(m-1, p) = 1$, then the mapping torus $\mathbb{T}(\Gamma(T_{m,p}))$ is homeomorphic to M_{magic} (Corollary 3.28). Otherwise $\Gamma(T_{m,p})$ is reducible. We set

$$\mathcal{M}^n_{\text{magic}} = \{\phi \in \mathcal{M}(\Sigma_{0,n}) \mid \mathbb{T}(\phi) \text{ is homeomorphic to } M_{\text{magic}}\}.$$

Let us define an integral polynomial

$$f_{(x,y,z)}(t) = t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1.$$

The following, our main theorem, states that for all n but 6 and 8, the minimum among the dilatations of $\phi \in \mathcal{M}^n_{\mathrm{magic}}$ is realized by $\Gamma(T_{n-1,p})$ for some $p = p(n)$ and it is computed as the largest real root of one of the polynomials $f_{(x,y,z)}(t)$.

Theorem 1.1. *For each $n \geq 4$, the minimum among the dilatations of $\phi \in \mathcal{M}^n_{\mathrm{magic}}$ is realized by:*

(1) $\Gamma(T_{2k,2})$ *in case $n = 2k+1$ for $k \geq 2$. The dilatation $\lambda(T_{2k,2})$ equals the largest real root of*

$$f_{(k-1,k,0)}(t) = t^{2k-1} - 2(t^{k-1} + t^k) + 1.$$

(2-i) $\Gamma(\sigma_1\sigma_2^2\sigma_3\sigma_4)$ *in case $n = 6$. The dilatation $\lambda(\sigma_1\sigma_2^2\sigma_3\sigma_4) \approx 2.08102$ equals the largest real root of*

$$f_{(3,2,1)}(t) = t^4 - t^3 - 2t^3 - t + 1.$$

(2-ii) $\Gamma(T_{4k+1,2k-1})$ *in case $n = 4k+2$ for $k \geq 2$. The dilatation $\lambda(T_{4k+1,2k-1})$ equals the largest real root of*

$$f_{(2k+1,2k-1,0)}(t) = t^{4k} - 2(t^{2k-1} + t^{2k+1}) + 1.$$

(3a-i) $\Gamma(T_{3,1})$ *in case $n = 4$. The dilatation $\lambda(T_{3,1}) \approx 3.73205$ equals the largest real root of*

$$f_{(1,1,0)}(t) = t^2 - 4t + 1.$$

(3a-ii) $\Gamma(T_{8k+3,2k+1})$ *in case $n = 8k+4$ for $k \geq 1$. The dilatation $\lambda(T_{8k+3,2k+1})$ equals the largest real root of*

$$f_{(4k-1,4k+3,0)}(t) = t^{8k+2} - 2(t^{4k-1} + t^{4k+3}) + 1.$$

(3b-i) $\Gamma(b)$ *in case $n = 8$, where*

$$b = \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1} \in B_7.$$

The dilatation $\lambda(b) \approx 1.72208$ equals the largest real root of

$$f_{(5,3,2)}(t) = t^6 - t^5 - 2t^3 - t + 1.$$

(3b-ii) $\Gamma(T_{8k+7,2k+1})$ *in case $n = 8(k+1)$ for $k \geq 1$. The dilatation $\lambda(T_{8k+7,2k+1})$ equals the largest real root of*

$$f_{(4k+5,4k+1,0)}(t) = t^{8k+6} - 2(t^{4k+1} + t^{4k+5}) + 1.$$

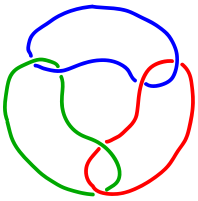

Figure 1: 3 chain link $\mathcal{C}_3$.

Moreover the above mapping class realizing the minimal dilatation among elements of $\mathcal{M}^n_{\rm magic}$ is unique up to conjugacy.

Forgetting the 1st strand of $T_{m,p}$, one obtains the $(m-1)$-braid, call it $T'_{m,p}$. For example, $T'_{6,2} = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_4^{-1}$. The families of braids $\{T'_{m,p}\}$ and $\{T_{m,p}\}$ contain examples of ℓ strands ($\ell = 3, 4, \cdots, 8$) with the smallest dilatation. (See Section 4.1.) Hironaka-Kin (resp. Venzke) found candidates with the smallest dilatation $\lambda(D_n)$ for n odd (resp. n even), see [12] (resp. [31]). All the braids in Theorem 1.1(1)(2-ii)(3a-ii)(3b-ii) relate to those examples. The braid $T'_{2k,2}$ (with odd strands) is conjugate to the braid $\sigma_{(k)}$ with the smallest known dilatation found by Hironaka-Kin. (See Theorem 1.1(1).) For the braid $T'_{m,p}$ (with even strands) obtained from $T_{m,p}$ in (2-ii), (3a-ii) or (3b-ii) of Theorem 1.1, the mapping class $\Gamma(T'_{m,p})$ is conjugate to the one given by Venzke. (See Section 4.1.)

Work of Farb-Leininger-Margalit [4] together with a result in [12] implies that there exists a complete, noncompact, finite volume, hyperbolic 3-manifold M' with the following property: there exist Dehn fillings of M' giving an infinite sequence of fiberings over S^1, with fibers D_{n_i} having n_i punctures with $n_i \to \infty$, and with the monodromy $\Phi_i : D_{n_i} \to D_{n_i}$ so that $\lambda(D_{n_i}) = \lambda(\Phi_i)$. The magic manifold is a potential example which could satisfy this property.

In [1, 11, 17], one can find pseudo-Anosovs on closed surfaces Σ_g of genus g with small dilatation which occur as monodromies on fibers for Dehn fillings of $M_{\rm magic}$. Using those pseudo-Anosovs, Hironaka [11], Aaber-Dunfield [1], and the authors [17] independently proved that

$$\lim_{g\to\infty} \sup g \log \lambda(\Sigma_g) \leq \log(\tfrac{3+\sqrt{5}}{2}).$$

This paper is organized as follows. Section 2 reviews basic facts. Section 3 contains the proof of Theorem 1.1. For the proof, we first compute the Teichmüler polynomial, introduced by McMullen [24], which determines the entropy function for M_{magic} (Theorem 3.4). Then we find all the homology classes whose representatives are genus 0 fiber surfaces (Corollary 3.10). We study the asymptotic behaviors of the normalized entropy function $\overline{\mathrm{ent}}(\cdot) = X_T(\cdot)\mathrm{ent}(\cdot)$ (Theorem 3.11). This tells us which class realizes the minimal dilatation among homology classes whose representatives are genus 0 fiber surfaces with n punctures (Proposition 3.12). We finally describe the monodromies for these fiber surfaces by using braids (Propositions 3.29 and 3.32). In Section 4, we discuss pseudo-Anosov braids with small dilatation. We also find a relation between the horseshoe map and the braids $T_{m,p}$ (Proposition 4.14).

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2 Notation and basic facts

2.1 Mapping class group

The *mapping class group* $\mathcal{M}(\Sigma)$ is the group of isotopy classes of orientation preserving homeomorphisms of Σ, where the group operation is induced by composition of homeomorphisms. An element of the mapping class group is called a *mapping class.*

A homeomorphism $\Phi : \Sigma \to \Sigma$ is *pseudo-Anosov* if there exists a constant $\lambda = \lambda(\Phi) > 1$ called the *dilatation of* Φ and there exists a pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that

$$\Phi(\mathcal{F}^s) = \tfrac{1}{\lambda}\mathcal{F}^s \text{ and } \Phi(\mathcal{F}^u) = \lambda\mathcal{F}^u.$$

In this case the mapping class $\phi = [\Phi]$ is called pseudo-Anosov. We define the dilatation of ϕ, denoted by $\lambda(\phi)$, to be the dilatation of Φ.

The (topological) entropy $\mathrm{ent}(f)$ is a measure of the complexity of a continuous self-map f on a compact manifold, see [32]. For a pseudo-Anosov homeomorphism $\Phi : \Sigma \to \Sigma$, the equality

$$\mathrm{ent}(\Phi) = \log(\lambda(\Phi))$$

holds and $\mathrm{ent}(\Phi)$ attains the minimal entropy among all homeomorphisms which are isotopic to Φ, see [5, Exposé 10]. We denote by $\mathrm{ent}(\phi)$ this characteristic number. Using the Euler characteristic $\chi(\Sigma)$, we define the *normalized dilatation* $\overline{\lambda}(\phi)$ and *normalized entropy* $\overline{\mathrm{ent}}(\phi)$ of ϕ by $\overline{\lambda}(\phi) = \lambda(\phi)^{|\chi(\Sigma)|}$ and $\overline{\mathrm{ent}}(\phi) = \log \overline{\lambda}(\phi) = |\chi(\Sigma)| \mathrm{ent}(\phi)$.

We recall the surjective homomorphism

$$\Gamma : B_n \to \mathcal{M}(D_n)$$

which sends the Artin generator σ_i for $i \in \{1, \cdots, n-1\}$ (see Figure 2(left)) to $\hat{t}_i$, where $\hat{t}_i$ is the mapping class which represents the positive half twist about the arc from the ith puncture to the $(i+1)$st puncture. The kernel of Γ is the center of B_n which is generated by the full twist $(\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$. By replacing the boundary of D_n with the $(n+1)$st puncture, the injective homomorphism from $\mathcal{M}(D_n)$ to $\mathcal{M}(\Sigma_{0,n+1})$ is induced. In the rest of the paper we regard an element of $\mathcal{M}(D_n)$ as an element of $\mathcal{M}(\Sigma_{0,n+1})$.

We say that a braid $b \in B_n$ is *pseudo-Anosov* if $\Gamma(b) \in \mathcal{M}(D_n)$ is pseudo-Anosov. In this case, $\mathrm{vol}(\Gamma(b))$ equals the hyperbolic volume of the exterior of the link $\overline{b}$ in S^3, where $\overline{b}$ is a union of the closed braid of b and the braid axis. Our convention of the orientation of $\overline{b}$ is given by Figure 2(right).

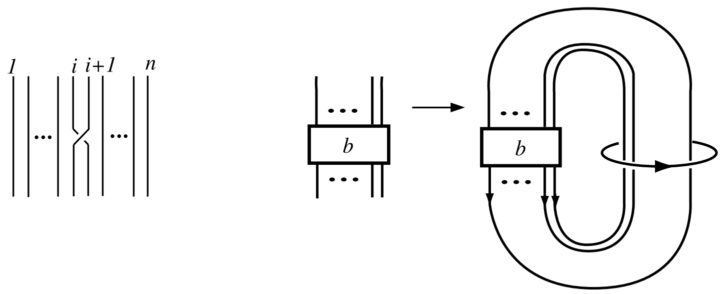


Figure 2: (left) generator σ_i. (right) braid $b \to$ braided link $\overline{b}$.

2.2 Roots of polynomials

Let $f(t)$ be an integral polynomial of degree d. The reciprocal of $f(t)$ is $f_*(t) = t^d f(1/t)$. We denote by $\lambda(f)$, the maximal absolute value of the roots of $f(t)$.

Let $R(t)$ be a monic integral polynomial and let $S(t)$ be an integral polynomial. We set

$$Q_{n,\pm}(t) = t^n R(t) \pm S(t)$$

for each integer $n \geq 1$. In case $S(t) = R_*(t)$, we call $Q_{n,\pm}(t) = t^n R(t) \pm R_*(t)$ the *Salem-Boyd polynomial associated to $R(t)$*.

Lemma 2.1. *Let $Q_{n,\pm}(t) = t^n R(t) \pm S(t)$. Suppose that $R(t)$ has a root outside the unit circle. Then, the roots of $Q_{n,\pm}(t)$ outside the unit circle converge to those of $R(t)$ counting multiplicity as n goes to ∞. In particular,*

$$\lambda(R) = \lim_{n\to\infty} \lambda(Q_{n,\pm}).$$

The proof of Lemma 2.1 can be found in [16, Lemma 2.5].

2.3 Hyperbolic surface bundle over the circle

Let M be an irreducible, atoroidal and oriented 3-manifold with boundary ∂M (possibly $\partial M = \emptyset$). Thurston discovered a norm function $X_T : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ (see [28]). In case M is a surface bundle over the circle, he described a relation between X_T and fibrations of M which we record as Theorem 2.2 below.

2.3.1 Thurston norm

The norm function $X_T : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ has the property that for any integral class $a \in H_2(M, \partial M; \mathbb{R})$,

$$X_T(a) = \min_F\{-\chi(F)\},$$

where the minimum is taken over all oriented surface F embedded in M, satisfying $a = [F]$, with no components of non-negative Euler characteristic. The surface F which realizes this minimum is called a *minimal representative* of a. For a rational class $a \in H_2(M, \partial M; \mathbb{R})$, take a rational number r so that ra is an integral class. Then $X_T(a)$ is defined to be

$$X_T(a) = \tfrac{1}{|r|} X_T(ra).$$

The function X_T defined on rational classes admits a unique continuous extension to $H_2(M, \partial M; \mathbb{R})$ which is linear on the ray though the origin. The unit ball $U = \{a \in H_2(M, \partial M; \mathbb{R}) \mid X_T(a) \leq 1\}$ is a compact, convex polyhedron [28].

The following notations are needed to describe how fibrations of M are related to the Thurston norm.

- A top dimensional face in the boundary ∂U of the unit ball U is denoted by Δ, and its open face is denoted by $int(\Delta)$.
- The open cone with the origin over Δ is denoted by $int(C_\Delta)$.

- The set of integral classes of $int(C_\Delta)$ is denoted by $int(C_\Delta(\mathbb{Z}))$, and the set of rational classes of $int(C_\Delta)$ is denoted by $int(C_\Delta(\mathbb{Q}))$.

Theorem 2.2 ([28]). *Suppose that M is a surface bundle over the circle and let F be a fiber. Then there exists a top dimensional face Δ satisfying the following.*

(1) $[F] \in int(C_\Delta(\mathbb{Z}))$.

(2) *For any $a \in int(C_\Delta(\mathbb{Z}))$, a minimal representative E of a is a fiber of fibrations of M.*

The face Δ in Theorem 2.2 is called the *fiber face*. For the fiber face Δ, it follows that $a \in int(C_\Delta(\mathbb{Z}))$ is a primitive integral class if and only if a minimal representative E of a is connected.

It is known that if $a_0 \in H_2(M, \partial M; \mathbb{Z})$ has a representative F which is a fiber of the fibration of M, then any incompressible surface which represents a_0 is isotopic to the fiber F, see [28]. In particular F is a minimal representative of a_0. Thus a minimal representative of a_0 is unique up to isotopy.

2.3.2 Entropy function

Suppose that M is a hyperbolic surface bundle over the circle. We fix a fiber face Δ for M. The *entropy function* $\mathrm{ent}(\cdot) : int(C_\Delta(\mathbb{Q})) \to \mathbb{R}$ introduced by Fried in [6] is defined as follows. The minimal representative F_a for $a \in int(C_\Delta(\mathbb{Z}))$ is a fiber of fibrations of M. Let $\Phi_a : F_a \to F_a$ be the monodromy. Since M is a hyperbolic manifold, the mapping class $\phi_a = [\Phi_a]$ must be pseudo-Anosov. The entropy $\mathrm{ent}(a)$ and dilatation $\lambda(a)$ are defined as the entropy and dilatation of ϕ_a, respectively. For a rational number r and an integral class a, the entropy $\mathrm{ent}(ra)$ is defined by $\frac{1}{|r|}\mathrm{ent}(a)$. Notice that $X_T(\cdot)\mathrm{ent}(\cdot) : int(C_\Delta(\mathbb{Q})) \to \mathbb{R}$ is constant on each ray through the origin. We call $X_T(a)\mathrm{ent}(a)$ and $X_T(a)\lambda(a)$ the *normalized entropy* and *normalized dilatation* of a.

We recall an important property of the entropy function proved by Matsumoto and independently McMullen.

Theorem 2.3 ([22, 24]). *The function $\frac{1}{\mathrm{ent}(\cdot)} : int(C_\Delta(\mathbb{Q})) \to \mathbb{R}$ is strictly concave.*

By Theorem 2.3, the function $\mathrm{ent}(\cdot)$ on $int(C_\Delta(\mathbb{Q}))$ admits a unique continuous extension to $\mathrm{ent}(\cdot) : int(C_\Delta) \to \mathbb{R}$.

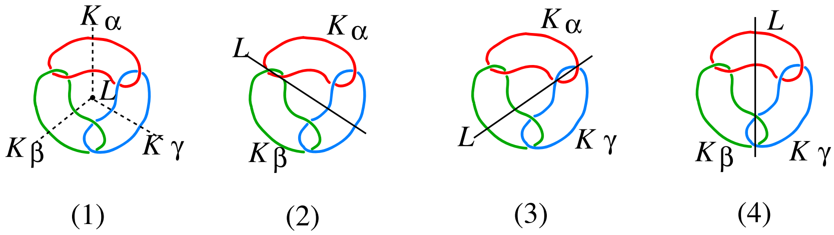


Figure 3: axis L for periodic map

Since $\mathrm{ent}(a)$ goes to ∞ as a goes to a point on the boundary $\partial\Delta$ (see [6]), Theorem 2.3 implies the normalized entropy function

$$\overline{\mathrm{ent}}(\cdot) = X_T(\cdot)\mathrm{ent}(\cdot) : int(C_\Delta) \to \mathbb{R}$$

has the minimum at a unique ray through the origin. In other words $\mathrm{ent}(\cdot)$ has the minimum at a unique point of $int(\Delta)$. The following question was posed by McMullen [24, p 542].

Problem 2.4. *On which ray in $int(C_\Delta)$ does $\overline{ent}(\cdot)$ attain the minimum? Is the minimum attained on a rational class of $int(\Delta)$?*

We solve this problem for the magic manifold in Section 3.2.

3 Magic manifold

3.1 Fiber face

Let L be one of the four lines in S^3 depicted in Figure 3(1),(2),(3) and (4). For each L, there exist an integer n and a periodic map $f : (S^3, \mathcal{C}_3) \to (S^3, \mathcal{C}_3)$ such that f is a $2\pi/n$ rotation with respect to L. Such symmetry of $\mathcal{C}_3$ is reflected in the shape of the Thurston unit ball. Let K_α, K_β and K_γ be the components of $\mathcal{C}_3$ such that K_α (resp. K_β, K_γ) bounds the oriented twice-punctured disk F_α (resp. F_β, F_γ) in M_{magic} whose normal direction is indicated as in Figure 4(right). Those oriented surfaces induce the orientation of $\mathcal{C}_3$. Let $\alpha = [F_\alpha]$, $\beta = [F_\beta]$, and $\gamma = [F_\gamma]$. In [28], Thurston computes the unit ball U which is the the parallelepiped with vertices $\pm\alpha = (\pm 1, 0, 0)$, $\pm\beta = (0, \pm 1, 0)$, $\pm\gamma = (0, 0, \pm 1)$, $\pm(\alpha + \beta + \gamma)$, see Figure 4(left). The set $\{\alpha, \beta, \gamma\}$ is a basis of $H_2(M_{\mathrm{magic}}, \partial M_{\mathrm{magic}}; \mathbb{Z})$.

The magic manifold is a surface bundle over the circle as we will see later. The symmetry of $\mathcal{C}_3$ tells us that every top dimensional face is a fiber

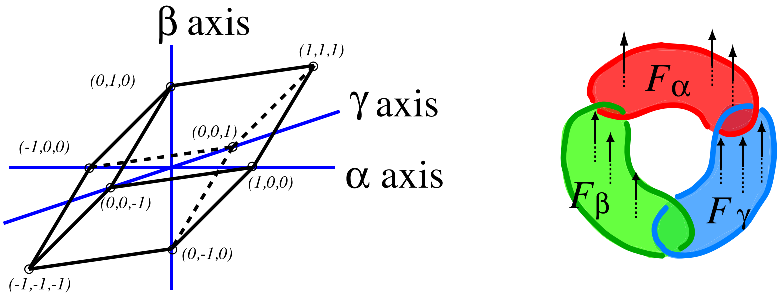


Figure 4: (left) Thurston unit ball. (right) F_α, F_β, F_γ. (arrows indicate the normal direction of oriented surfaces.)

face. We (arbitrarily) pick the shaded fiber face Δ as in Figure 5(left) with vertices $\alpha = (1,0,0)$, $\alpha+\beta+\gamma = (1,1,1)$, $\beta = (0,1,0)$ and $-\gamma = (0,0,-1)$. The open face $int(\Delta)$ is written by

$$int(\Delta) = \{x\alpha + y\beta + z\gamma \mid x + y - z = 1,\ x > 0,\ y > 0,\ x > z,\ y > z\}. \quad (3.1)$$

For $x\alpha + y\beta + z\gamma \in int(C_\Delta)$,

$$X_T(x\alpha + y\beta + z\gamma) = x + y - z.$$

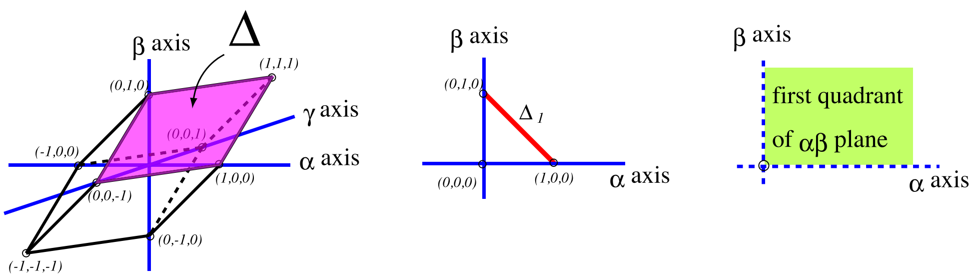


Figure 5: (left) fiber face Δ. (center) $\Delta_1 \subset \Delta$. (right) $C_{\Delta_1} \subset int(C_\Delta)$.

Let $\mathcal{N}(L)$ be the regular neighborhood of a link L in S^3. We denote the tori $\partial\mathcal{N}(K_\alpha)$, $\partial\mathcal{N}(K_\beta)$, $\partial\mathcal{N}(K_\gamma)$ by T_α, T_β, T_γ respectively. Let $x\alpha+y\beta+z\gamma$ be a primitive integral class in $int(C_\Delta)$. We denote by $F_{x\alpha+y\beta+z\gamma}$ or $F_{(x,y,z)}$, the minimal representative of $x\alpha + y\beta + z\gamma$. Let us set

$$\partial_\alpha F_{(x,y,z)} = \partial F_{(x,y,z)} \cap T_\alpha$$

which consists of the parallel simple closed curves on T_α. We define the subsets $\partial_\beta F_{(x,y,z)}$, $\partial_\gamma F_{(x,y,z)} \subset \partial F_{(x,y,z)}$ in the same manner. We denote by $\Phi_{(x,y,z)} : F_{(x,y,z)} \to F_{(x,y,z)}$, the monodromy on a fiber $F_{(x,y,z)}$. It is clear that $\Phi_{(x,y,z)}$ permutes elements of each of the sets $\partial_\alpha F_{(x,y,z)}$, $\partial_\beta F_{(x,y,z)}$ and $\partial_\gamma F_{(x,y,z)}$ cyclically. Let $\mathcal{F}_{(x,y,z)}$ be the stable foliation for the pseudo-Anosov $\Phi_{(x,y,z)}$.

Lemma 3.1. *Let $x\alpha + y\beta + z\gamma \in int(C_\Delta)$ be a primitive integral class. The number of the boundary components $\sharp(\partial F_{(x,y,z)})$ is equal to the sum of the three greatest common divisors*

$$\gcd(x, y+z) + \gcd(y, z+x) + \gcd(z, x+y),$$

where $\gcd(0,w)$ *is defined by* $|w|$. *More precisely*

(1) $\sharp(\partial_\alpha F_{(x,y,z)}) = \gcd(x, y+z)$,

(2) $\sharp(\partial_\beta F_{(x,y,z)}) = \gcd(y, z+x)$,

(3) $\sharp(\partial_\gamma F_{(x,y,z)}) = \gcd(z, x+y)$.

Proof. We prove (1). The proof of (2),(3) is similar. We have the meridian and longitude basis $\{m_\alpha, \ell_\alpha\}$ for T_α. Similarly we have the bases $\{m_\beta, \ell_\beta\}$ for T_β and $\{m_\gamma, \ell_\gamma\}$ for T_γ. We consider the long exact sequence of the homology groups of the pair $(M_{\text{magic}}, \partial M_{\text{magic}})$. The boundary map is given by

$$\begin{array}{rcl} \partial_* : H_2(M_{\text{magic}}, \partial M_{\text{magic}}; \mathbb{R}) & \to & H_1(\partial M_{\text{magic}}; \mathbb{R}), \\ \alpha & \mapsto & \ell_\alpha - m_\beta - m_\gamma, \\ \beta & \mapsto & \ell_\beta - m_\gamma - m_\alpha, \\ \gamma & \mapsto & \ell_\gamma - m_\alpha - m_\beta. \end{array}$$

Hence

$$\partial_*(x\alpha + y\beta + z\gamma) = x\ell_\alpha - (y+z)m_\alpha + y\ell_\beta - (z+x)m_\beta + z\ell_\gamma - (x+y)m_\gamma. \quad (3.2)$$

Since $F_{(x,y,z)}$ is the minimal representative, the set $\partial_\alpha F_{(x,y,z)}$ is a union of oriented parallel simple closed curves on T_α whose homology class equals $x\ell_\alpha - (y+z)m_\alpha \in H_1(T_\alpha; \mathbb{R})$, see (3.2). Thus the number of the components of $\partial_\alpha F_{(x,y,z)}$ equals $\gcd(x, y+z)$. $\square$

In Section 4.1, we will use the following to see $\lambda(T'_{m,p})$ is equal to $\lambda(T_{m,p})$.

Lemma 3.2 ([17]). *Let $x\alpha+y\beta+z\gamma$ be as in Lemma 3.1. The stable foliation $\mathcal{F}_{(x,y,z)}$ has*

(1) $\frac{x}{\gcd(x,y+z)}$ *prongs at each element of $\partial_\alpha F_{(x,y,z)}$,*

(2) $\frac{y}{\gcd(y,x+z)}$ *prongs at each element of $\partial_\beta F_{(x,y,z)}$,*

(3) $\frac{x+y-2z}{\gcd(z,x+y)}$ *prongs at each element of $\partial_\gamma F_{(x,y,z)}$, and*

(4) *no singularities in the interior of $F_{(x,y,z)}$.*

3.2 Teichmüler polynomial

The Teichmüler polynomial, defined in [24], can be used to computer the entropy function. We compute the Teichmüler polynomial $P = P_\Delta$ with respect to Δ.

Remark 3.3. *Oertel obtained similar polynomial with respect to each fiber face as the Teichmüler polynomial, see [25].*

A fiber $F = F_{(1,1,0)}$ is homeomorphic to a sphere with 4 boundary components. We now see that the monodromy $\Phi = \Phi_{(1,1,0)}$ on F is represented by the 3-braid $b = \sigma_2\sigma_1^{-1}\sigma_2$. A homeomorphism $H : S^3 \setminus \mathcal{N}(\mathcal{C}_3) \to S^3 \setminus \mathcal{N}(\overline{b})$ is given as follows. The link illustrated in Figure 6(left) is isotopic to $\mathcal{C}_3$. We consider the exterior $S^3 \setminus \mathcal{C}_3$ and we open the twice-punctured disk F_α bounded by K_α. Let F'_α and F''_α be the resulting twice-punctured disks obtained from F_α. Reglue F'_α and F''_α by twisting one of the disks by 360 degrees in the clockwise direction. Then we obtain the braided link $\overline{b}$ whose exterior $S^3 \setminus \overline{b}$ is homeomorphic to $S^3 \setminus \mathcal{C}_3$, see Figure 6.

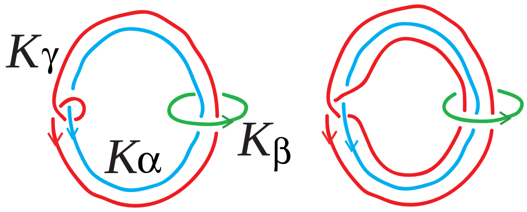


Figure 6: (left) $\mathcal{C}_3$. (right) $\overline{b}$. (this figure explains how to obtain H.)

Let u be the meridian of the component of $\overline{b}$ which is the braid axis. Let t_2 (resp. t_1) be the meridian of the component of $\overline{b}$ which is the closure of the second strand of b (resp. which is the closure of the rest of the strand of

b). By using the argument in [24, Section 11], one sees that the Teichmüler polynomial $P(t_1, t_2, u)$ is given by

$$P(t_1, t_2, u) = \det(uI - \sigma_2^{-1}(t_2)\sigma_1(t_1)\sigma_2^{-1}(t_1)),$$

where $\sigma_1(t) = \begin{pmatrix} t & t \\ 0 & 1 \end{pmatrix}$ and $\sigma_2^{-1}(t) = \begin{pmatrix} 1 & 0 \\ t^{-1} & t^{-1} \end{pmatrix}$. (Note that our convention of the sign of braids is different from the one in [24].) Hence

$$P(t_1, t_2, u) = \tfrac{1}{t_2} - u - t_1 u - \tfrac{u}{t_2} - \tfrac{u}{t_1 t_2} + u^2.$$

Now we transform this to the polynomial using our basis. Let $\{\alpha^*, \beta^*, \gamma^*\}$ be the dual basis for $H_1(M_{\text{magic}}; \mathbb{Z})$. We set $s_1 = \alpha^*$, $s_2 = \beta^*$, $s_3 = \gamma^*$. By the construction of the homeomorphism H, one observes that $t_1 = s_3$, $t_2 = s_1 s_2^{-1} s_3^{-1}$ and $u = s_2$. We obtain

$$P(s_1, s_2, s_3) = -s_1 - s_2 + s_3 + s_1 s_2 - s_1 s_3 - s_2 s_3.$$

Theorem 3.4. *The dilatation of $x\alpha + y\beta + z\gamma \in int(C_\Delta(\mathbb{Z}))$ is the largest real root of*

$$P(t^x, t^y, t^z) = -t^x - t^y + t^z + t^{x+y} - t^{x+z} - t^{y+z}.$$

In particular, the dilatation of $x\alpha + y\beta \in int(C_\Delta(\mathbb{Z}))$ is the largest real root of $t^{x+y} - 2(t^x + t^y) + 1$.

Proof. We identify $H_2(M_{\text{magic}}, \partial M_{\text{magic}})$ with $H^1(M_{\text{magic}}, \partial M_{\text{magic}})$. By [24, Section 1], the dilatation of $a = x\alpha + y\beta + z\gamma$ is equal to the largest real root of

$$P(t^{a(\alpha^*)}, t^{a(\beta^*)}, t^{a(\gamma^*)}) = P(t^x, t^y, t^z) = -t^x - t^y + t^z + t^{x+y} - t^{x+z} - t^{y+z}.$$

This completes the proof. $\square$

Since

$$P(t^x, t^y, t^z) = t^z(t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1),$$

the dilatation of $x\alpha + y\beta + z\gamma$ is the largest real root of

$$f_{(x,y,z)}(t) = t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1.$$

Lemma 3.5. *For $x \geq y > 0$ and $z > 0$,*

$$\lambda(x\alpha + y\beta - z\gamma) = \lambda((x+z)\alpha + (y+z)\beta + z\gamma).$$

Proof. $f_{(x,y,-z)}(t) = f_{(x+z,y+z,z)}(t) = t^{x+y+z} - t^{x+z} - t^{y+z} - t^x - t^y + 1$. $\square$

Theorem 3.6. *The homology class $\alpha + \beta$ realizes the minimal normalized entropy with respect to Δ, i.e, the ray through $\alpha + \beta$ attains the minimum of $\overline{\mathrm{ent}}(\cdot) = X_T(\cdot)\mathrm{ent}(\cdot) : int(C_\Delta) \to \mathbb{R}$.*

Proof. Clearly $x\alpha + y\beta + z\gamma$ is in $int(C_\Delta)$ if and only if $y\alpha + x\beta + z\gamma$ is in $int(C_\Delta)$. The equality $\overline{\mathrm{ent}}(x\alpha + y\beta + z\gamma) = \overline{\mathrm{ent}}(y\alpha + x\beta + z\gamma)$ holds, since these classes have the same entropy and the same Thurston norm. Thus if the minimum of $\overline{\mathrm{ent}}$ is realized by the ray though $x\alpha+y\beta+z\gamma$, then x must equal y.

On the other hand, $x\alpha + x\beta + \gamma$ and $(x-1)\alpha + (x-1)\beta - \gamma$ have the same Thurston norm $2x - 1$ if $x > 1$. By Lemma 3.5, they have the same entropy. Thus,

$$\overline{\mathrm{ent}}(x\alpha + x\beta + \gamma) = \overline{\mathrm{ent}}((x-1)\alpha + (x-1)\beta - \gamma).$$

Since $\overline{\mathrm{ent}}$ is constant on each ray, we have

$$\begin{aligned} \overline{\mathrm{ent}}(x\alpha + x\beta + \gamma) &= \overline{\mathrm{ent}}(\alpha + \beta + x^{-1}\gamma), \\ \overline{\mathrm{ent}}((x-1)\alpha + (x-1)\beta - \gamma) &= \overline{\mathrm{ent}}(\alpha + \beta - (x-1)^{-1}\gamma). \end{aligned}$$

The minimal ray does not pass through both $\alpha+\beta+x^{-1}\gamma$ and $\alpha+\beta-(x-1)^{-1}\gamma$, because the minimum is realized by a unique ray. Since $x(> 1)$ is arbitrary, the desired ray must pass through $\alpha + \beta$. $\square$

3.3 Fiber surface of genus 0

Let $x\alpha + y\beta + z\gamma \in int(C_\Delta)$ be an integral homology class. Recall that $F_{(x,y,z)}$ is connected if and only if $x\alpha + y\beta + z\gamma$ is primitive. Since $\{\alpha, \beta, \gamma\}$ is a basis of $H_2(M_{\mathrm{magic}}, \partial M_{\mathrm{magic}}; \mathbb{Z})$, we see that $x\alpha+y\beta+z\gamma$ is primitive if and only if $\gcd(x, y, z) = 1$, where $\gcd(x, y, 0)$ is defined to be $\gcd(x, y)$. The topological type of $F_{(x,y,z)}$ can be determined from the Thurston norm and Lemma 3.1. In this section, we find all the homology classes whose minimal representatives are connected and of genus 0.

By (3.1), if $x \geq y$, then $x\alpha+y\beta+z\gamma$ is in $int(C_\Delta)$ if and only if $x \geq y > 0$ and $y > z$. In this section, we consider those classes for simplicity.

Lemma 3.7. *Let $x, y, z \in \mathbb{Z}$. Suppose that $x \geq y > 0$, $y > z$ and $\gcd(x, y, z) = 1$. Then*

$$\gcd(x, y+z) + \gcd(y, z+x) + \gcd(z, x+y) - x - y + z - 2 \leq 0. \quad (3.3)$$

If (x, y, z) satisfies the equality in (3.3), then $z \geq 0$.

Proof. Note that

$$-\chi(F_{(x,y,z)}) = -(2 - 2g - \sharp(\partial F_{(x,y,z)})) = x + y - z,$$

where g denotes the genus of $F_{(x,y,z)}$. Hence

$$g = \tfrac{x+y-z+2-\sharp(\partial F_{(x,y,z)})}{2} \geq 0. \tag{3.4}$$

By substituting $\sharp(\partial F_{(x,y,z)}) = \gcd(x, y + z) + \gcd(y, x + z) + \gcd(z, x + y)$ for (3.4), we have the desired inequality. Suppose that $x \geq y > 0 > z = -z'$ $(z' > 0)$. Then

$$\begin{aligned} & \gcd(x, y - z') + \gcd(y, -z' + x) + \gcd(-z', x + y) - x - y - z' - 2 \\ \leq \ & x + y + z' - x - y - z' - 2 \\ = \ & -2. \end{aligned}$$

This completes the proof. □

Proposition 3.8. *Let $x, y, z \in \mathbb{Z}$. Suppose that $x \geq y > z \geq 0$ and $\gcd(x, y, z) = 1$. Then the equality of (3.3) holds for (x, y, z) if and only if (x, y, z) is either*

(1) *$z = 0$ and $\gcd(x, y) = 1$,*

(2) *$(x, y, z) = (n + 1, n, n - 1)$ for $n \not\equiv 0 \pmod 3$ and $n \geq 2$, or*

(3) *$(x, y, z) = (2n + 1, n + 1, n)$ for $n \geq 1$.*

The following proof was shown to the authors by Shigeki Akiyama.

Proof. The equality of (3.3) holds for $(x, y, 0)$ if and only if $\gcd(x, y) = 1$, and hence we may suppose that $x \geq y > z > 0$ by Lemma 3.7. It is easy to see that if (x, y, z) is of either type (2) or type (3), then it satisfies the equality of (3.3). To prove the "only if" part, we first show:

Claim 3.9. *Let $x, y, z \in \mathbb{N}$. Suppose that $\gcd(x, y, z) = 1$. Then*

$$\{\gcd(N, x), \gcd(N, y), \gcd(N, z)\}$$

is pairwise coprime, where $N = x + y + z$.

Proof of Claim 3.9. We set $\gcd(\gcd(N, x), \gcd(N, y)) = k$. Then k is a divisor of three integers N, x and y. It is also a divisor of $z(= N - x - y)$. Since $\gcd(x, y, z) = 1$, the integer k must be 1. This completes the proof the claim.

Notice that the inequality of (3.3) is equivalent to the inequality

$$N - \gcd(N,x) - \gcd(N,y) - \gcd(N,z) \geq 2z - 2. \tag{3.5}$$

For all $N \leq 79$, one can check that the statement of Proposition 3.8 is valid. We may suppose that $N \geq 80$. Since $x > \frac{N}{3}$ and $z < \frac{N}{3}$, we have $x \geq \frac{N+1}{3}$ and $z \leq \frac{N-1}{3}$. Hence $x \geq 27$. Let p, q and r be natural numbers so that $p > q > r$ and

$$\{p, q, r\} = \{\gcd(N,x), \gcd(N,y), \gcd(N,z)\}.$$

By Claim 3.9, $\{p, q, r\}$ is pairwise coprime, and p, q and r are divisors of N. Therefore $pqr \leq N$. This shows that

$$\tfrac{N-p-q-r}{N} \geq \tfrac{pqr-p-q-r}{pqr} = 1 - \tfrac{1}{qr} - \tfrac{1}{p}(\tfrac{1}{q} + \tfrac{1}{r}). \tag{3.6}$$

If $r \geq 2$, then $q \geq 3$ and $p \geq 5$. Hence

$$N - p - q - r \geq N(1 - \tfrac{1}{qr} - \tfrac{1}{p}(\tfrac{1}{q} + \tfrac{1}{r})) = \tfrac{5}{6}N(1 - \tfrac{1}{p}) \geq \tfrac{2}{3}N > 2z.$$

Thus, no (x, y, z) satisfies the equality of (3.5) in this case.

We may suppose that $r = 1$. If $q \geq 4$, then

$$\tfrac{N-p-q-r}{N} \geq \tfrac{3}{4} - \tfrac{5}{4p}$$

by (3.6). Since $z \leq \frac{N-1}{3}$, we obtain

$$N - p - q - r - 2z + 2 \geq N(\tfrac{3}{4} - \tfrac{5}{4p}) - \tfrac{2N-8}{3} = \tfrac{8}{3} + \tfrac{N(p-15)}{12p}.$$

If $p > 15$, then $\frac{8}{3} + \frac{N(p-15)}{12p} > 0$, which implies that no (x, y, z) satisfies the equality of (3.5) in this case. If $p \leq 14$, then $q \leq 13$. We have $N - 2z = x + y - z > x > 26 = -1 + 14 + 13$. Thus

$$N - p - q - 1 \geq N - 14 - 13 - 1 > 2z - 2,$$

which implies that no (x, y, z) satisfies the equality of (3.5) in this case.

We may suppose that $q \leq 3$. It is enough to consider the equality of (3.5) in case $(q, r) = (3, 1), (2, 1)$ and $(1, 1)$. Take $w \in \{x, y, z\}$ so that $p = \gcd(N, w)$.

(1) Case $(q, r) = (2, 1)$ or $(3, 1)$.

Then $N - \gcd(N, w) - 2 - 1 = 2z - 2$ or $N - \gcd(N, w) - 3 - 1 = 2z - 2$. We set $\gcd(N, w) = \frac{N}{k}$ $(k > 1)$. Then $N(1 - \frac{1}{k}) \leq \frac{2(N-1)}{3} + 2$. Since we assume that $N \geq 80 > 16$, k must be 2 or 3.

(i) Case $k = 2$.

Then $x = \frac{N}{2}$. If $(q, r) = (2, 1)$, then $(x, y, z) = (\frac{N}{2}, \frac{N+2}{4}, \frac{N-2}{4})$. We set $n = \frac{N-2}{4}$. We obtain $(x, y, z) = (2n + 1, n + 1, n)$, and such (x, y, z) satisfies the equality (3.5). If $(q, r) = (3, 1)$, then $(x, y, z) = (\frac{N}{2}, \frac{N+4}{4}, \frac{N-4}{4})$. In this case, $\gcd(N, \frac{N\pm4}{4}) = \gcd(4, \frac{N\pm4}{4})$, which is a divisor of 4. This does not occur since $(q, r) = (3, 1)$.

(ii) Case $k = 3$.

If $(q, r) = (2, 1)$, then $z = \frac{N}{3} - \frac{1}{2}$, which can not be an integer. Let $(q, r) = (3, 1)$. Then $z = \frac{N}{3} - 1$. Since $\gcd(N, w) = \frac{N}{3}$, we see that $w = \frac{N}{3}$ or $\frac{2N}{3}$. If $w = \frac{2N}{3}$, then $(x, y, z) = (\frac{2N}{3}, 1, \frac{N}{3} - 1)$. This is a contradiction since $y > z$. If $w = \frac{N}{3}$, then $(x, y, z) = (\frac{N}{3}+1, \frac{N}{3}, \frac{N}{3}-1)$. We set $n = \frac{N}{3}$. Then $(x, y, z) = (n+1, n, n-1)$. If $n \equiv 0 \pmod 3$, then $\gcd(3n, n\pm1) = 1$, which is a contradiction since $(q, r) = (3, 1)$. Otherwise, such (x, y, z) satisfies the equality of (3.5).

(2) Case $(q, r) = (1, 1)$.

Then $N - \gcd(N, w) - 1 - 1 = 2z - 2$. We have $N - \gcd(N, w) = 2z < \frac{2N}{3}$. This implies that $\gcd(N, w) = \frac{N}{2}$. Thus, $(x, y, z) = (\frac{N}{2}, \frac{N}{4}, \frac{N}{4})$, which does not occur since $y > z$.

This completes the proof of Proposition 3.8. □

By Proposition 3.8 and Lemma 3.1, we immediately obtain the following which characterizes integral homology classes in $int(C_\Delta)$ whose minimal representatives are spheres with punctures.

Corollary 3.10. *Let $x\alpha + y\beta + z\gamma \in int(C_\Delta)$ be an integral homology class. Suppose that $x \geq y$ and $\gcd(x, y, z) = 1$. Then the genus of $F_{(x,y,z)}$ is 0 if and only if (x, y, z) satisfies either*

(1) $z = 0$ *and* $\gcd(x, y) = 1$,

(2) $(x, y, z) = (n + 1, n, n - 1)$ *for* $n \not\equiv 0 \pmod 3$ *and* $n \geq 2$, *or*

(3) $(x, y, z) = (2n + 1, n + 1, n)$ *for* $n \geq 1$.

In case (1),

$$\sharp(\partial_\alpha F_{(x,y,z)}) = \sharp(\partial_\beta F_{(x,y,z)}) = 1 \text{ and } \sharp(\partial_\gamma F_{(x,y,z)}) = x + y.$$

In case (2),

$$\{\sharp(\partial_\alpha F_{(x,y,z)}), \sharp(\partial_\gamma F_{(x,y,z)})\} = \{1,3\} \text{ and } \sharp(\partial_\beta F_{(x,y,z)}) = n.$$

In case (3),

$$\sharp(\partial_\alpha F_{(x,y,z)}) = 2n+1 \text{ and } \{\sharp(\partial_\beta F_{(x,y,z)}), \sharp(\partial_\gamma F_{(x,y,z)})\} = \{1,2\}.$$

Corollary 3.10 implies that each mapping class $\phi_{(x,y,z)} = [\Phi_{(x,y,z)}]$ can be described by a braid since $\Phi_{(x,y,z)}$ fixes one boundary component of $F_{(x,y,z)}$.

Theorem 3.11.

(1) $\lim\limits_{n,m\to\infty} \overline{\mathrm{ent}}(n\alpha + m\beta) = 2\log(2+\sqrt{3})$ *if* $\lim\limits_{n,m\to\infty} \frac{n}{m} = 1.$

(2) $\lim\limits_{n\to\infty} \overline{\mathrm{ent}}((n+1)\alpha + n\beta + (n-1)\gamma) = \infty.$

(3) $\lim\limits_{n\to\infty} \overline{\mathrm{ent}}((2n+1)\alpha + (n+1)\beta + n\gamma) = \infty.$

Proof. (1) We see that $\overline{\mathrm{ent}}(n\alpha+m\beta)$ goes to $\overline{\mathrm{ent}}(\alpha+\beta)$ as m,n go to ∞ with the condition $\lim\limits_{n,m\to\infty} \frac{n}{m} = 1$. By Theorem 3.4, $\overline{\mathrm{ent}}(\alpha+\beta) = 2\log(2+\sqrt{3})$.

(2) We see that

$$\lim_{n\to\infty} \overline{\mathrm{ent}}(\tfrac{(n+1)\alpha+n\beta+(n-1)\gamma}{n}) = \overline{\mathrm{ent}}(\alpha+\beta+\gamma) = \infty$$

since $\alpha+\beta+\gamma \in \partial\Delta$. The proof of (3) is similar to the proof of (2). $\square$

3.4 Proposition 3.12

Let $\mathcal{H}_n$ be the set of homology classes $x\alpha + y\beta + z\gamma \in int(C_\Delta(\mathbb{Z}))$, $x \geq y$ such that their minimal representatives are n-punctured spheres. By Corollary 3.10, one can determine elements of $\mathcal{H}_n$. This section is devoted to prove:

Proposition 3.12. *The homology class which achieves the minimal dilatation among elements of $\mathcal{H}_n$ is as follows.*

(1) $k\alpha + (k-1)\beta$ *in case* $n = 2k+1$ *for* $k \geq 2$.

(2) $3\alpha + 2\beta + \gamma$ *in case* $n = 6$ *and* $(2k+1)\alpha + (2k-1)\beta$ *in case* $n = 4k+2$ *for* $k \geq 2$.

(3a) $\alpha + \beta$ *in case* $n = 4$ *and* $(4k+3)\alpha + (4k-1)\beta$ *in case* $n = 8k+4$ *for* $k \geq 1$.

(3b) $5\alpha+3\beta+2\gamma$ *in case* $n = 8$ *and* $(4k+5)\alpha+(4k+1)\beta$ *in case* $n = 8k+8$ *for* $k \geq 1$.

Lemma 3.13. *Let* $m > n > 0$.

(1) $\overline{\mathrm{ent}}((m+1)\alpha + m\beta + (m-1)\gamma) > \overline{\mathrm{ent}}((n+1)\alpha + n\beta + (n-1)\gamma)$.

(2) $\overline{\mathrm{ent}}((2m+1)\alpha + (m+1)\beta + m\gamma) > \overline{\mathrm{ent}}((2n+1)\alpha + (n+1)\beta + n\gamma)$.

Proof. Let us consider homology classes $x\alpha + y\beta + z\gamma$ with $(x, y, z) = (\frac{n+1}{n+2}, \frac{n}{n+2}, \frac{n-1}{n+2})$ for each $n > 0$. These classes are in the open face $int(\Delta)$ and pass through the line $x = \frac{1}{3}t + 1$, $y = \frac{2}{3}t + 1$, $z = t + 1$. Note that

$$\overline{\mathrm{ent}}((n+1)\alpha + n\beta + (n-1)\gamma) = \mathrm{ent}((\tfrac{n+1}{n+2})\alpha + (\tfrac{n}{n+2})\beta + (\tfrac{n-1}{n+2})\gamma),$$

and it goes to ∞ as n goes to ∞ by Proposition 3.11(2). We have

$$\frac{1}{3\mathrm{ent}(2\alpha+\beta+0\gamma)} \approx \frac{1}{2.887} > \frac{1}{4\mathrm{ent}(3\alpha+2\beta+\gamma)} \approx \frac{1}{2.931}.$$

Since $\frac{1}{\mathrm{ent}(\cdot)} : int(C_\Delta(\mathbb{Q})) \to \mathbb{R}$ is a strictly concave function, we have for all $m > n \geq 3$,

$$\frac{1}{4\mathrm{ent}(3\alpha+2\beta+\gamma)} > \frac{1}{(n+2)\mathrm{ent}((n+1)\alpha+n\beta+(n-1)\gamma)} > \frac{1}{(m+2)\mathrm{ent}((m+1)\alpha+m\beta+(m-1)\gamma)}.$$

This implies (1). The proof of (2) is similar. □

We set

- $\Delta_1 = \{x\alpha + y\beta \mid x + y = 1,\ x > 0,\ y > 0\} \subset int(\Delta)$ (see Figure 5(center)).
- $C_{\Delta_1} = \{x\alpha + y\beta \mid x > 0,\ y > 0\}$ (see Figure 5(right)).
- $C_{\Delta_1}(\mathbb{Z}) = \{a \mid a \in C_{\Delta_1}$ is an integral class$\}$.
- $C_{\Delta_1}(\mathbb{Q}) = \{a \mid a \in C_{\Delta_1}$ is a rational class$\}$.

Lemma 3.14.

(1) *For* $x, y \in \mathbb{N}$ *such that* $\gcd(x, y) = 1$, *the monodromy for* $x\alpha + y\beta$ *is conjugate to the inverse of the monodromy for* $y\alpha + x\beta$.

(2) *For* $x, y > 0$, *we have* $\mathrm{ent}(x\alpha + y\beta) = \mathrm{ent}(y\alpha + x\beta)$.

Proof. The existence of a π rotation $f : (S^3, \mathcal{C}_3) \to (S^3, \mathcal{C}_3)$ with respect to the line $\mathtt{L}$ of Figure 3(2) implies that the monodromy for $x\alpha+y\beta$ is conjugate to the one for $-y\alpha - x\beta$. This implies (1). The claim (2) is immediate from the expression for $f_{(x,y,z)}(t)$. $\square$

Fixing $n \in \mathbb{N}$, we set

- $\Delta_n = \{x\alpha + y\beta \mid x > 0,\ y > 0,\ x + y = n\} \subset C_{\Delta_1}$.
- $\Delta_n(\mathbb{N}) = \{x\alpha + y\beta \in \Delta_n \mid x, y \in \mathbb{N},\ \gcd(x, y) = 1\}$.

Lemma 3.15. $\mathrm{ent}\big(\frac{n\alpha+n\beta}{2}\big) = \min\{\mathrm{ent}(a) \mid a \in \Delta_n\}$ *for each* $n \in \mathbb{N}$.

Proof. Recall that the restriction of $\frac{1}{\mathrm{ent}(\cdot)}$ on Δ_n is strictly concave. Note that $\mathrm{ent}(a) \to \infty$ as $a \to n\alpha$ or $n\beta$. Thus $\mathrm{ent}(\cdot)|_{\Delta_n} : \Delta_n \to \mathbb{R}$ has the unique minimum. By Lemma 3.14(2), $\frac{n\alpha+n\beta}{2}$ attains the minimum. $\square$

Lemma 3.16. *For* $m \geq 3$, $\min\{\mathrm{ent}(a) \mid a \in \Delta_{m-1}(\mathbb{N})\}$ *is realized by:*

(1) $(k-1)\alpha + k\beta$ *and* $k\alpha + (k-1)\beta$ *in case* $m = 2k$.

(2) $(2k-1)\alpha + (2k+1)\beta$ *and* $(2k+1)\alpha + (2k-1)\beta$ *in case* $m = 4k+1$.

(3a) $\alpha+\beta$ *in case* $m = 3$, *and* $(4k-1)\alpha+(4k+3)\beta$ *and* $(4k+3)\alpha+(4k-1)\beta$ *in case* $m = 8k+3$ *for* $k \geq 1$.

(3b) $(4k+1)\alpha + (4k+5)\beta$ *and* $(4k+5)\alpha + (4k+1)\beta$ *in case* $m = 8k+7$.

Proof. The concavity of $\frac{1}{\mathrm{ent}(\cdot)}|_{\Delta_n} : \Delta_n \to \mathbb{R}$ and Lemma 3.15 tell us that if $|x - y| < |x' - y'|$ for $x\alpha + y\beta$, $x'\alpha + y'\beta \in \Delta_n$, then

$$\mathrm{ent}(x\alpha + y\beta) < \mathrm{ent}(x'\alpha + y'\beta). \tag{3.7}$$

This implies that the minimal entropy among elements of $\Delta_{m-1}(\mathbb{N})$ is realized by $(k-1)\alpha + k\beta$ or $k\alpha + (k-1)\beta$ if $m = 2k$. (See Figure 7.) The proof for other cases can be shown in a similar way. $\square$

Lemma 3.17. *For* $k' > k > 0$, *we have the following.*

(1) $\overline{\mathrm{ent}}(k\alpha + (k+c)\beta) > \overline{\mathrm{ent}}(k'\alpha + (k'+c)\beta)$.

(2) $\mathrm{ent}(k\alpha + (k+c)\beta) > \mathrm{ent}(k'\alpha + (k'+c)\beta)$.

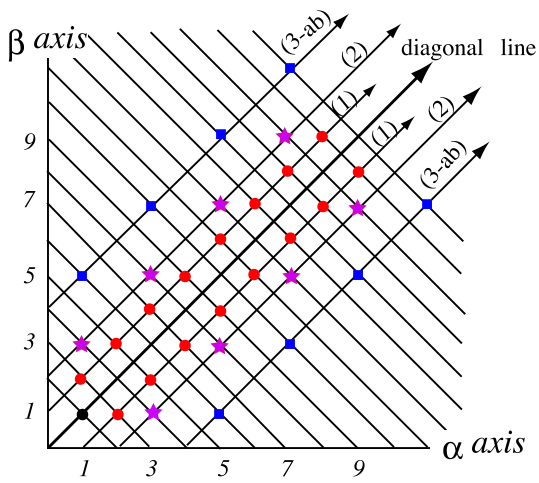


Figure 7: the first quadrant of $\alpha\beta$ plane. homology classes in (1) (resp. (2), (3a, 3b) of Prop. 3.29) lie on the lines $\xrightarrow{(1)}$ (resp. $\xrightarrow{(2)}$ and $\xrightarrow{(3-ab)}$).

Proof. For any $k > 0$, we have

$$\overline{\mathrm{ent}}(\tfrac{k\alpha+(k+c)\beta}{2k+c}) = \mathrm{ent}(\tfrac{k\alpha+(k+c)\beta}{2k+c}) = (2k+c)\mathrm{ent}(k\alpha+(k+c)\beta).$$

If $0 < k < k'$, then $\frac{c}{2k+c} > \frac{c}{2k'+c}$. By (3.7), we see that

$$\mathrm{ent}(\tfrac{k\alpha+(k+c)\beta}{2k+c}) > \mathrm{ent}(\tfrac{k'\alpha+(k'+c)\beta}{2k'+c}).$$

This implies (1). By (1),

$$\begin{aligned} & \mathrm{ent}(\tfrac{k\alpha+(k+c)\beta}{2k+c}) = (2k+c)\mathrm{ent}(k\alpha+(k+c)\beta) \\ > \ & \mathrm{ent}(\tfrac{k'\alpha+(k'+c)\beta}{2k'+c}) = (2k'+c)\mathrm{ent}(k'\alpha+(k'+c)\beta) \\ > \ & (2k+c)\mathrm{ent}(k'\alpha+(k'+c)\beta). \end{aligned}$$

Thus, $\mathrm{ent}(k\alpha+(k+c)\beta) > \mathrm{ent}(k'\alpha+(k'+c)\beta)$. This completes the proof of (2). □

Proof of Proposition 3.12. (1) We consider the case $n = 2k+1$. For $k = 2$, we see that $\mathcal{H}_5 = \{2\alpha+\beta\}$. If $k \neq 2$ and $2k \equiv 0 \pmod 3$, $\mathcal{H}_{2k+1}$ is the set of homology classes of type (1) of Corollary 3.10, that is

$$\mathcal{H}_{2k+1} = \{x\alpha + y\beta \mid x\alpha+y\beta \in \Delta_{2k-1}(\mathbb{N}),\ x \geq y\}.$$

In this case, $k\alpha + (k-1)\beta$ reaches the minimal entropy among elements of $\mathcal{H}_{2k+1}$ by Lemma 3.16(1). Otherwise (i.e, $2k \not\equiv 0 \pmod 3$), $\mathcal{H}_{2k+1}$ is the union of homology classes of type (1) and (2) of Corollary 3.10:

$$\{(2k-2)\alpha+(2k-3)\beta+(2k-4)\gamma\}\cup\{x\alpha+y\beta \mid x\alpha+y\beta \in \Delta_{2k-1}(\mathbb{N}),\ x \geq y\}.$$

One needs to compare the entropy for $(2k-2)\alpha+(2k-3)\beta+(2k-4)\gamma$ with the one for $k\alpha + (k-1)\beta$. In case $k = 4$,

$$\lambda(4\alpha + 3\beta) \approx 1.46557 < \lambda(6\alpha + 5\beta + 4\gamma) \approx 1.72208.$$

By Lemmas 3.17 and 3.13, for $k > 4$, we have

$$\begin{array}{rl} & (2k-1)\mathrm{ent}(k\alpha + (k-1)\beta) \\ \leq & 7\mathrm{ent}(4\alpha + 3\beta) \\ < & 7\mathrm{ent}(6\alpha + 5\beta + 4\gamma) \\ < & (2k-1)\mathrm{ent}((2k-2)\alpha + (2k-3)\beta + (2k-4)\gamma). \end{array}$$

Thus, $\mathrm{ent}(k\alpha + (k-1)\beta) < \mathrm{ent}((2k-2)\alpha + (2k-3)\beta + (2k-4)\gamma)$. This completes the proof.

(2) Let us consider the case $n = 4k+2$. For $k = 1$, $\mathcal{H}_6 = \{3\alpha+\beta, 3\alpha+2\beta+\gamma\}$. We have

$$\lambda(3\alpha + 2\beta + \gamma) \approx 2.08102 < \lambda(3\alpha + \beta) \approx 2.29663.$$

For $k = 2$, $\mathcal{H}_{10} = \{7\alpha + 4\beta + 3\gamma, 5\alpha + 3\beta, 7\alpha + \beta\}$. We have inequalities

$$\lambda(5\alpha+3\beta) \approx 1.41345 < \lambda(7\alpha+4\beta+3\gamma) \approx 1.55603 \text{ and } \lambda(5\alpha+3\beta) < \lambda(7\alpha+\beta).$$

For $k = 3$, $\mathcal{H}_{14} = \{11\alpha + 6\beta + 5\gamma, 7\alpha + 5\beta, 11\alpha + \beta, 11\alpha + 10\beta + 9\gamma\}$. We have $\lambda(7\alpha + 5\beta) < \lambda(11\alpha + \beta)$ by Lemma 3.16(2) and

$$\begin{array}{rcl} \lambda(7\alpha + 5\beta) \approx 1.25141 & < & \lambda(11\alpha + 6\beta + 5\gamma) \approx 1.39241 \\ & < & \lambda(11\alpha + 10\beta + 9\gamma) \approx 1.62913. \end{array}$$

By using the same arguments as in the case $n = 2k+1$, we have for all $k > 3$, $(4k)\mathrm{ent}((2k+1)\alpha + (2k-1)\beta) < 12\mathrm{ent}(7\alpha + 5\beta)$ and

$$\begin{array}{rcl} 12\mathrm{ent}(11\alpha + 6\beta + 5\gamma) & < & (4k)\mathrm{ent}((4k-1)\alpha + 2k\beta + (2k-1)\gamma), \\ 12\mathrm{ent}(11\alpha + 10\beta + 9\gamma) & < & (4k)\mathrm{ent}((4k-1)\alpha + (4k-2)\beta + (4k-3)\gamma). \end{array}$$

Thus, for all $k \geq 2$, $(2k+1)\alpha+(2k-1)\beta$, which realizes the minimal entropy among elements of $\Delta_{4k}(\mathbb{N})$, reaches the minimal entropy among elements of $\mathcal{H}_{4k+2}$.

(3a) The proof for the case $n = 8k+4$ is shown in a similar way.

(3b) Let us consider the case $n = 8(k+1)$. For $k = 0$, we see that $\mathcal{H}_8 = \{5\alpha+3\beta+2\gamma, 5\alpha+\beta, 5\alpha+4\beta+3\gamma\}$, and

$$\lambda(5\alpha+3\beta+2\gamma) \approx 1.72208 < \lambda(5\alpha+4\beta+3\gamma) \approx 1.78164 < \lambda(5\alpha+\beta) \approx 2.08102.$$

For $k \geq 1$, one can show that $(4k+5)\alpha + (4k+1)\beta$ reaches the minimal dilatation among elements of $\mathcal{H}_{8k+4}$. $\square$

3.5 Monodromy

The braid $\Theta = \Theta_m = (\sigma_1^2\sigma_2\sigma_3\cdots\sigma_{m-1})^{m-1} = (\sigma_1\sigma_2\cdots\sigma_{m-1})^m \in B_m$ is the full twist. Hence we have:

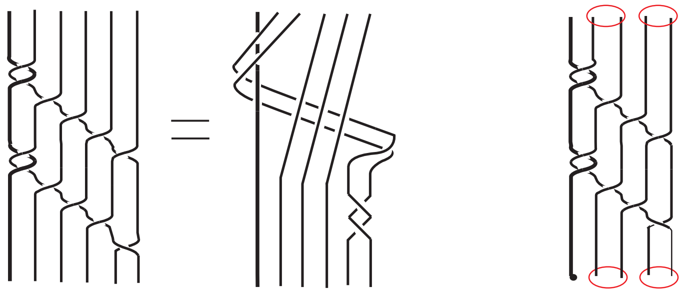


Figure 8: (left) braid $T_{6,2}$. (right) braid $T_{5,2}$.

Lemma 3.18. *If $p \equiv p'$ (mod $m-1$), then there exists an integer k such that $T_{m,p} = T_{m,p'}\Theta^k$ and $\Gamma(T_{m,p}) = \Gamma(T_{m,p'}) \in \mathcal{M}(D_m)$.*

Let us consider the braid $T_{m,p}$ in case $\gcd(m-1,p) \neq 1$. For example, $T_{5,2}$ is a reducible braid, since a disjoint union of two simple closed curves in D_5 is invariant under $\Gamma(T_{5,2})$, see Figure 8(right). It is not hard to see the following.

Lemma 3.19. *If $\gcd(m-1,p) \neq 1$, then $T_{m,p}$ is a reducible braid.*

Theorem 3.20. *Suppose that $x\alpha + y\beta \in \Delta_{m-1}(\mathbb{N})$ for $m \geq 3$. Then there exists $p = p(x,y)$ such that the braid $T_{m,p}$ is the monodromy on a fiber which is the minimal representative of $x\alpha+y\beta$.*

The rest of section is devoted to proving Theorem 3.20 and explaining how to compute $p = p(x, y)$.

3.5.1 Fiber surface

The aim of this section is to find fibers $F_{m(q),p(q)}$ for the magic manifold associated to sequences of natural numbers q whose homology class $[F_{m(q),p(q)}]$ is in $C_{\Delta_1}(\mathbb{Z})$.

Let L be a link in S^3. Let E_1 be an oriented disk with punctures which is embedded in the exterior $E(L) = S^3 \setminus \mathcal{N}(L)$ of L and let E_2 be any embedded, oriented surface in $E(L)$ as in Figure 9(1). The oriented surface $E_1 + E_2$, which depends on the orientation of E_1 and E_2, is either of type (3) or type (4) in Figure 9. The front (resp. back) of E_1+E_2 is dark-colored (resp. light-colored) in the figure.

Suppose that E_1+E_2 is of type (3) (resp. (4)). Now, open $E(L)$ along E_1, and let E' and E'' be the resulting punctured disks obtained from E_1. Reglue E' and E'' by twisting one of the disks by $360 \times N$ degrees in the clockwise (resp. counterclockwise) direction for some $N \in \mathbb{N}$. Then we obtain a new link, call it L' such that $E(L') \simeq E(L)$ (i.e, $E(L')$ is homeomorphic to $E(L)$). Let (E_1', E_2') be the ordered pair of the embedded, oriented surfaces in $E(L)$ which are obtained from the ordered pair (E_1, E_2) (see Figure 9(2)). The orientations of E_1' and E_2' are induced from E_1 and E_2 respectively.

Lemma 3.21. *Let L, L' be the links, E_i, E_i' $(i = 1, 2)$ be the surfaces as above and let $N \in \mathbf{N}$ as above. There exists an orientation preserving homeomorphism $f : E(L) \to E(L')$ such that*

(1) $f(yE_1 + xE_2) = rE_1' + xE_2'$ *and*

(2) $f(E_1) = E_1'$,

where $y = xN + r$ for $x, y \in \mathbb{N}$ and $0 \le r < x$. In particular,

(1') $f(NE_1 + E_2) = E_2'$.

Proof. The construction of L' implies the existence of a homeomorphism $f : E(L) \to E(L')$ with the properties (1) and (2). (By using Figure 9(3) and (4), one easily sees that $f(E_1 + E_2) = E_2'$ and $f(E_1) = E_1'$. One can generalize the first equality to the claim (1).) □

Note that by Lemma 3.21, $([E_1'], [E_2']) = ([E_1], [NE_1 + E_2])$.

Let us consider the exterior of the braided link $E(\overline{T}_{m,p})$. Now, we shall define two oriented surfaces $\widehat{F}_{m,p}$, $F_{m,p} \hookrightarrow E(\overline{T}_{m,p})$ whose orientations are

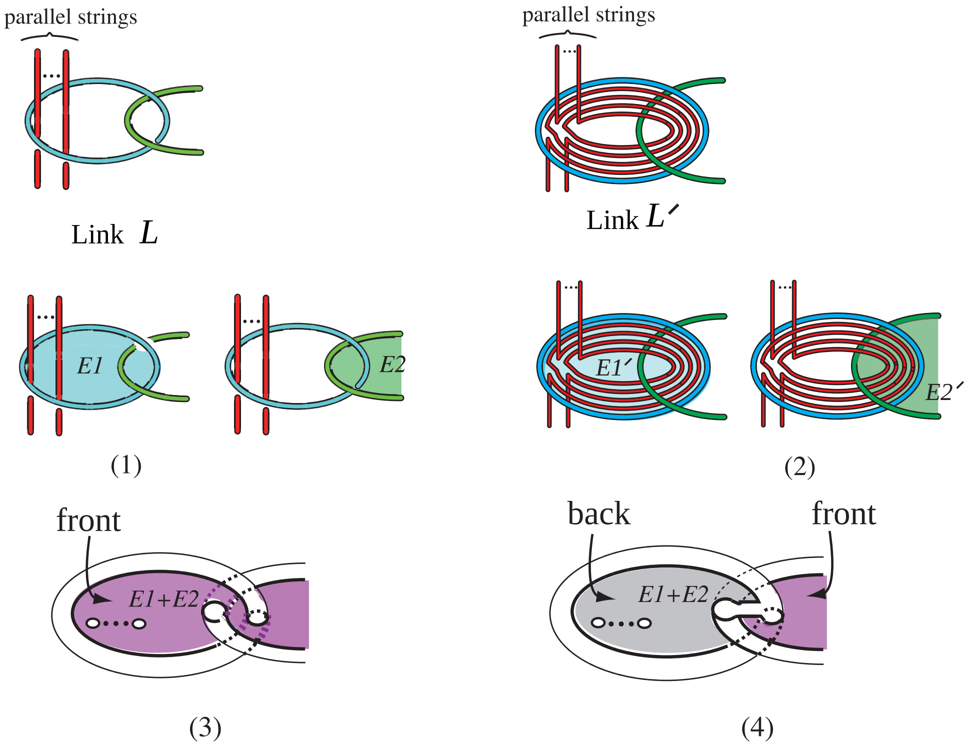


Figure 9: (1) $E_1,\ E_2 \hookrightarrow E(L)$. (2) $E_1',\ E_2' \hookrightarrow E(L')$ (in case $N=2$ in Lemma 3.21). (3) and (4) $E_1 + E_2 \hookrightarrow E(L)$.

induced by the oriented link $\overline{T}_{m,p}$. (Recall that the orientation of $\overline{T}_{m,p}$ is given as in Figure 2(right).) The oriented surface $F_{m,p}$ is an m-punctured disk which is bounded by the braid axis of $T_{m,p}$, see Figure 10(left). Clearly, $F_{m,p}$ is a fiber for $E(\overline{T}_{m,p})$ with the monodromy $T_{m,p}$. The oriented surface $\widehat{F}_{m,p}$ is a $(p+1)$-punctured disk which is bounded by $K_{m,p}$, where $K_{m,p}$ is the knot which is the closing the 1st strand of $T_{m,p}$, see Figure 10(right).

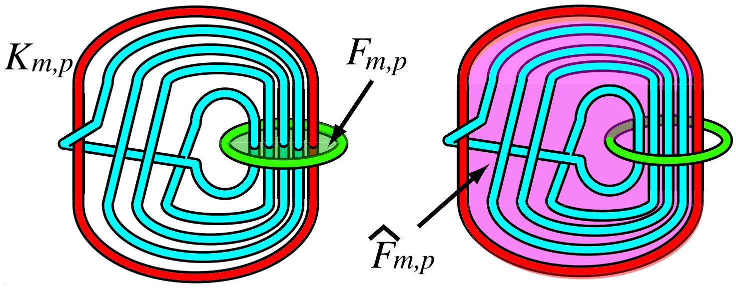


Figure 10: $F_{m,p}$, $\widehat{F}_{m,p} \hookrightarrow E(\overline{T}_{m,p})$ in case $(m,p) = (5,1)$. (one sees that $\widehat{F}_{5,1}$ is a twice-punctured disk.)

Given $m, p \in \mathbb{N}$ and $(k, \ell) \in \mathbb{N} \times \mathbb{N}$, the following construction enables us to see another fiber $F_{m',p'} \hookrightarrow E(\overline{T}_{m,p})$ with the monodromy $T_{m',p'}$.

(Construction of fibers.)

Step 1. Apply Lemma 3.21 for the link $\overline{T}_{m,p}$, the ordered pair $(E_1, E_2) = (F_{m,p}, \widehat{F}_{m,p})$ and $N = k$. (Note that $F_{m,p}$ is a disk with punctures and hence one can apply Lemma 3.21.) Then we obtain the ordered pair of embedded surfaces in $E(\overline{T}_{m,p})$;

$$(F_{m,k(m-1)+p}, \widehat{F}_{m,k(m-1)+p}) = (F_{m,p}, kF_{m,p} + \widehat{F}_{m,p}).$$

(Notice that $\Theta^k = (\sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_{m-1})^{k(m-1)}$.)

Step 2. Apply Lemma 3.21 for the link $\overline{T}_{m,k(m-1)+p}$, the ordered pair $(E_1, E_2) = (\widehat{F}_{m,k(m-1)+p}, F_{m,k(m-1)+p})$ and $N = \ell$. (Note that $\widehat{F}_{m,k(m-1)+p}$ is a disk with punctures and hence one can apply the lemma.) Then it turns out that the new link $L' = L'(\overline{T}_{m,k(m-1)+p})$ is isotopic to $\overline{T}_{m',p'}$, and we obtain the ordered pair of embedded surfaces in $E(\overline{T}_{m,p})$;

$$(\widehat{F}_{m',p'}, F_{m',p'}) = (\widehat{F}_{m,k(m-1)+p}, \ell\widehat{F}_{m,k(m-1)+p} + F_{m,k(m-1)+p}), \tag{3.8}$$

where

$$\begin{aligned} m' &= \ell(k(m-1)+p)+m, \\ p' &= k(m-1)+p. \end{aligned}$$

Thus we have another fiber $F_{m',p'}$ for $E(\overline{T}_{m,p})$ with the monodromy $T_{m'p'}$. (End of the construction.)

We sometimes denote m' and p' by $m'(m,p,k,\ell)$ and $p'(m,p,k,\ell)$.

For example, in case $T_{m,p} = T_{5,1}$ and $(k,\ell) = (1,2)$, we have $T_{m',p'} = T_{15,5}$, see Figure 13(2),(3),(4). ((3) explains Step 1 and (4) explains Step 2.)

By using Lemma 3.21, it is easy to see the following.

Proposition 3.22. *Let $x, y \in \mathbb{N}$. Suppose that $0 < x < y$ and $y \not\equiv 0 \pmod{x}$. Take k, ℓ, r_1, r_2 such that*

$$\begin{aligned} y &= xk + r_1 \ (0 < r_1 < x,\ k \in \mathbb{N}), \\ x &= r_1\ell + r_2 \ (0 \leq r_2 < r_1,\ \ell \in \mathbb{N}). \end{aligned}$$

We apply the construction of a fiber for a given $m, p \in \mathbb{N}$ and such a pair (k,ℓ). Then there exists an orientation preserving homeomorphism $f : E(\overline{T}_{m,p}) \to E(\overline{T}_{m',p'})$ such that

$$f(x\widehat{F}_{m,p} + yF_{m,p}) = r_2\widehat{F}_{m',p'} + r_1F_{m',p'}.$$

Let $q = (q_1, q_2, \cdots, q_t)$ be a sequence of natural numbers. The number t in the sequence, denoted by $|q|$, is called the *length* of q. For $q = (k_1, \ell_1, \cdots, k_j, \ell_j)$ with even length, let $q|_i = (k_1, \ell_1, \cdots, k_i, \ell_i)$ for $i \leq j$. For $q = (\ell_0, k_1, \ell_1, \cdots, k_j, \ell_j)$ with odd length, let $q|_i = (\ell_0, k_1, \ell_1, \cdots, k_i, \ell_i)$ for $i \leq j$. Note that $q = q|_j$.

We will define a fiber $F_{m(q),p(q)}$ for $E(\mathcal{C}_3)$ with the monodromy $T_{m(q),p(q)}$ associated to q such that its homology class $[F_{m(q),p(q)}]$ is in $C_{\Delta_1}(\mathbb{Z})$. To do so, we define a fiber $F_{m(q|_i),p(q|_i)}$ for $E(\mathcal{C}_3)$ with the monodromy $T_{m(q|_i),p(q|_i)}$ inductively as follows. Another oriented diagram of $\mathcal{C}_3$ is given in Figure 11(left). The oriented twice-punctured disk E_α (resp. E_β) bounded by K_α (K_β), whose orientation is induced by $\mathcal{C}_3$ is a representative of α (resp. β), see Figure 11(center, right). We first consider a sequence q with even length.

Case 1 (even). Suppose that $q = (k_1, \ell_1)$. First, apply Lemma 3.21 for $L = \mathcal{C}_3$, the ordered pair $(E_1, E_2) = (E_\beta, E_\alpha)$ and $N = k_1$. Let $(E_\beta, k_1E_\beta + E_\alpha)$ be the ordered pair of embedded surface in $E(\mathcal{C}_3) \simeq E(L')$

induced from (E_β, E_α). Second, apply Lemma 3.21 for L', the ordered pair $(k_1 E_\beta + E_\alpha, E_\beta)$ and $N = \ell_1$. Then we have the ordered pair of embedded surfaces

$$(k_1 E_\beta + E_\alpha, \ell_1(k_1 E_\beta + E_\alpha) + E_\beta) \tag{3.9}$$

in $E(\mathcal{C}_3) \simeq E(L'')$, where $L'' = (L')'$. We see that L'' is a braided link of $T_{m(q),p(q)} = T_{(k_1+1)\ell_1+2,k_1+1}$, and

$$(\widehat{F}_{m(q),p(q)}, F_{m(q),p(q)}) = (k_1 E_\beta + E_\alpha, \ell_1(k_1 E_\beta + E_\alpha) + E_\beta) \tag{3.10}$$

by (3.9). Therefore $F_{m(q),p(q)}$ is a fiber for $E(\mathcal{C}_3)$ with the monodromy $T_{m(q),p(q)}$, and by (3.10),

$$\left[F_{m(q),p(q)}\right] = \ell_1\alpha + (\ell_1 k_1 + 1)\beta \in C_{\Delta_1}(\mathbb{Z})$$

since $\alpha = [E_\alpha]$ and $\beta = [E_\beta]$. For example in case $q = (1,1)$, we have $T_{m(q),p(q)} = T_{4,2}$, see Figure 12.

Suppose that $q = (k_1, \ell_1, \cdots, k_j, \ell_j)$, $j > 1$. For $i = 1$, we have defined a fiber $F_{m(q|_1),p(q|_1)}$ for $E(\mathcal{C}_3)$ with the monodromy $T_{m(q|_1),p(q|_1)}$ as above. Suppose that we have a fiber $F_{m(q|_i),p(q|_i)}$ for $E(\mathcal{C}_3) \simeq E(\overline{T}_{m(q|_i),p(q|_i)})$ with the monodromy $T_{m(q|_i),p(q|_i)}$. Apply the construction of a fiber for $\overline{T}_{m(q|_i),p(q|_i)}$ and the pair (k_{i+1}, ℓ_{i+1}). Then we have the ordered pair of embedded surfaces $(\widehat{F}_{m(q|_{i+1}),p(q|_{i+1})},\ F_{m(q|_{i+1}),p(q|_{i+1})})$ in $E(\mathcal{C}_3)$ (given in Step 2 in the construction) which is defined by

$$(\widehat{F}_{m(q|_{i+1}),p(q|_{i+1})}, F_{m(q|_{i+1}),p(q|_{i+1})}) = (\widehat{F}_{m',p'}, F_{m',p'}),$$

where $m' = m'(m(q|_i), p(q|_i), k_{i+1}, \ell_{i+1})$, $p' = p'(m(q|_i), p(q|_i), k_{i+1}, \ell_{i+1})$, see (3.8). The surface $F_{m(q|_{i+1}),p(q|_{i+1})}$ is a fiber for $E(\mathcal{C}_3)$ with the monodromy $T_{m(q|_{i+1}),p(q|_{i+1})}$. By induction, it is shown that $\left[F_{m(q|_{i+1}),p(q|_{i+1})}\right] \in C_{\Delta_1}(\mathbb{Z})$.

Next, let us consider a sequence q with odd length.

Case 2 (odd). Suppose that $q = (\ell_0)$. Applying Lemma 3.21 for $L = \mathcal{C}_3$, the ordered pair $(E_1, E_2) = (E_\alpha, E_\beta)$ (<u>not</u> $(E_1, E_2) = (E_\beta, E_\alpha)$ as in Case 1 (even)) and $N = \ell_0$, we obtain the ordered pair of embedded surfaces $(E_\alpha, \ell_0 E_\alpha + E_\beta)$ in $E(\mathcal{C}_3) \simeq E(L')$. We see that L' is a braided link of $T_{m(q),p(q)} = T_{\ell_0+2,1}$. We have

$$(\widehat{F}_{m(q),p(q)}, F_{m(q),p(q)}) = (E_\alpha, \ell_0 E_\alpha + E_\beta).$$

Therefore $F_{m(q),p(q)}$ is a fiber for $E(\mathcal{C}_3)$ with the monodromy $T_{m(q),p(q)}$, and

$$\left[F_{m(q),p(q)}\right] = \ell_0\alpha + \beta \in C_{\Delta_1}(\mathbb{Z}).$$

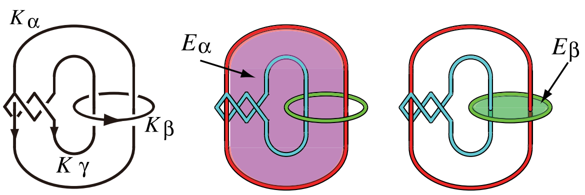
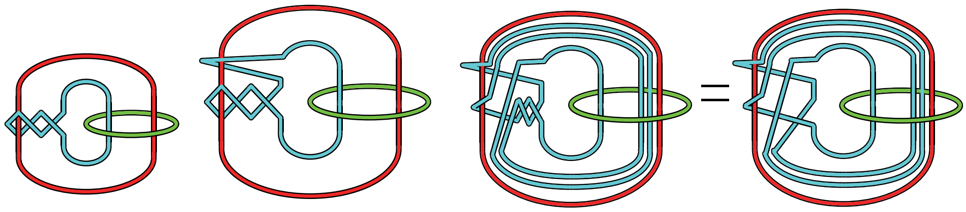


Figure 11: (left) $\mathcal{C}_3$. (center), (right) E_α, $E_\beta \hookrightarrow E(\mathcal{C}_3)$.



(1) (2) (3)

Figure 12: fiber $F_{4,2}$ with monodromy $T_{4,2}$ associated to $(1,1)$. (2) and (3) describe the first and second part of Case 1 (even) respectively.

In case $\ell_0 = 3$, see Figure 13(2).

Suppose that $q = (\ell_0, k_1, \ell_1, \cdots, k_j, \ell_j)$, $2j+1 > 1$. For $i = 0$, we have defined a fiber $F_{m(q|_0),p(q|_0)}$ for $E(\mathcal{C}_3) \simeq E(\overline{T}_{m(0),p(0)})$ with the monodromy $T_{m(q|_0),p(q|_0)}$ as above. For $i \geq 1$, in the same manner as Case 1 (even), a fiber $F_{m(q|_i),p(q|_i)}$ for $E(\mathcal{C}_3)$ with the monodromy $T_{m(q|_i),p(q|_i)}$ is given inductively, and we see that $\left[F_{m(q),p(q)}\right] \in C_{\Delta_1}(\mathbb{Z})$. In case $q = (3,1,2)$, see Figure 13.

3.5.2 Continued fraction

Let us consider a continued fraction with length j

$$w_1 + \frac{1}{w_2 +} \frac{1}{w_3 +} \cdots + \frac{1}{w_{j-1} +} \frac{1}{w_j} := w_1 + \cfrac{1}{w_2 + \cfrac{1}{w_3 + \cdots \cfrac{1}{w_{j-1} + \cfrac{1}{w_j}}}}$$

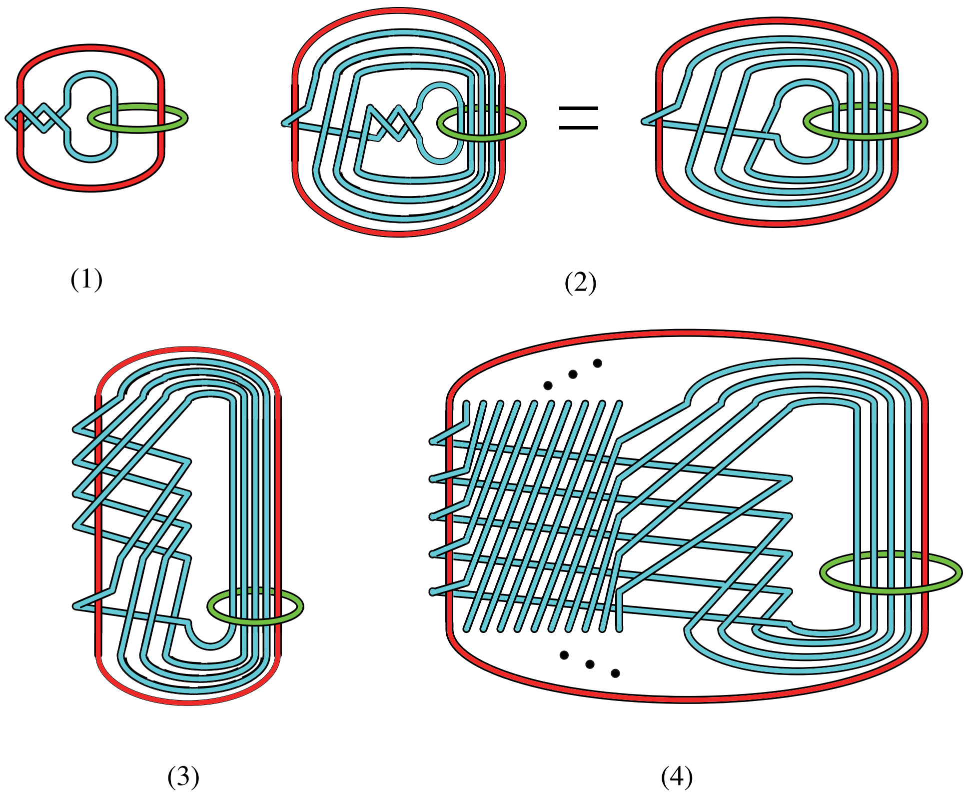


Figure 13: construction of fiber $F_{15,5}$ associated to $(3,1,2)$. (1) $\mathcal{C}_3$. (2) $\overline{T}_{5,1}$. (fiber $F_{5,1}$ associated to $q=(3)$.) (3) $\overline{T}_{5,5}$. (4) $\overline{T}_{15,5}$.

for $w_i \in \mathbb{N}$. We define $[w_1, w_2, \cdots, w_j] \in \mathbb{N}$ inductively as follows.

$$\begin{aligned}
[w_1] &= w_1, \\
[w_1, w_2] &= w_1 w_2 + 1, \\
[w_1, w_2, \cdots, w_i] &= [w_1, w_2, \cdots, w_{i-1}]\, w_i + [w_1, w_2, \cdots, w_{i-2}]\, .
\end{aligned}$$

The following is elementary and well-known.

Lemma 3.23.

(1) $w_1 + \frac{1}{w_2} \genfrac{}{}{0pt}{}{}{+} \frac{1}{w_3} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{1}{w_{j-1}} \genfrac{}{}{0pt}{}{}{+} \frac{1}{w_j} = \frac{[w_1, w_2, \cdots, w_j]}{[w_2, w_3, \cdots, w_j]}$.

(2) $[w_1, w_2, \cdots, w_j] = [w_j, w_{j-1}, \cdots, w_1]$.

Definition 3.24. *Suppose that* $\gcd(u, v) = 1$ *for* $u, v \in \mathbb{N}$. *We define two sequences of non-negative integers* $r = (r_0, r_1, \cdots, r_{j+1})$ *and* $q = (q_1, q_2, \cdots, q_j)$ *associated to* $\{u, v\}$ *(according to the Euclidean algorithm). We set* $r_0 = \max\{u, v\}$ *and* $r_1 = \min\{u, v\}$. *Write* $r_0 = r_1 q_1 + r_2$ $(0 \leq r_2 < r_1)$.

- *If* $r_2 = 0$, *then* r_1 *must be* 1 *since* $\gcd(u, v) = 1$. *We set*

$$r = (r_0, r_1 = 1, r_2 = 0) \text{ and } q = (q_1).$$

- *Suppose that* $r_2 \neq 0$. *We define* $q_2, q_3, \cdots$ *and* $r_3, r_4, \cdots$ *inductively as follows. Let* $q_i > 0$ *and* $r_{i+1} \geq 0$ *such that* $r_{i-1} = r_i q_i + r_{i+1}$ $(0 \leq r_{i+1} < r_i)$. *Since* $r_0 > \cdots > r_i > r_{i+1} \geq 0$, *there exists* j *such that* $r_{j+1} = 0$. *(Then* r_j *must be* 1 *since* $\gcd(u, v) = 1$.*) We set*

$$r = (r_0, r_1, \cdots, r_j = 1, r_{j+1} = 0) \text{ and } q = (q_0, \cdots, q_j).$$

By using the sequence q, the fraction $\frac{\max\{u,v\}}{\min\{u,v\}}$ can be expressed by the following two kinds of continued fractions.

$$\frac{\max\{u, v\}}{\min\{u, v\}} = q_1 + \frac{1}{q_2} \genfrac{}{}{0pt}{}{}{+} \frac{1}{q_3} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{1}{q_{j-1}} \genfrac{}{}{0pt}{}{}{+} \frac{1}{q_j}\,, \tag{3.11}$$

$$\frac{\max\{u, v\}}{\min\{u, v\}} = q_1 + \frac{1}{q_2} \genfrac{}{}{0pt}{}{}{+} \frac{1}{q_3} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{1}{q_{j-1}} \genfrac{}{}{0pt}{}{}{+} \frac{1}{(q_j - 1)} \genfrac{}{}{0pt}{}{}{+} \frac{1}{1} \tag{3.12}$$

with length j and $j + 1$ respectively. We can choose the one with odd/even length among those continued fractions.

Proof of Theorem 3.20. Let $x\alpha + y\beta \in \Delta_{m-1}(\mathbb{N})$. (By the definition of $\Delta_{m-1}(\mathbb{N})$, x and y are relatively prime.) From the continued fractions of

$\frac{\max\{x,y\}}{\min\{x,y\}}$ of the forms in (3.11) and (3.12) (constructed by one of the sequence $(q_1, q_2, \cdots) = (w_1, w_2, \cdots)$ in Definition 3.24 associated to $\{x, y\}$), we choose the one with odd length if $x > y$ (resp. even length if $x < y$):

$$\frac{\max\{x,y\}}{\min\{x,y\}} = w_1 + \frac{1}{w_2 +}\frac{1}{w_3 + \cdots +}\frac{1}{w_{j-1} +}\frac{1}{w_j}\,. \tag{3.13}$$

Now, we take $s = (s_0, s_1, \cdots, s_{j+1})$ which is defined by

$$\begin{aligned} s_0 &= \max\{x, y\}, \\ s_1 &= \min\{x, y\}, \\ s_{i+1} &= s_{i-1} - s_i w_i \text{ for } i \geq 1. \end{aligned}$$

Notice that $s_j = 1$ and $s_{j+1} = 0$. (If the continued fraction in (3.13) is of type (3.11), then s equals r in Definition 3.24.)

Suppose that $x < y$. (In this case, the continued fraction of (3.13) has even length.) Let us write $q = (w_1, w_2, \cdots, w_j) = (k_1, \ell_1, \cdots, k_{j/2}, \ell_{j/2})$. It is enough to show that a fiber $F_{m(q),p(q)}$ for $E(\mathcal{C}_3)$ associated to q is a representative of $x\alpha + y\beta$. (If this is the case, $F_{m(q),p(q)}$ is the minimal representative of $x\alpha+y\beta$ since $F_{m(q),p(q)}$ is a fiber.) By using Proposition 3.22 repeatedly, we have

$$\begin{aligned} x\alpha + y\beta &= [xE_\alpha + yE_\beta]\,(= [s_1 E_\alpha + s_0 E_\beta]) \\ &= \left[s_3 \widehat{F}_{m(q|_1),p(q|_1)} + s_2 F_{m(q|_1),p(q|_1)}\right] \\ &\;\;\vdots \\ &= \left[s_{j+1} \widehat{F}_{m(q|_{j/2}),p(q|_{j/2})} + s_j F_{m(q|_{j/2}),p(q|_{j/2})}\right] \\ &= \left[0 \widehat{F}_{m(q|_{j/2}),p(q|_{j/2})} + 1 F_{m(q|_{j/2}),p(q|_{j/2})}\right] \\ &= \left[F_{m(q),p(q)}\right]. \end{aligned}$$

Since the minimal representative of $x\alpha + y\beta$ is an $(x + y + 2)$-punctured sphere, $m(q)$ equals $x + y + 1 (= m)$.

The proof for the case $x > y$ is similar. $\square$

3.5.3 Computation of $p = p(x, y)$ in Theorem 3.20

In this section we give a recipe to compute p in Theorem 3.20. In Example 3.25, we explain how the number p is related to the pair (x, y).

Recall that $K_{m,p}$ is the knot obtained by the closing the 1st strand of $T_{m,p}$. Let $K^{\diamondsuit}_{m,p}$ be the knot obtained by the closing the rest of strands, i.e,

$K^{\diamondsuit}_{m,p}$ equals the closed braid of $T_{m,p}$ with $K_{m,p}$ removed. For the braided link $\overline{T}_{m,p}$, we have a pair of natural numbers

$$(i(\widehat{F}_{m,p}, K^{\diamondsuit}_{m,p}), i(F_{m,p}, K^{\diamondsuit}_{m,p})) = (p, m-1),$$

where $i(S, K)$ is the intersection number between the surface S and the knot K. For $\mathcal{C}_3$, we have

$$(i(E_\alpha, K_\gamma), i(E_\beta, K_\gamma)) = (1, 1),$$

see Figure 11.

Example 3.25. *By the proof of Theorem 3.20 (see also the argument in Case 2 (odd) in Section 3.5.1 and Figure 13), $T_{15,5}$ is the monodromy on a fiber which is the minimal representative for $11\alpha + 3\beta$. We explain why $p = 5$ is derived from $q = (q_1, q_2, q_3) = (3, 1, 2)$ and $r = (r_0, r_1, r_2, r_3, r_4) = (11, 3, 2, 1, 0)$ of Definition 3.24 associated to $\{3, 11\}$. We have*

$$[11E_\alpha + 3E_\beta] = \left[2\widehat{F}_{5,1} + 3F_{5,1}\right] = \left[2\widehat{F}_{5,5} + 1F_{5,5}\right] = \left[0\widehat{F}_{15,5} + 1F_{15,5}\right].$$

The following is a simple description of these equalities.

$$\begin{array}{ccccccc} (r_0, r_1) & & (r_2, r_1) & & (r_2, r_3) & & (r_4, r_3) \\ \| & & \| & & \| & & \| \\ (11, 3) & \xrightarrow[q_1=3]{} & (2, 3) & \xrightarrow[q_2=1]{} & (2, 1) & \xrightarrow[q_3=2]{} & (0, 1) \end{array} \qquad (3.14)$$

In the process to find the fiber associated to $q = (3, 1, 2)$, we can find a sequence of pairs of intersection numbers $(1, 1)$, $(1, 4)$, $(5, 4)$, $(5, 14)$ obtained from $\mathcal{C}_3$, $\overline{T}_{5,1}$, $\overline{T}_{5,5}$, $\overline{T}_{15,5}$ respectively which is described from left to right as follows.

$$(1, 1) \xleftarrow[q_1=3]{} (1, 4) \xleftarrow[q_2=1]{} (5, 4) \xleftarrow[q_3=2]{} (5, 14) \qquad (3.15)$$

Hence we can compute the number $p = 5$ from the sequence $q = (3, 1, 2)$. To describe the number p explicitly, we extend the sequence of (3.14) to the left according to the Euclidean algorithm:

$$\begin{array}{ccccccccc} (r_0, r_0 + r_1) & & (r_0, r_1) & & (r_2, r_1) & & (r_2, r_3) & & (r_4, r_3) \\ \| & & \| & & \| & & \| & & \| \\ (11, 14) & \xrightarrow[q_0=1]{} & (11, 3) & \xrightarrow[q_1=3]{} & (2, 3) & \xrightarrow[q_2=1]{} & (2, 1) & \xrightarrow[q_3=2]{} & (0, 1) \end{array}$$

In the same way, we extend the sequence of (3.15) to the left:

$$(0,1) \xleftarrow[q_0=1]{} (1,1) \xleftarrow[q_1=3]{} (1,4) \xleftarrow[q_2=1]{} (5,4) \xleftarrow[q_3=2]{} (5,14)$$

These show that

$$\begin{aligned}
\frac{14}{11} &= 1+\frac{1}{3}+\frac{1}{1}+\frac{1}{2} = \frac{[1,3,1,2]}{[3,1,2]} = \frac{[q_0,q_1,q_2,q_3]}{[q_1,q_2,q_3]},\\
\frac{14}{5} &= 2+\frac{1}{1}+\frac{1}{3}+\frac{1}{1} = \frac{[2,1,3,1]}{[1,3,1]} = \frac{[q_3,q_2,q_1,q_0]}{[q_2,q_1,q_0]}.
\end{aligned}$$

Thus the number $p(=5)$ in the question equals $[q_2, q_1, q_0]$.

Proposition 3.26. *Let $T_{m,p(x,y)}$ be the braid as in Theorem 3.20.*

(1) *Let $\frac{\max\{x,y\}}{\min\{x,y\}} = w_1 + \frac{1}{w_2} + \frac{1}{w_3} + \cdots + \frac{1}{w_{j-1}} + \frac{1}{w_j}$ be the continued fraction chosen in (3.13). Then*

$$p = p(x,y) = [w_{j-1}, w_{j-2}, \cdots, w_1, w_0 = 1].$$

(2) *$p = p(x,y)$ satisfies*

$$p \times \max\{x,y\} \equiv (-1)^j \pmod{x+y},$$

where j is the length of the continued fraction of $\frac{\max\{x,y\}}{\min\{x,y\}}$ in (1).

Proof. (1) We have

$$\frac{x+y}{\max\{x,y\}} = w_0 + \frac{1}{w_1} + \frac{1}{w_2} + \cdots + \frac{1}{w_{j-1}} + \frac{1}{w_j},$$

where $w_0 = 1$. It is not hard to show (1) by using the argument in Example 3.25.

(2) By induction, one can show that

$$\begin{aligned}
&\begin{pmatrix} w_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} w_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} w_j & 1 \\ 1 & 0 \end{pmatrix}\\
=& \begin{pmatrix} [w_0,w_1,\cdots,w_j] & [w_0,w_1,\cdots,w_{j-1}] \\ [w_1,w_2,\cdots,w_j] & [w_1,w_2,\cdots,w_{j-1}] \end{pmatrix}.
\end{aligned}$$

Taking the determinant, one has

$$\begin{aligned}
(-1)^{j+1} &\equiv -[w_0,w_1,\cdots,w_{j-1}]\,[w_1,w_2,\cdots,w_j] \pmod{[w_0,w_1,\cdots,w_j]}\\
&\equiv -[w_{j-1},w_{j-2},\cdots,w_0]\,[w_1,w_2,\cdots,w_j] \pmod{[w_0,w_1,\cdots,w_j]}.
\end{aligned}$$

Note that $x+y = [w_0, w_1, \cdots, w_j]$, $p = [w_{j-1}, w_{j-2}, \cdots, w_0]$, and $\max\{x, y\} = [w_1, w_2, \cdots, w_j]$. Thus,

$$(-1)^{j+1} \equiv -p \times \max\{x, y\} \pmod{x+y}.$$

This implies (2). □

We show the converse of Theorem 3.20.

Theorem 3.27. *Suppose that* $\gcd(p, m-1) = 1$ *for* $p \geq 1$ *and* $m \geq 3$. *Then there exist* $x, y \in \mathbb{N}$ *such that* $T_{m,p}$ *is the monodromy on a fiber which is the minimal representative of* $x\alpha + y\beta \in \Delta_{m-1}(\mathbb{N})$.

Proof. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be the Euler function. The number of braids $T_{m,p}$ satisfying $1 \leq p \leq m-1$ and $\gcd(p, m-1) = 1$ equals $\varphi(m-1)$. Also, the number of elements $x\alpha + y\beta \in \Delta_{m-1}(\mathbb{N})$ equals $\varphi(m-1)$. Let $x\alpha + y\beta$ and $x'\alpha + y'\beta$ be distinct elements of $\Delta_{m-1}(\mathbb{N})$. By Theorem 3.20, it is enough to show that $p(x, y) \neq p(x', y')$ since we may assume that $1 \leq p(x, y), p(x', y') \leq m-1$ (see Lemma 3.18).

Suppose that $(x, y) \neq (y', x')$. The concavity of $\mathrm{ent}(\cdot)|_{\Delta_{m-1}} : \Delta_{m-1} \to \mathbb{R}$ and Lemma 3.14 imply that $\mathrm{ent}(x\alpha + y\beta) \neq \mathrm{ent}(x'\alpha + y'\beta)$, and hence $T_{m,p(x,y)} \neq T_{m,p(x',y')}$ which implies that $p(x, y) \neq p(x', y')$.

Suppose that $(x, y) = (y', x')$. (In this case, $\mathrm{ent}(x\alpha + y\beta) = \mathrm{ent}(x'\alpha + y'\beta)$.) By Proposition 3.26(2), we see that

$$\begin{aligned} & p(x, y) \times \max\{x, y\} + p(y, x) \times \max\{x, y\} \\ = \; & (p(x, y) + p(y, x)) \times \max\{x, y\} \equiv 0 \pmod{x+y}. \end{aligned} \tag{3.16}$$

Since $\gcd(\max\{x, y\}, x+y) = 1$, we have $p(x, y) + p(y, x) \equiv 0 \pmod{x+y}$. Thus, $p(x, y) \not\equiv p(y, x) \pmod{x+y}$ which implies that $p(x, y) \neq p(y, x) (= p(x', y'))$. This completes the proof. □

Theorem 3.27 immediately gives:

Corollary 3.28. *Suppose that* $\gcd(p, m-1) = 1$ *for* $p \geq 1$ *and* $m \geq 3$. *Then* $S^3 \setminus \overline{T}_{m,p}$ *is homeomorphic to* $S^3 \setminus \mathcal{C}_3$.

Proposition 3.29. *Let* $m \geq 3$. *The following shows homology classes realizing* $\min\{\mathrm{ent}(a) \mid a \in \Delta_{m-1}(\mathbb{N})\}$ *and their monodromies.*

(1) *If* $m = 2k$, *then* $(k-1)\alpha + k\beta$ *and* $k\alpha + (k-1)\beta$ *realize the minimum and their monodromies are given by* $T_{2k,2}$ *and* $T_{2k,2k-3}$ *respectively.*

(2) *If $m = 4k+1$, then $(2k-1)\alpha + (2k+1)\beta$ and $(2k+1)\alpha + (2k-1)\beta$ realize the minimum and their monodromies are given by $T_{4k+1,2k+1}$ and $T_{4k+1,2k-1}$ respectively.*

(3a) *If $m = 3$, then $\alpha + \beta$ realize the minimum and its monodromy is given by $T_{3,1}$. If $m = 8k+3$ $(k \geq 1)$, then $(4k-1)\alpha + (4k+3)\beta$ and $(4k+3)\alpha + (4k-1)\beta$ realize the minimum and their monodromies are given by $T_{8k+3,2k+1}$ and $T_{8k+3,6k+1}$ respectively.*

(3b) *If $m = 8k+7$, then $(4k+1)\alpha + (4k+5)\beta$ and $(4k+5)\alpha + (4k+1)\beta$ realize the minimum and their monodromies are given by $T_{8k+7,6k+5}$ and $T_{8k+7,2k+1}$ respectively.*

Proof. We show the claim in case $m = 2k$. Other cases can be shown in a similar way. By Lemma 3.16, the homology classes $a = (k-1)\alpha + k\beta$ and $a' = k\alpha + (k-1)\beta$ realize the minimum. Let us consider the monodromies $T_{m,p(k-1,k)}$ and $T_{m,p(k,k-1)}$. Let $(x,y) = (k-1,k)$. Since $x < y$, the continued fraction which is chosen in (3.13) is $\frac{y}{x} = w_1 + \frac{1}{w_2}$, where $w_1 = 1$ and $w_2 = k-1$. By Proposition 3.26(1), $p(k-1,k) = [w_1, w_0] = [1,1] = 2$. By (3.16),

$$p(k-1,k) + p(k,k-1) \equiv 0 \pmod{2k-1}.$$

Hence $p(k,k-1) = 2k-3$. By Lemma 3.14(1), $T_{2k,2}$ or $T_{2k,2k-3}$ gives the monodromy for a and a'. $\Box$

3.6 Proof of Theorem 1.1

In Propositions 3.12 and 3.29, we have proved Theorem 1.1 except $n = 6, 8$. To complete the proof, we shall describe monodromies for two homology classes $3\alpha + 2\beta + \gamma$ and $5\alpha + 3\beta + 2\gamma$ in Proposition 3.32.

Lemma 3.30.

(1) *The 5-braided link $\overline{\sigma_1\sigma_2^2\sigma_3\sigma_4}$ and the 4-braided link $\overline{T}_{4,2}$ are isotopic to the $(-2,4,6)$-pretzel link.*

(2) *The braided link $\overline{b}$ for the 7-braid b as in Theorem 1.1(3b-i) is isotopic to the 5-braided link $\overline{\sigma_1\sigma_2^2\sigma_3\sigma_4\Theta_5^{-1}}$.*

Proof. (1) This is an easy exercise and we leave the proof for the readers. (Note: $T_{4,2}$ is conjugate to the 4-braid $\sigma_1^2\sigma_2\sigma_3\sigma_1^2$, and it might be easier to see $\overline{\sigma_1^2\sigma_2\sigma_3\sigma_1^2}$ is isotopic to the $(-2,4,6)$-pretzel link.)

(2) Let β be an n-braid. By deforming the axis of β, the braided link

$\overline{\beta}$ can be represented by the closed braid $\widehat{\beta'}$ of $\beta' \in B_{n+2}$, where $\beta' = \sigma_{n+1}^{\varepsilon_1}\beta\sigma_n^{\varepsilon_2}\sigma_{n-1}^{\varepsilon_2}\cdots\sigma_1^{\varepsilon_2}\sigma_1^{\varepsilon_2}\sigma_2^{\varepsilon_2}\cdots\sigma_n^{\varepsilon_2}$ $(\varepsilon_1, \varepsilon_2 \in \{-1, 1\})$, see Figure 14. By using this method, $\overline{\sigma_1\sigma_2^2\sigma_3\sigma_4\Theta_5^{-1}}$ is represented by the closed 7-braid $\widehat{a'}$, where

$$a' = \sigma_6^{-1}(\sigma_1\sigma_2^2\sigma_3\sigma_4\Theta_5^{-1})\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}.$$

On the other hand, the braided link $\overline{b}$ (Figure 15(left)) can be represented by a closed 6-braid as in Figure 15(center) whose link type equals a closed 7-braid as in Figure 15(right). Namely, $\overline{b}$ is isotopic to the closure of the 7-braid b':

$$b' = \underline{6}\ \underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{1}\ \underline{2}\ \underline{3}\ \underline{1}\ \underline{2}\ \underline{5}^4\ \underline{4}\ \underline{3}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}\ \underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{5},$$

where $\underline{i}$ stands for σ_i^{-1}. We see that a' is conjugate to b', since the super summit set for a' is equal to the one for b'. (The super summit set is a complete conjugacy invariant, see [3].) In fact, the super summit set consists of 4 elements $\Theta_7^{-1}1234321543654321$, $\Theta_7^{-1}1213432543654321$, $\Theta_7^{-1}1232145432654321$ and $\Theta_7^{-1}1232143254654321$, where i stands for σ_i. (One can use the computer program "Braiding" by González-Meneses for a computation of the super summit set [8].) Thus, the link types of $\overline{b}$ and $\overline{\sigma_1\sigma_2^2\sigma_3\sigma_4\Theta_5^{-1}}$ are the same. This completes the proof. □

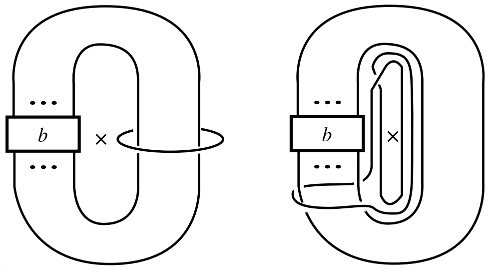


Figure 14: (left) braided link $\overline{\beta}$. (right) closed braid representing $\overline{\beta}$.

Lemma 3.30 together with Corollary 3.28 implies:

Corollary 3.31.

(1) $S^3 \setminus \overline{\sigma_1\sigma_2^2\sigma_3\sigma_4}$ *is homeomorphic to* $S^3 \setminus \mathcal{C}_3$.

(2) $S^3 \setminus \overline{b}$ *is homeomorphic to* $S^3 \setminus \mathcal{C}_3$.

Proposition 3.32.

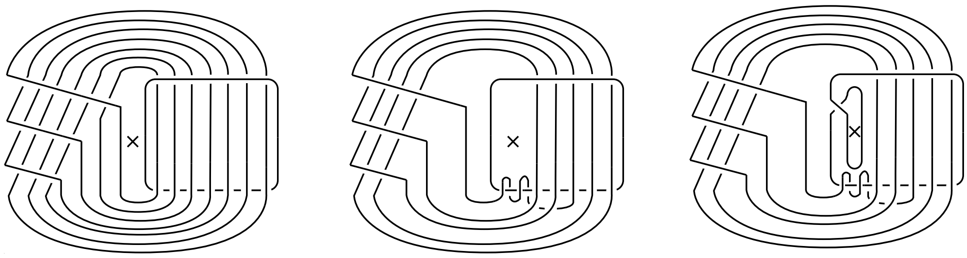


Figure 15: (left) braided link $\bar{b}$. (center) closed 6-braid representing $\bar{b}$. (right) closed 7-braid $\widehat{b'}$ representing $\bar{b}$.

(1) $\Gamma(\sigma_1\sigma_2^2\sigma_3\sigma_4)$ *is the monodromy on a fiber which represents* $3\alpha+2\beta+\gamma$.

(2) $\Gamma(b)$ *is the monodromy on a fiber which represents* $5\alpha+3\beta+2\gamma$.

Proof. (1) We see that $\mathcal{H}_6 = \{3\alpha+2\beta+\gamma, 3\alpha+\beta\}$, see the proof of Proposition 3.12. By Corollary 3.10, the monodromy for $3\alpha+\beta$ permutes 4 punctures cyclically and fixes two 1 punctures. On the other hand, the monodromy for $3\alpha+2\beta+\gamma$ permutes 3 punctures cyclically, and the mapping class $\Gamma(\sigma_1\sigma_2^2\sigma_3\sigma_4)$ permutes 3 punctures cyclically. By Corollary 3.31(1), we complete the proof.

(2) We see that $\mathcal{H}_8 = \{5\alpha+\beta, 5\alpha+3\beta+2\gamma, 5\alpha+4\beta+3\gamma\}$. The mapping class $\Gamma(b)$ permutes 5 punctures cyclically, 2 punctures cyclically and fixes the other 1 puncture. Among elements of $\mathcal{H}_8$, $5\alpha+3\beta+2\gamma$ is the only class whose monodromy permutes 2 punctures cyclically. By Corollary 3.31(2), we complete the proof. $\square$

4 Further discussion

4.1 Pseudo-Anosov braids with small dilatation

We consider the braids $T'_{m,p}$ defined in the introduction. The braid $T'_{m,p}$ may not be pseudo-Anosov, even though $T_{m,p}$ is so if $\gcd(p, m-1) = 1$ (Corollary 3.28). The inequality $\lambda(T'_{m,p}) \leq \lambda(T_{m,p})$ holds in case $T'_{m,p}$ is pseudo-Anosov. The following, which is clear by the definition of pseudo-Anosovs, says when the equality holds.

Lemma 4.1. *Suppose that* $\gcd(p, m-1) = 1$. *Let* $\Phi_{m,p}$ *be the pseudo-Anosov homeomorphism which represents* $\Gamma(T_{m,p}) \in \mathcal{M}(D_m)$. *Corresponding to the* 1*st strand of* $T_{m,p}$, *there exists a puncture, say* $a_{m,p}$, *which is fixed by* $\Phi_{m,p}$.

Suppose that the invariant foliation associated to $\Phi_{m,p}$ has no 1*-pronged singularity at $a_{m,p}$. Then $T'_{m,p}$ is pseudo-Anosov such that*

$$\lambda(T'_{m,p}) = \lambda(T_{m,p}).$$

The families of braids $\{T'_{m,p}\}$ and $\{T_{m,p}\}$ contain examples with minimal dilatation. The following braids realize the minimal dilatation.

- $T'_{4,1} = \sigma_1\sigma_2^{-1} \in B_3$, see Matsuoka [23].
- $T'_{5,1} = \sigma_1\sigma_2\sigma_3^{-1} \in B_4$, see Ko-Los-Song [18] and Ham-Song [10].
- $T'_{6,2} \sim \sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2 \in B_5$, see Ham-Song [10].
- $T_{6,3} \sim (\sigma_2\sigma_1\sigma_2\sigma_1(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^2)^{-1}\Theta \in B_6$, see Lanneau-Thiffeault [19].
- $T'_{8,2} \sim \sigma_4^{-2}(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6)^2 \in B_7$, see Lanneau-Thiffeault [19].
- $T'_{9,5} \sim \sigma_2^{-1}\sigma_1^{-1}(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7)^5 \in B_8$, see Lanneau-Thiffeault [19].

Here $b \sim b'$ means that b is conjugate to b'.

All the braids in Proposition 3.29 have been studied from the view point of their dilatations. Hironaka-Kin studied a family of braids

$$\sigma_{(k)} = \sigma_1\sigma_2\cdots\sigma_{2k-2}\sigma_1\sigma_2\cdots\sigma_{2k-4} \in B_{2k-1} \ \ (k \geq 3)$$

with odd strands [12]. It is easy to see that $\sigma_{(k)} \sim T'_{2k,2}$ (cf. Proposition 3.29(1)). Each braid $\sigma_{(k)} \in B_{2k-1}$ has the smallest known dilatation. Venzke found a family of braids $\{\psi_n\}$ with small dilatation [31].

$$\begin{array}{rcll}
\psi_n &=& L_n^2\sigma_1^{-1}\sigma_2^{-1} & \text{if } n = 2k-1 \ (k \geq 3),\\
\psi_n &=& L_n^{2k+1}\sigma_1^{-1}\sigma_2^{-1} & \text{if } n = 4k \ (k \geq 2),\\
\psi_n &=& L_n^{2k+1}\sigma_1^{-1}\sigma_2^{-1} & \text{if } n = 8k+2 \ (k \geq 1),\\
\psi_n &=& L_n^{6k+5}\sigma_1^{-1}\sigma_2^{-1} & \text{if } n = 8k+6 \ (k \geq 1),\\
\psi_6 &=& \sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_5\sigma_4, &
\end{array}$$

where $L_n = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1 \in B_n$. It is not hard to see that $\psi_{2k-1} \sim T'_{2k,2}$, $\psi_{4k} \sim T'_{4k+1,2k+1}$, $\psi_{8k+2} \sim T'_{8k+3,2k+1}$, $\psi_{8k+6} \sim T'_{8k+7,6k+5}$, and $\psi_6 \sim T_{6,2}$ (cf. Proposition 3.29(2)(3a)(3b)). By using Lemma 3.2 and Proposition 3.29 together with Lemma 4.1, we verify that

$$\begin{array}{rcl}
\lambda(\psi_{2k-1}) &=& \lambda(T'_{2k,2}) = \lambda(T_{2k,2}),\\
\lambda(\psi_{4k}) &=& \lambda(T'_{4k+1,2k+1}) = \lambda(T_{4k+1,2k+1}),\\
\lambda(\psi_{8k+2}) &=& \lambda(T'_{8k+3,2k+1}) = \lambda(T_{8k+3,2k+1}),\\
\lambda(\psi_{8k+6}) &=& \lambda(T'_{8k+7,6k+5}) = \lambda(T_{8k+7,6k+5}).
\end{array}$$

Let $T_{(m)} \in B_m$ be either of the two braids realizing the minimum in Proposition 3.29. For example, $T_{(2k)} = T_{2k,2}$ or $T_{2k,2k-3}$. Let $T'_{(m)} \in B_{m-1}$ be the braid obtained from $T_{(m)}$ by forgetting the 1st strand of $T_{(m)}$. By using Lemmas 4.1, 3.2 and Proposition 3.29, one has $\lambda(T_{(m)}) = \lambda(T'_{(m)})$. By Theorem 3.4 and Proposition 3.29, we have the following.

Corollary 4.2.

(1) $\lambda(T'_{(2k)})$ *equals the largest real root of*

$$f_{(k-1,k,0)}(t) = t^{2k-1} - 2(t^{k-1} + t^k) + 1.$$

(2) $\lambda(T'_{(4k+1)})$ *equals the largest real root of*

$$f_{(2k-1,2k+1,0)}(t) = t^{4k} - 2(t^{2k-1} + t^{2k+1}) + 1.$$

(3a) $\lambda(T'_{(8k+3)})$ *equals the largest real root of*

$$f_{(4k-1,4k+3,0)}(t) = t^{8k+2} - 2(t^{4k-1} + t^{4k+3}) + 1.$$

(3b) $\lambda(T'_{(8k+7)})$ *equals the largest real root of*

$$f_{(4k+1,4k+5,0)}(t) = t^{8k+6} - 2(t^{4k+1} + t^{4k+5}) + 1.$$

We now discuss the monotonicity of the dilatation of braids $T_{(m)}$. The following proposition is a corollary of Lemma 3.17 and Proposition 3.29.

Proposition 4.3.

(1) $\lambda(T_{(2k)}) > \lambda(T_{(2(k+1))})$.

(2) $\lambda(T_{(4k+1)}) > \lambda(T_{(4(k+1)+1)})$.

(3a) $\lambda(T_{(8k+3)}) > \lambda(T_{(8(k+1)+3)})$.

(3b) $\lambda(T_{(8k+7)}) > \lambda(T_{(8(k+1)+7)})$.

One can prove the following by using the argument in the proof of Lemma 3.17.

Lemma 4.4. $\lambda(T_{(2k-1)}) > \lambda(T_{(2k)})$.

In contrast to Lemma 4.4, it is not true that $\lambda(T_{(2k)}) > \lambda(T_{(2k+1)})$ for all k. For example,

$$\begin{aligned} \lambda(T_{(6)}) &< \lambda(T_{(7)}), \\ \lambda(T_{(10)}) &< \lambda(T_{(11)}). \end{aligned}$$

See the computation of $\lambda(T_{(m)})$ and $\mathrm{ent}(T_{(m)})$ in the following table. We shall show $\lambda(T_{(2k)}) > \lambda(T_{(2k+1)})$ is true for other cases in the next.

m	$T_{(m)}$	$\lambda(T_{(m)})$	$\mathrm{ent}(T_{(m)})$
3	$T_{3,1}$	3.73205	1.31696
4	$T_{4,2}$ or $T_{4,1}$	2.61803	0.962424
5	$T_{5,3}$ or $T_{5,1}$	2.29663	0.831443
6	$T_{6,2}$ or $T_{6,3}$	1.72208	0.543535
7	$T_{7,1}$ or $T_{7,5}$	2.08102	0.732858
8	$T_{8,2}$ or $T_{8,5}$	1.46557	0.382245
9	$T_{9,5}$ or $T_{9,3}$	1.41345	0.346031
10	$T_{10,2}$ or $T_{10,7}$	1.34372	0.295442
11	$T_{11,3}$ or $T_{11,7}$	1.35293	0.302271
12	$T_{12,2}$ or $T_{12,9}$	1.27248	0.240965
13	$T_{13,7}$ or $T_{13,5}$	1.25141	0.224273
14	$T_{14,2}$ or $T_{14,11}$	1.22572	0.203526
15	$T_{15,3}$ or $T_{15,11}$	1.22257	0.200958
16	$T_{16,2}$ or $T_{16,13}$	1.19267	0.176191
17	$T_{17,9}$ or $T_{17,7}$	1.18129	0.166609
18	$T_{18,2}$ or $T_{18,15}$	1.16806	0.155345
19	$T_{19,5}$ or $T_{19,13}$	1.16432	0.152136
20	$T_{20,2}$ or $T_{20,17}$	1.14903	0.13892
21	$T_{21,11}$ or $T_{21,9}$	1.14192	0.132708
22	$T_{22,2}$ or $T_{22,19}$	1.13388	0.125641
23	$T_{23,5}$ or $T_{23,17}$	1.13071	0.122845
24	$T_{24,2}$ or $T_{24,21}$	1.12152	0.114683
25	$T_{25,13}$ or $T_{25,11}$	1.11665	0.11033
26	$T_{26,2}$ or $T_{26,23}$	1.11125	0.105485
27	$T_{27,7}$ or $T_{27,19}$	1.10869	0.103176
28	$T_{28,2}$ or $T_{28,25}$	1.10258	0.0976543
29	$T_{29,15}$ or $T_{29,13}$	1.09904	0.0944354
30	$T_{30,2}$ or $T_{30,27}$	1.09517	0.0909069
31	$T_{31,7}$ or $T_{31,23}$	1.09309	0.0890074
32	$T_{32,2}$ or $T_{32,29}$	1.08875	0.0850323
33	$T_{33,17}$ or $T_{33,15}$	1.08606	0.0825554
34	$T_{34,2}$ or $T_{34,31}$	1.08315	0.0798714
35	$T_{35,9}$ or $T_{35,25}$	1.08144	0.0782958
36	$T_{36,2}$ or $T_{36,33}$	1.07821	0.0753015
37	$T_{37,19}$ or $T_{37,17}$	1.07609	0.0733366
38	$T_{38,2}$ or $T_{38,35}$	1.07382	0.0712265
39	$T_{39,9}$ or $T_{39,29}$	1.07241	0.0699047

Lemma 4.5. $\lambda(T_{(2k)}) > \lambda(T_{(2k+1)})$ *for all* $k \geq 2$ *but* $k = 3, 5$.

The following is used for the proof of Lemma 4.5.

Lemma 4.6. *Let* $x' > x > 1$ *and let* y' *be the positive number such that*

$$X_T(x'\alpha + y'\beta)(= x' + y') = X_T(x\alpha + (x-1)\beta) + 1(= 2x).$$

If $\lambda(x'\alpha + y'\beta) < \lambda(x\alpha + (x-1)\beta)$, *then*

$$\lambda((x' + \tfrac{1}{2})\alpha + (y' + \tfrac{1}{2})\beta) < \lambda((x + \tfrac{1}{2})\alpha + (x - \tfrac{1}{2})\beta).$$

Proof. One can show the claim by using the same argument as in [17, Proposition 4.17]. □

Proof of Lemma 4.5. One has $\lambda(3\alpha + \beta) < \lambda(2\alpha + \beta)$. This together with Lemma 4.6 implies that

$$\lambda((2k+1)\alpha + (2k-1)\beta) < \lambda(2k\alpha + (2k-1)\beta) \text{ for all } k \geq 1.$$

One has another inequality $\lambda(9\alpha + 5\beta) < \lambda(7\alpha + 6\beta)$. Hence by Lemma 4.6, for all $k \geq 2$, one has

$$\begin{aligned} \lambda((4k+3)\alpha + (4k-1)\beta) &< \lambda((4k+1)\alpha + 4k\beta), \\ \lambda((4k+5)\alpha + (4k+1)\beta) &< \lambda((4k+3)\alpha + (4k+2)\beta). \end{aligned}$$

This together with Proposition 3.29 completes the proof. □

As a corollary of Lemmas 4.4 and 4.5 together with the equality $\lambda(T_{(m)}) = \lambda(T'_{(m)})$, one has:

Proposition 4.7.

(1) $\lambda(T_{(2k-1)}) > \lambda(T'_{(2k)})$ *for all* $k \geq 2$.

(2) $\lambda(T_{(6)}) < \lambda(T'_{(7)})$ *and* $\lambda(T_{(10)}) < \lambda(T'_{(11)})$. *For all* $k \geq 2$ *but* $k = 3, 5$, $\lambda(T_{(2k)}) > \lambda(T'_{(2k+1)})$.

In particular, $T_{(10)} \in B_{10}$ has smaller dilatation than the Venzke's conjectural minimum $\lambda(\psi_{10})(= \lambda(T'_{(11)}))$.

We turn to the asymptotic behavior of the normalized entropy of the braid $T_{(m)}$. By Theorem 3.11(1) and Proposition 3.29, we obtain the following.

Corollary 4.8. *The normalized entropy of $T_{(m)}$ goes to the minimal normalized entropy with respect to Δ as m goes to ∞, i.e,*

$$\lim_{m\to\infty} \overline{\mathrm{ent}}(T_{(m)}) = \overline{\mathrm{ent}}(\alpha+\beta) = 2\log(2+\sqrt{3}).$$

Finally, we propose a conjecture on the minimal dilatation of braids of ℓ strands for $\ell \geq 9$.

Conjecture 4.9.

(1) *The braid $T'_{(2k)}$ realizes the minimal dilatation among $(2k-1)$-braids for all $k \geq 5$.*

(2) *The braid $T_{(10)}$ realizes the minimal dilatation among 10-braids. The braid $T'_{(2k+1)}$ realizes the minimal dilatation among $2k$-braids for all $k \geq 6$.*

4.2 Asymptotic behavior of entropy function

We consider asymptotic behaviors of the entropy function for a family of homology classes in Proposition 3.8.

Theorem 4.10. *Let $x\alpha + y\beta \in C_{\Delta_1}$.*

(1) $\lim\limits_{x,y\to\infty} \mathrm{ent}(x\alpha+y\beta) = 0.$

(2) $\lim\limits_{y\to\infty} \mathrm{ent}(x\alpha+y\beta) = \frac{\log 2}{x}.$

Of course, $\lim\limits_{x\to\infty} \mathrm{ent}(x\alpha+y\beta) = \frac{\log 2}{y}$ by symmetry.

Proof. (1) We may suppose that $x \leq y$. By [20, Theorem 3.5], we have an inequality

$$\mathrm{ent}(a+b) \leq \min\{\mathrm{ent}(a), \mathrm{ent}(b)\}$$

for $a, b \in int(C_\Delta)$. Hence for all $\varepsilon > 0$ so that $x-\varepsilon > 0$ and for all $\delta > 0$,

$$\mathrm{ent}(x\alpha + (x+\delta)\beta) \leq \min\{\mathrm{ent}((x-\varepsilon)\alpha + x\beta), \mathrm{ent}(\varepsilon\alpha+\delta\beta)\}.$$

Notice that $\mathrm{ent}(\varepsilon\alpha+\delta\beta)$ goes to ∞ as ε goes to 0. If one takes $\varepsilon > 0$ sufficiently small, then one may assume that

$$\mathrm{ent}(x\alpha + (x+\delta)\beta) \leq \mathrm{ent}((x-\varepsilon)\alpha + x\beta).$$

Since ent$(\cdot)$ is continuous, we have ent$(x\alpha+(x+\delta)\beta) \leq$ ent$(x\alpha+x\beta)$. Thus,

$$\lim_{x\to\infty} \mathrm{ent}(x\alpha + (x+\delta)\beta) \leq \lim_{x\to\infty} \mathrm{ent}(x\alpha + x\beta) = \lim_{x\to\infty} \frac{1}{x}\mathrm{ent}(\alpha+\beta) = 0.$$

Since $\delta > 0$ is arbitrary, the proof is completed.

(2) By Theorem 3.4, the dilatation of $x\alpha + y\beta + 0\gamma \in C_{\Delta_1}(\mathbb{Z})$ is the largest real root of

$$P(t^x, t^y, t^0) = P(t^x, t^y, 1) = t^y R_x(t) + (R_x)_*(t),$$

where $R_x(t) = t^x - 2$. By Lemma 2.1, the largest real root of $P(t^x, t^y, 1)$ converges to $2^{1/x}$, which is the unique real root of $R_x(t)$, as $y \to \infty$. This claim can be extended to homology classes of $C_{\Delta_1}(\mathbb{Q})$, that is the dilatation of $x\alpha+y\beta \in C_{\Delta_1}(\mathbb{Q})$ converges to $2^{1/x}$ as $y \to \infty$. Since the entropy function on $C_{\Delta_1}(\mathbb{Q})$ can be extended to C_{Δ_1} uniquely, the proof is completed. □

Proposition 4.11. *The entropy of* $(n+1)\alpha + n\beta + (n-1)\gamma \in int(C_\Delta)$ *converges to the logarithm of the golden mean* $\frac{1+\sqrt{5}}{2}$ *as* n *goes to* ∞.

Proof. We have

$$P(t^{n+1}, t^n, t^{n-1}) = t^{n-1}\big(t^n(t^2-t-1) + (t^2-t-1)_*\big).$$

If $(n+1)\alpha + n\beta + (n-1)\gamma$ is an integral class, then its dilatation λ_n is the largest real root of $t^n(t^2-t-1) + (t^2-t-1)_*$. The polynomial t^2-t-1 has the real root $\frac{1+\sqrt{5}}{2} > 1$. By Lemma 2.1, λ_n converges to $\frac{1+\sqrt{5}}{2}$ as $n \in \mathbb{N}$ goes to ∞. Since ent$(\cdot)$ is continuous on $int(C_\Delta)$, the proof is completed. □

4.3 Relation between horseshoe braid and braid $T_{m,p}$

The horseshoe map was discovered by Smale around 1960. This map is well-known to be a simple factor possessing chaotic dynamics ([26, Section 8.4.2] for example). For $\epsilon > 0$, any $C^{1+\epsilon}$ surface diffeomorphism with positive topological entropy "contains a horseshoe" in some iterate, see [14] for more details. This tells us that the features of the horseshoe map is universal for chaotic dynamical systems. In this section, we relate monodromies for homology classes in $C_{\Delta_1}(\mathbb{Z})$ to the horseshoe map.

The horseshoe map $H : D \to D$ is an orientation preserving diffeomorphism of the disk D defined as follows. The action of H on the rectangle R and two half disks S_0, S_1 is given as in Figure 16. More precisely, the restriction $H|_{R_i}$ for $i \in \{0, 1\}$ is an affine map such that H contracts R_i vertically and stretches horizontally, and $H|_{S_0 \cup S_1} : S_0 \cup S_1 \to S_0 \cup S_1$ is a contraction map. Then H can be extended over the rest of D without producing any new periodic points.

The set $\Omega = \bigcap_{j \in \mathbb{Z}} H^j(R)$ is invariant under H. The map $H|_\Omega : \Omega \to \Omega$ can be described by using the symbolic dynamics as follows. We set $\mathcal{S} = \{0,1\}^{\mathbb{Z}}$, that is $\mathcal{S}$ is the the set of all two sided infinite sequences $s = (\cdots s_{-1}s_0|s_1 \cdots)$ of 0 and 1, where we put the symbol $|$ between the 0th element and the 1st element. We introduce the metric on $\mathcal{S}$ as follows.

$$d(s,t) = \sum_{i \in \mathbb{Z}} \tfrac{|s_i - t_i|}{2^{|i|}},$$

where $s = (\cdots s_{-1}s_0|s_1s_2 \cdots)$ and $t = (\cdots t_{-1}t_0|t_1t_2 \cdots)$.

Theorem 4.12 (Smale)**.** *Let $\mathtt{s} : \mathcal{S} \to \mathcal{S}$ be the shift map, i.e, $\mathtt{s}$ is a homeomorphism such that*

$$\mathtt{s}(\cdots s_0|s_1s_2 \cdots) = (\cdots s_0s_1|s_2 \cdots).$$

The restriction $H|_\Omega : \Omega \to \Omega$ is conjugate to the shift map $\mathtt{s} : \mathcal{S} \to \mathcal{S}$. The conjugacy $\mathcal{K} : \Omega \to \mathcal{S}$ is given by

$$\mathcal{K}(x) = (\cdots \mathcal{K}_{-1}(x)\mathcal{K}_0(x)|\mathcal{K}_1(x) \cdots), \quad \textit{where}$$

$$\mathcal{K}_j(x) = \begin{cases} 1 & \textit{if} \quad H^j(x) \in R_1, \\ 0 & \textit{if} \quad H^j(x) \in R_0. \end{cases}$$

If x is a periodic point with the least period k for H, then $\mathcal{K}(x)$ is a periodic sequence. The word $\mathcal{K}_0(x)\mathcal{K}_1(x) \cdots \mathcal{K}_{k-1}(x)$ is called the *code* for x. Such word (modulo cyclic permutation) is said to be the *code* for the periodic orbit $\mathcal{O}_H(x) = \{x, H(x), \cdots, H^{k-1}(x)\}$.

Remark 4.13.

(1) *Theorem 4.12 asserts that there exists a one to one correspondence between the set of periodic points for $H|_\Omega$ and the set of periodic sequences in $\mathcal{S}$.*

(2) *By using Theorem 4.12, one can show that the set of periodic points of $H|_\Omega$ is dense on Ω.*

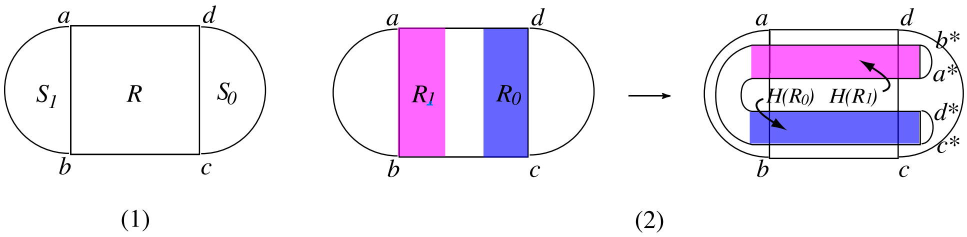


Figure 16: (1) R, S_0, $S_1 \subset D$. (2) horseshoe map H. (a^* is the image of a under H, for example.)

Let Q be a set of n points consisting of periodic orbits of $H|_\Omega$. We take an isotopy $\{H_t\}_{t\in I=[0,1]}$ such that H_0 = identity map on D and $H_1 = H$. Then

$$b(Q; \{H_t\}_{t\in I}) = \bigcup_{t\in I} H_t(Q) \times \{t\} \subset D \times I$$

is an n-braid. This depends on the choice of the isotopy, but it is determined uniquely up to a power of the full twist $\Theta = (\sigma_1 \cdots \sigma_{n-1})^n$. Consider the suspension flow on the mapping torus by using a "natural" isotopy $\{H_t\}$, see Figure 17(left). For this isotopy, we denote the braid $b(Q; \{H_t\}_{t\in I})$ by b_Q. By the definition of H, one can collapse the image of the vertical lines of R_0 and R_1 under the isotopy to build the *horseshoe template* $\mathcal{T}$ as in Figure 17(center). (For the template theory, see [7].) In this case the template is equipped with the semiflow induced by the suspension flow. It is easy to see that there exists a one to one correspondence between the set of periodic orbits of $H|_\Omega$ and the set of periodic orbits of the semiflow on $\mathcal{T}$. Each braid b_Q can be embedded in $\mathcal{T}$ so that the closed braid of b_Q becomes a finite union of periodic orbits of the semiflow on $\mathcal{T}$. Simply, we write b_Q for the image of $b_Q \hookrightarrow \mathcal{T}$ when there exists no confusion.

Now, we define horseshoe mapping classes and horseshoe braids. Let A_n be a set of n points which lie on the horizontal line through the origin in the round disk D. We set an n-punctured disk $D_n = D \setminus A_n$. We say that $\phi \in \mathcal{M}(D_n)$ is a *horseshoe mapping class* if there exists a set of n points Q consisting of periodic orbits of $H|_\Omega$ and there exists an orientation preserving homeomorphism $g : D \setminus Q \to D_n$ such that ϕ is conjugate to the mapping class $[g \circ H|_{D\setminus Q} \circ g^{-1}] \in \mathcal{M}(D_n)$. A braid $\beta \in B_n$ is a *horseshoe braid* if the mapping class $\Gamma(\beta) \in \mathcal{M}(D_n)$ is a horseshoe mapping class. In other words, β is a horseshoe braid if there exists an integer k and there exists a set of n points consisting of a finite union of periodic orbits of $H|_\Omega$, denoted by Q, such that $\beta\Theta^k$ is conjugate to the braid b_Q. In this case, there exists a braid $\gamma \in B_n$ such that $\gamma\beta\gamma^{-1}\Theta^k$ can be embedded in $\mathcal{T}$. However the converse is not true. For example, the 4-braid of Figure 17(right) is not a horseshoe braid since there exists exactly one periodic orbit with the least period 2 for $H|_\Omega$ whose code is 01. By Remark 4.13(1), one can show that a braid β embedded in $\mathcal{T}$ (ignoring the semiflow) is a horseshoe braid if and only if no strings of the braid are parallel. (See Figure 17(right).)

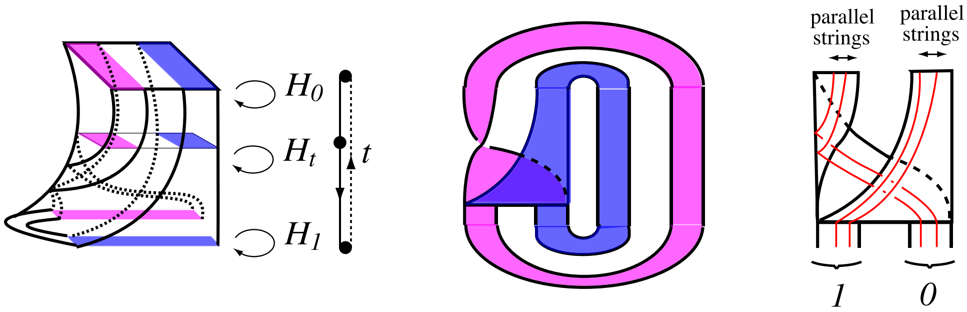


Figure 17: (left) suspension of horseshoe map. (center) horseshoe template $\mathcal{T}$. (right) non-horseshoe 4-braid embedded in $\mathcal{T}$. (In this case, two strings of the braid are parallel.)

Proposition 4.14. *Suppose that* $\gcd(p, m-1) = 1$*. If* $1 < p \leq \frac{m-1}{2}$*, then* $T_{m,p} \in B_m$ *is a horseshoe braid.*

Obviously, if the braid b is written by $b = cb'$, then b is conjugate to $b'c$. This is used for the proof of Proposition 4.14. Before proving the proposition, we first see that $T_{12,4}$ is a horseshoe braid by using Figure 18.

Example 4.15. *The first braid of Figure 18 is a representative of* $T_{12,4}$*. We slide the last crossing in the small circle to the top, see the second braid. Then it is conjugate to the third braid of Figure 18. We repeat to slide the last crossing in the small circle of the third braid to the top. We see that it is conjugate to the fourth braid. The crossings in the large circle of the fourth braid can slide to the top, and then we see that the fourth braid is conjugate to the fifth braid which is isotopic to the sixth braid. Finally, it is easy to see that the sixth braid is conjugate to the seventh braid which can be embedded in* $\mathcal{T}$*. (In fact, the braid* $\sigma_1\sigma_2\sigma_3\sigma_4$ *is a conjugacy.) Since no strings of the latter braid are parallel, one concludes that* $T_{12,4}$ *is a horseshoe braid.*

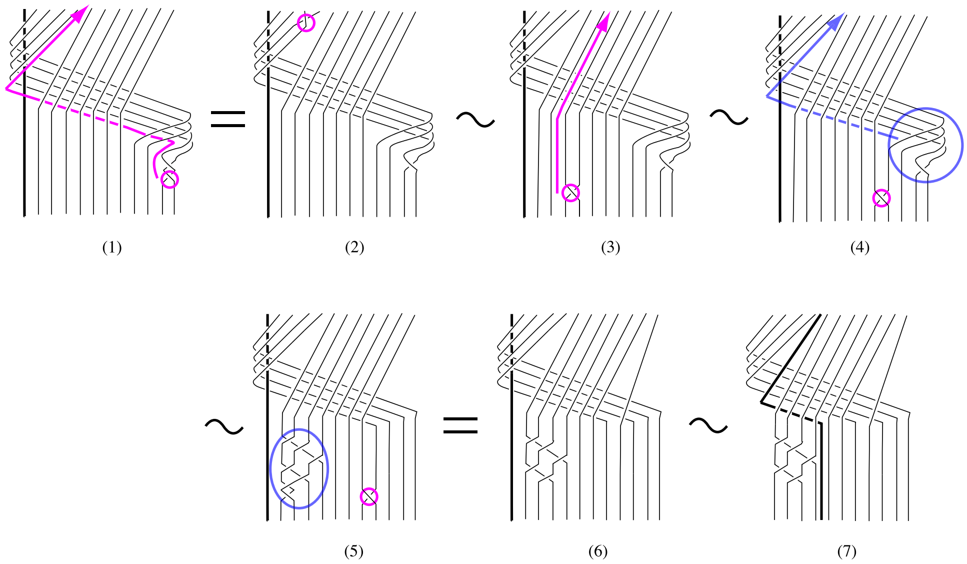


Figure 18: conjugate braid of $T_{12,4}$. (1) $T_{12,4}$. (7) braid embedded in $\mathcal{T}$.

Proof of Proposition 4.14. We consider a representative of $T_{m,p}$ as in the first braid of Figure 19. (See also Figure 8(left).) By using the slide technique in Example 4.15, we see that $T_{m,p}$ is conjugate to the second braid or the third of Figure 19. (For example, $T_{12,4}$ is conjugate to the second type and $T_{12,5}$ is conjugate to the third type.)

First, we show that the second braid is a horseshoe braid by using Figure 20. This braid is conjugate to the first braid of Figure 20 which is equal to the second braid of Figure 20. (See the fifth and sixth braid of Figure 18.) The second braid of Figure 20 is conjugate to the third braid in Figure 20 which can be embedded in $\mathcal{T}$.

Second, we show the third braid of Figure 19 is a horseshoe braid by using Figure 21. This braid is conjugate to the first braid of Figure 21. (For example, $T_{12,5}$ is conjugate to the third braid of Figure 21.) It is easy to see that the first braid of Figure 21 is conjugate to the second braid of Figure 21 which can be embedded in $\mathcal{T}$. □

4.4 Alternative proof of Theorem 4.10(2)

In this section, we give an alternative proof of Theorem 4.10(2).

Proof of Theorem 4.10(2). By Proposition 3.26, we have seen that $T_{m,1}$

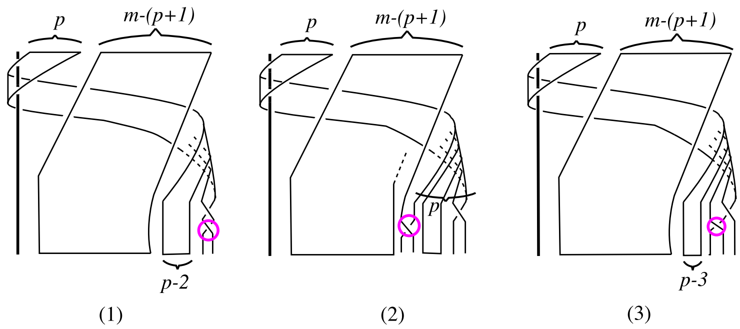


Figure 19: (1) $T_{m,p}$. ($T_{m,p}$ is conjugate to either the braid drawn in (2) or the one drawn in (3).)

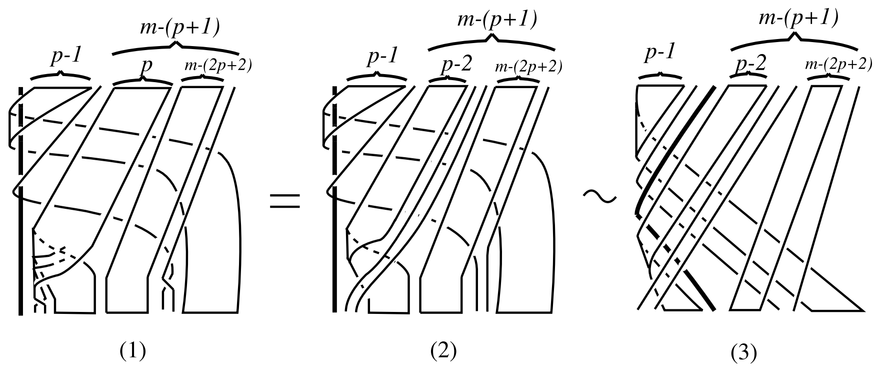


Figure 20: conjugate braid of $T_{m,p}$.

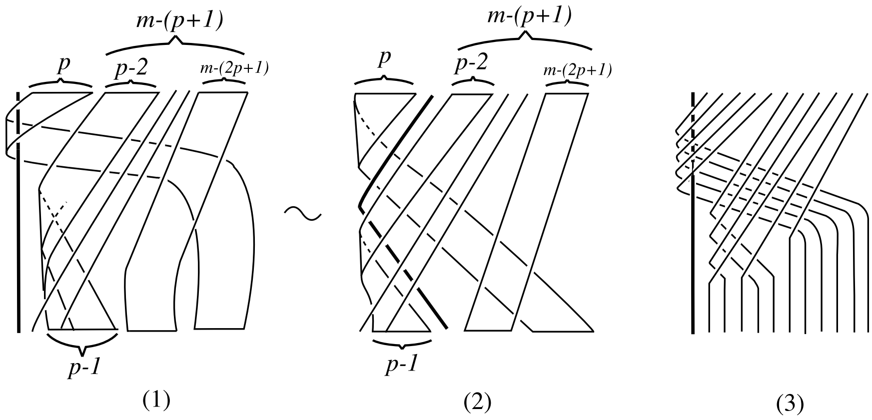


Figure 21: (1,2) conjugate braid of $T_{m,p}$. (3) conjugate braid of $T_{12,5}$.

represents the monodromy of $(m-2)\alpha+\beta \in C_{\Delta_1}(\mathbb{Z})$. For the proof, it is enough to show that

$$\lim_{m(\in\mathbb{N})\to\infty} \mathrm{ent}((m-2)\alpha+\beta) = \log 2. \tag{4.1}$$

The reason is as follows. The equality in (4.1) implies that

$$\lim_{x\to\infty} \mathrm{ent}(x\alpha+\beta) = \log 2$$

by the continuity of $\mathrm{ent}(\cdot)$. Therefore

$$\mathrm{ent}(x\alpha+y\beta) = \tfrac{1}{y}\mathrm{ent}(\tfrac{x}{y}\alpha+\beta) \to \tfrac{\log 2}{y} \text{ as } x\to\infty.$$

Now we show (4.1). Let $\beta_{(m_1,m_2,\cdots,m_{k+1})}$ be a family of braids depicted in Figure 22 for each integer $k\geq 1$ and each integer $m_i \geq 1$. By [16, Theorem 1.2], these braids are all pseudo-Anosov and the dilatation of $\beta_{(m_1,\cdots,m_{k+1})}$ is the largest real root of the Salem-Boyd polynomial

$$t^{m_{k+1}}R_{(m_1,\cdots,m_k)}(t) + (-1)^{k+1}R_{(m_1,\cdots,m_k)_*}(t),$$

where $R_{(m_1,\cdots,m_i)}(t)$ is given inductively as follows: For $2\leq i\leq k$,

$$\begin{aligned} R_{(m_1)}(t) &= t^{m_1+1}(t-1)-2t, \\ R_{(m_1,\cdots,m_i)}(t) &= t^{m_i}(t-1)R_{(m_1,\cdots,m_{i-1})}(t) + (-1)^i 2tR_{(m_1,\cdots,m_{i-1})_*}(t). \end{aligned}$$

In particular, the dilatation of $\beta_{(1,m-3)} = \sigma_1^{-1}\sigma_2\sigma_3\cdots\sigma_{m-2} \in B_{m-1}$ is the largest root of

$$t^{m-2}R_{(1)}(t) + (R_{(1)})_*(t), \tag{4.2}$$

where $R_{(1)}(t) = t(t+1)(t-2)$. By Lemma 2.1, the dilatation of $\beta_{(1,m-3)}$ converges to 2 as $m \to \infty$. The polynomial (4.2) comes from the graph map shown in Figure 23(center). This is the induced graph map for $\beta_{(1,m-3)} \in B_{m-1}$. The polynomial (4.2) is the characteristic polynomial of the transition matrix for the graph map. The smoothing of the graph gives rise to the train track associated to $\beta_{(1,m-3)}$ (Figure 23(right)). Since the train track contains an $(m-2)$-gon, a pseudo-Anosov homeomorphism $\Phi_{\beta_{(1,m-3)}}$ which represents the mapping class $\beta_{(1,m-3)}$ has an $(m-2)$-pronged singularity, say p, in the interior of the punctured disk. By puncturing the point p, one obtains a pseudo-Anosov homeomorphism $\widehat{\Phi}_{\beta_{(1,m-3)}}$. It is easy to see that the mapping class $[\widehat{\Phi}_{\beta_{(1,m-3)}}]$ is given by

$$\widehat{\beta}_{(1,m-3)} = \sigma_1^{-1}\sigma_2\sigma_3\cdots\sigma_{m-2}\sigma_{m-1}^2 \in B_m$$

with the same dilatation as $\beta_{(1,m-3)}$. Since $\widehat{\beta}_{(1,m-3)}$ is conjugate to the braid $T_{m,1}$, the dilatation $\lambda(T_{m,1})$ converges to 2 as m goes to ∞. This completes the proof. □

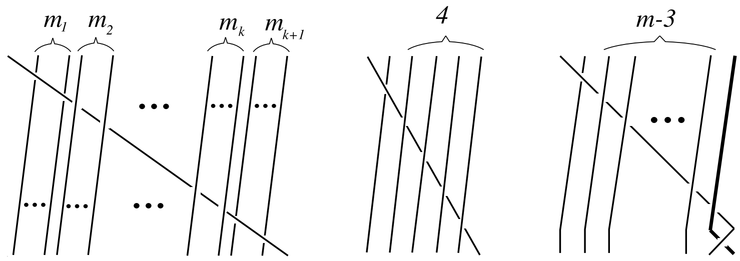


Figure 22: (left) $\beta_{(m_1,m_2,\cdots,m_{k+1})}$. (center) $\beta_{(1,4)}$. (right) $\widehat{\beta}_{(1,m-3)}$.

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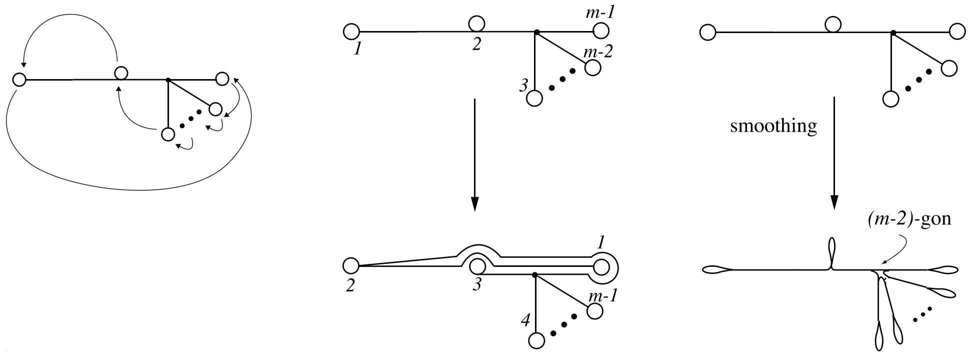

Figure 23: (left) transition of peripheral edge. (center) graph map. (right) train track.

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