ENTROPY VERSUS VOLUME FOR PSEUDO-ANOSOVS

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ABSTRACT. We discuss a comparison of the entropy of pseudo-Anosov maps and the volume of their mapping tori. Recent study of Weil-Petersson geometry of the Teichmüller space tells us that they admit linear inequalities for both directions under some bounded geometry condition. Based on the experiments, we present various observations on the relation between minimal entropies and volumes, and on bounding constants for the entropy over the volume from below. We also provide explicit bounding constants for a punctured torus case.

1. INTRODUCTION

Let $\Sigma = \Sigma_{g,p}$ be an orientable surface of genus g with p punctures and $\mathcal{M}(\Sigma)$ the mapping class group of Σ. Assume that $3g - 3 + p \geq 1$. According to Thurston [23], the elements of $\mathcal{M}(\Sigma)$ are classified into three types: periodic, pseudo-Anosov and reducible. A pseudo-Anosov element ϕ of $\mathcal{M}(\Sigma)$ defines two natural numerical invariants. One is the entropy $\mathrm{ent}(\phi)$ which is the logarithm of the stretching factor of the invariant foliation of ϕ (often called the dilatation of ϕ). The other is the volume $\mathrm{vol}(\phi)$ of its mapping torus,

$$\mathbb{T}(\phi) = \Sigma \times [0,1]/\sim$$

with respect to the hyperbolic metric of which the existence is due to Thurston [24] and the uniqueness to Mostow rigidity. Here, $\sim$ identifies $(x, 1)$ with $(f(x), 0)$ for some representative f of ϕ.

Our study is motivated by the experiments of the last author, illustrated in Figure 1, in his 2000 thesis [21] comparing $\mathrm{ent}(\phi)$ and $\mathrm{vol}(\phi)$. To see this more precisely, we let $\mathcal{M}^{\mathrm{pA}}(\Sigma)$ be the set of pseudo-Anosov mapping classes of $\mathcal{M}(\Sigma)$ and put

$$\mathcal{E}(\Sigma) = \{(\mathrm{vol}(\phi), \mathrm{ent}(\phi)) \in \mathbb{R}^2 \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)\}.$$

Figure 1 is the plot of $\mathcal{E}(\Sigma_{2,0})$ for all pseudo-Anosov classes represented by words of length at most 7 with respect to the Lickorish generators. By this plot, one might suspect that the ratios $\mathrm{ent}(\phi)/\mathrm{vol}(\phi)$ are bounded for both directions, namely, that there is a constant C depending only on the topology of Σ satisfying

$$C^{-1}\mathrm{vol}(\phi) \leq \mathrm{ent}(\phi) \leq C\mathrm{vol}(\phi). \tag{1.1}$$

However, this is false in general since it has been known to the experts from Long-Morton [16] and Fathi [6] that there are many families of pseudo-Anosov maps

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whose entropies tend to infinity while volumes remain bounded. We will present more recent plots in which we can observe this fact in section 4.

Nevertheless, it is still reasonable to expect under some bounded geometry condition that the ratios $\mathrm{ent}(\phi)/\mathrm{vol}(\phi)$ are bounded for both directions, because the families in [16, 6] necessarily contain short geodesics asymptotically. As we will explain in section 3, our expectation turns out to be a consequence of the deep results by Minsky [18], Brock [3] and the recent work by Brock-Mazur-Minsky [4].

On the other hand, the theory above does not say very much about accurate value of bounding constants. The experiments should provide more practical working hypothesis. From computing viewpoints, it is rather easy to work with not closed surfaces but punctured disks since they have nice descriptions in terms of braid data. Let D_n be an n-punctured disk. See Figure 2 for a more accurate plot for the case of D_6.

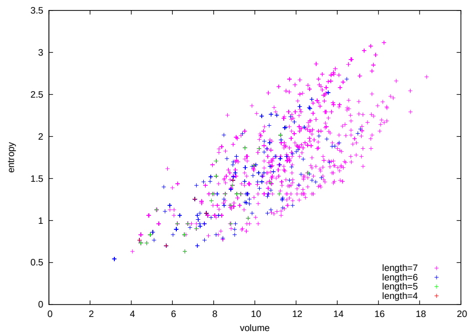

FIGURE 1. Entropy vs. volume for $\Sigma_{2,0}$.

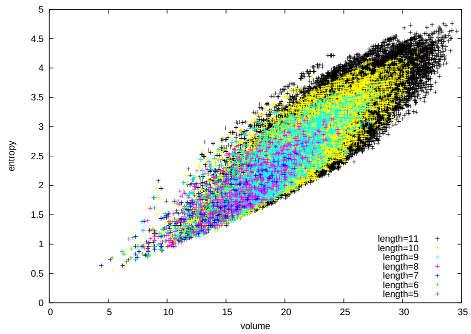

FIGURE 2. Entropy vs. volume for D_6.

The purpose of this paper is to present various observations and problems based on our experiments for the relation between minimal entropies and volumes, and for constants bounding the ratio ent/vol from below for pseudo-Anosov maps on punctured disks and torus. Moreover, we prove in Theorem 6.7 that

$$\frac{\mathrm{ent}(\phi)}{\mathrm{vol}(\phi)} > \frac{\log(\frac{3+\sqrt{5}}{2})}{2v_8} \approx 0.1313,$$

for any $\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma_{1,1})$, where $v_8 \approx 3.6638$ is the volume of a regular ideal octahedron. This bound is not best possible unfortunately. However, restricting our attention to mapping classes of *block length* 1, we obtain in Proposition 6.8 the best possible lower bound

$$\frac{\log(\frac{3+\sqrt{5}}{2})}{2v_3} \approx 0.4741,$$

where $v_3 \approx 1.0149$ is the volume of a regular ideal tetrahedron.

The organization of this paper is as follows. After recalling a basis of pseudo-Anosovs in the next section, we explain how the linear inequalities (1.1) are derived from recent studies of the Teichmüller space in section 3. Then choosing preferred generating sets of $\mathcal{M}(\Sigma)$ in section 4 for the experiments, we discuss the relation between minimal entropies and volumes in section 5, and lower bounds for ent/vol together with more accurate bounds of a special case in section 6.

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2. Pseudo-Anosovs

The *mapping class group* $\mathcal{M}(\Sigma)$ is the group of isotopy classes of orientation preserving homeomorphisms of Σ, where the group operation is induced by composition of homeomorphisms. An element of the mapping class group is called a *mapping class*.

A homeomorphism $\Phi : \Sigma \to \Sigma$ is said to be *pseudo-Anosov* if there exists a constant $\lambda = \lambda(\Phi) > 1$ called the *dilatation of* Φ and there exists a pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that

$$\Phi(\mathcal{F}^s) = \frac{1}{\lambda}\mathcal{F}^s \text{ and } \Phi(\mathcal{F}^u) = \lambda\mathcal{F}^u.$$

A mapping class which contains a pseudo-Anosov homeomorphism is called *pseudo-Anosov*. We define the dilatation of a pseudo-Anosov mapping class ϕ, denoted by $\lambda(\phi)$, to be the dilatation of a pseudo-Anosov representative in ϕ. It can be verified that $\lambda(\phi)$ does not depend on the choice of a representative.

Fixing Σ, the dilatation $\lambda(\phi)$ for $\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)$ is known to be an algebraic integer with a bounded degree depending only on Σ. Also, the number of conjugacy classes of $\mathcal{M}^{pA}(\Sigma)$ with dilatations bounded by some constant is finite. In particular, there exists a pseudo-Anosov with least dilatation, see [12].

The *topological entropy* $\mathrm{ent}(f)$ of a continuous self-map f on a compact metric space is a measure of the complexity, see for instance [25]. For a pseudo-Anosov homeomorphism Φ, the equality $\mathrm{ent}(\Phi) = \log(\lambda(\Phi))$ holds [7] and $\mathrm{ent}(\Phi)$ attains the minimal entropy among all homeomorphisms which are isotopic to Φ. Thus, we denote by $\mathrm{ent}(\phi)$ this characteristic number associated to a pseudo-Anosov mapping class ϕ.

Choosing a representative $f : \Sigma \to \Sigma$ of $\phi \in \mathcal{M}(\Sigma)$, we form a mapping torus

$$\mathbb{T}(\phi) = \Sigma \times [0, 1]/\sim,$$

where $\sim$ identifies $(x, 1)$ with $(f(x), 0)$. Then ϕ is pseudo-Anosov if and only if $\mathbb{T}(\phi)$ admits a complete hyperbolic structure of finite volume [24, 19]. Since such a structure is unique up to isometry by Mostow rigidity, it makes sense to speak of the volume $\mathrm{vol}(\phi)$ of ϕ, the hyperbolic volume of $\mathbb{T}(\phi)$.

Remark 2.1. Let ϕ be a pseudo-Anosov homeomorphism on Σ. Then the identities,

$$\mathrm{vol}(\phi^m) = m\,\mathrm{vol}(\phi) \quad \text{and} \quad \mathrm{ent}(\phi^m) = m\,\mathrm{ent}(\phi)$$

hold for any positive integer m. In particular, the line with the slope $\mathrm{ent}(\phi)/\mathrm{vol}(\phi)$ in $\mathbb{R}^2$ passing through the origin must intersect $\mathcal{E}(\Sigma)$ in infinitely many points.

3. Linear bounds for entropy versus volume

We here briefly describe what can be known about entropy versus volume for $\mathcal{M}^{\mathrm{pA}}(\Sigma)$ from the very recent theory. To see this, we introduce two norms for a pseudo-Anosov ϕ. Let $||\phi||_*$ be the minimal translation distance of the action of ϕ on the Teichmüller space with respect to the Teichmüller distance d_T or the Weil-Petersson distance d_{WP} according to whether $* = T$ or WP. Notice that

$$\text{ent}(\phi) = ||\phi||_T.$$

We start with the result of Brock in [3] which shows that there is a universal constant D depending only on the topology of Σ so that the inequalities

$$D^{-1}\,\text{vol}(\phi) \leq ||\phi||_{WP} \leq D\,\text{vol}(\phi) \tag{3.1}$$

hold for any pseudo-Anosov ϕ on Σ. To get (1.1), we want to replace $||\cdot||_{WP}$ by $||\cdot||_T$ under some bounded condition.

The Teichmüller distance is originally defined by using the infimum of dilatations of quasi-conformal maps between two Riemann surfaces. On the other hand, the Weil-Petersson distance is defined as the associated distance with the Riemannian part g_{WP} of the Weil-Petersson metric. Linch [15] succeeded to obtain a comparison of two distances directly. The modern treatment of the Teichmüller distance, which can be found for instance in [8], introduces an infinitesimal interpretation g_T of d_T. Then the infinitesimal form of Linch's inequality,

$$g_{WP} \leq -2\pi\chi(\Sigma)\,g_T, \tag{3.2}$$

can be derived simply from the Cauchy-Schwarz inequality between norms whose dual define infinitesimal forms of two distances, see for instance [20]. Now, choose a point x on the Teichmüller geodesic of ϕ, then since x may not be on the Weil-Petersson geodesic of ϕ, we have

$$\begin{aligned} ||\phi||_{WP} &\leq d_{WP}(x, \phi(x)) \\ &\leq -2\pi\chi(\Sigma)\,d_T(x, \phi(x)) \\ &= -2\pi\chi(\Sigma)\,||\phi||_T. \end{aligned}$$

This together with the left inequality of (3.1) immediately implies the left inequality in (1.1).

The right inequality in (1.1) dose not hold in general as we mentioned in the introduction. However, the deep analysis carried out for the Teichmüller distance by Minsky [18] and for the Weil-Petersson distance by Brock-Mazur-Minsky [4] implies, among many others, the following.

Theorem 3.1 (Minsky [18], Brock-Mazur-Minsky [4])**.** *For any $\varepsilon > 0$, there exists $\delta > 0$ such that both the Teichmüller and the Weil-Petersson geodesics invariant by the action of a pseudo-Anosov ϕ has no intersection with the subset of the Teichmüller space consisting of hyperbolic surfaces with closed geodesic of length $< \delta$ if $\mathbb{T}(\phi)$ contains no closed geodesics of length $< \varepsilon$.*

Now, since the part of the Teichmüller space by thick surfaces appeared above is invariant by the action of the mapping class group and moreover the quotient is compact by Mumford, we obtain the oppositely directed inequality to (3.2) within this region. Namely, there is some constant A such that the inequality,

$$g_T \leq A\,g_{WP}$$

holds in the region of the Teichmüller space consisting of surfaces without closed geodesics of length $< \delta$. Choose y on the Weil-Petersson geodesic of ϕ, then we have

$$\begin{aligned} ||\phi||_T &\leq d_T(x, \phi(x)) \\ &\leq A\, d_{WP}(x, \phi(x)) \\ &= A\, ||\phi||_{WP}. \end{aligned}$$

This together with the right inequality in (3.1) implies that of (1.1). Thus we have

Theorem 3.2 (Corollary to [3, 18, 4])**.** *There exists a constant $B = B(\Sigma)$ depending only on the topology of Σ such that the inequality,*

$$B\,\mathrm{vol}(\phi) \leq \mathrm{ent}(\phi)$$

holds for any pseudo-Anosov ϕ on Σ. Furthermore, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, \Sigma) > 1$ depending only on ε and the topology of Σ such that the inequality

$$\mathrm{ent}(\phi) \leq C\,\mathrm{vol}(\phi)$$

holds for any pseudo-Anosov ϕ on Σ whose mapping torus $\mathbb{T}(\phi)$ has no closed geodesics of length $< \varepsilon$.

Remark 3.3. The later half of Theorem 3.2 says that a sequence of pseudo-Anosov maps whose entropies diverge while volumes remain bounded must contain short geodesics asymptotically. The sequences which Long and Morton [16] and Fathi [6] found in fact have this property. The number of samples in the experiments plotted in Figure 1 was not sufficient enough to exhibit such a sequence.

4. Generating sets of mapping class groups

This section is to exhibit a preferred generating set which we use for the experimental plots of entropy versus volume for $\Sigma = D_n$ and $\Sigma_{n,p}$ where $(n, p) = (1, 1)$ or $(2, 0)$.

First of all, we introduce a generating set of the mapping class group $\mathcal{M}(D_n)$ by

$$\{h_{c_1}, \cdots, h_{c_{n-1}}\},$$

where h_{c_i} denotes the mapping class which represents the positive half twist about the arc c_i from the i^{th} puncture to the $(i+1)^{\text{st}}$, see Figure 3.

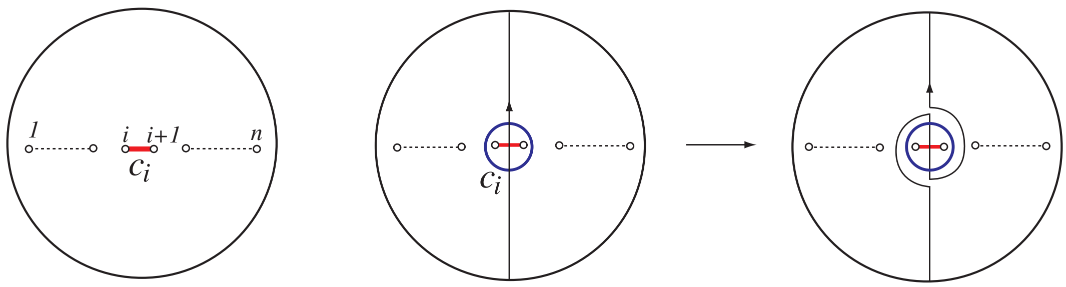

Figure 3. Arc c_i, and positive half twist h_{c_i}.

The n-braid group B_n and the mapping class group $\mathcal{M}(D_n)$ are related by the surjective homomorphism

$$\begin{array}{rcl} \Gamma : B_n & \to & \mathcal{M}(D_n) \\ \Cup & & \Cup \\ \sigma_i & \mapsto & h_{c_i} \end{array}$$

where σ_i for $i \in \{1, \cdots, n-1\}$ is the Artin generator, see Figure 4. The kernel of Γ is the center of B_n which is generated by the full twist braid $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$. Note that $\mathcal{M}(D_n)$ is isomorphic to a subgroup of $\mathcal{M}(\Sigma_{0,n+1})$ by identifying the boundary of D_n with the $(n+1)^{\text{st}}$ puncture. In the rest of the paper, we regard a mapping class in $\mathcal{M}(D_n)$ as a mapping class in $\mathcal{M}(\Sigma_{0,n+1})$ fixing the $(n+1)^{\text{st}}$ puncture. We

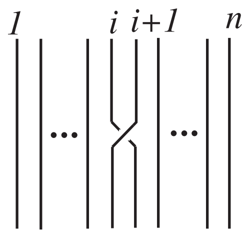

Figure 4. Generator σ_i.

say that a braid $b \in B_n$ is *pseudo-Anosov* if $\Gamma(b) \in \mathcal{M}(D_n)$ is pseudo-Anosov. When this is the case, $\text{vol}(\Gamma(b))$ equals the hyperbolic volume of the link complement $S^3 \setminus \overline{b}$ in the 3–sphere S^3, where $\overline{b}$ is the braided link of b which is a union of the closed braid of b and the braid axis, see Figure 5. Hereafter we represent a mapping class in $\mathcal{M}(D_n)$ by a braid and we denote $\Gamma(b) \in \mathcal{M}(D_n)$ by b confusingly. As for $\Sigma_{n,p}$

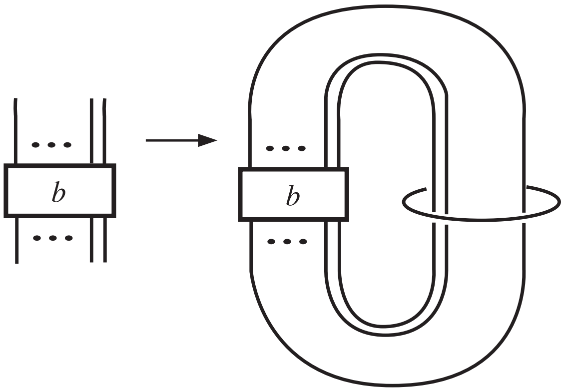

Figure 5. Link $\overline{b}$ obtained from a braid b.

where $(n, p) = (1, 1)$ or $(2, 0)$, we choose the set of Lickorish generators as a preferred generating set. Let us describe them more precisely. Let τ_1 and τ_2 be positive Dehn twists along a meridian and a longitude for $\Sigma_{1,1}$ respectively. Then, the set

$$\{\tau_1, \tau_2\}$$

will be a preferred generating set for $\mathcal{M}(\Sigma_{1,1})$. The elements τ_1, τ_2 are related with σ_1, σ_2 in $\mathcal{M}(D_3)$ through a double cover of D_3 branched along three punctures and one hole, where the hole corresponds to the puncture in $\Sigma_{1,1}$.

Let us choose a set of five essential simple closed curves $\alpha_1, \alpha_2, \cdots, \alpha_5$ on $\Sigma_{2,0}$ so that $\alpha_i \cap \alpha_j$ is one point if $|i-j|=1$ and empty otherwise, and τ_i a positive Dehn twist along α_i for $i = 1, 2, \cdots 5$. Then, the set

$$\{\tau_1, \tau_2, \cdots, \tau_5\}$$

will be our preferred generating set for $\mathcal{M}(\Sigma_{2,0})$. Again, the elements $\tau_1, \tau_2, \cdots, \tau_4$ are related with $\sigma_1, \sigma_2, \cdots, \sigma_4$ in $\mathcal{M}(D_5)$ through a double cover of D_5 branched along five punctures and one hole, where the hole corresponds to some point on $\Sigma_{2,0}$.

5. Minimal dilatation and minimal volume

5.1. **Experimental Data.** For the exposition of our experimental data, we introduce the notation $\mathcal{E}_k(\Sigma)$ for the subset of $\mathcal{E}(\Sigma)$ formed by pseudo-Anosovs of word length at most k with respect to the preferred generating set. Namely,

$$\mathcal{E}_k(\Sigma) = \{(\mathrm{vol}(\phi), \mathrm{ent}(\phi)) \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma) \text{ of word length} \leq k\}.$$

Figures 6 – 9 are the plots of $\mathcal{E}_{15}(D_3)$, $\mathcal{E}_{12}(D_4)$, $\mathcal{E}_{10}(D_5)$ and $\mathcal{E}_{11}(D_6)$ respectively. We use the program by T. Hall [10] for the computation of braid dilatations, and "SnapPea" by J. Weeks [26] for the computation of volumes of links in the 3-sphere S^3.

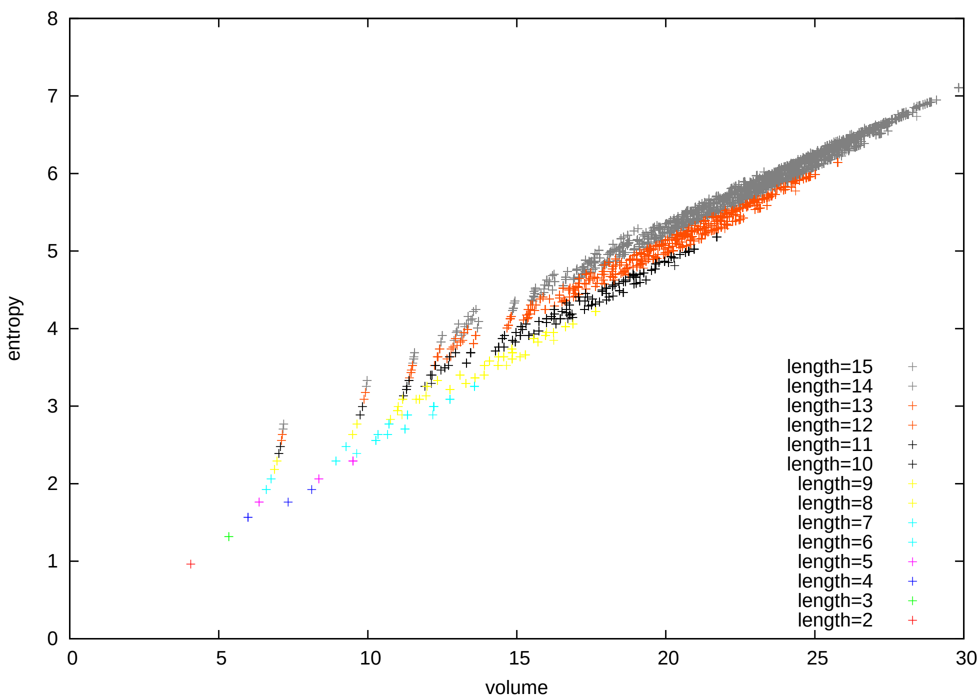

Figure 6. $\mathcal{E}_{15}(D_3)$.

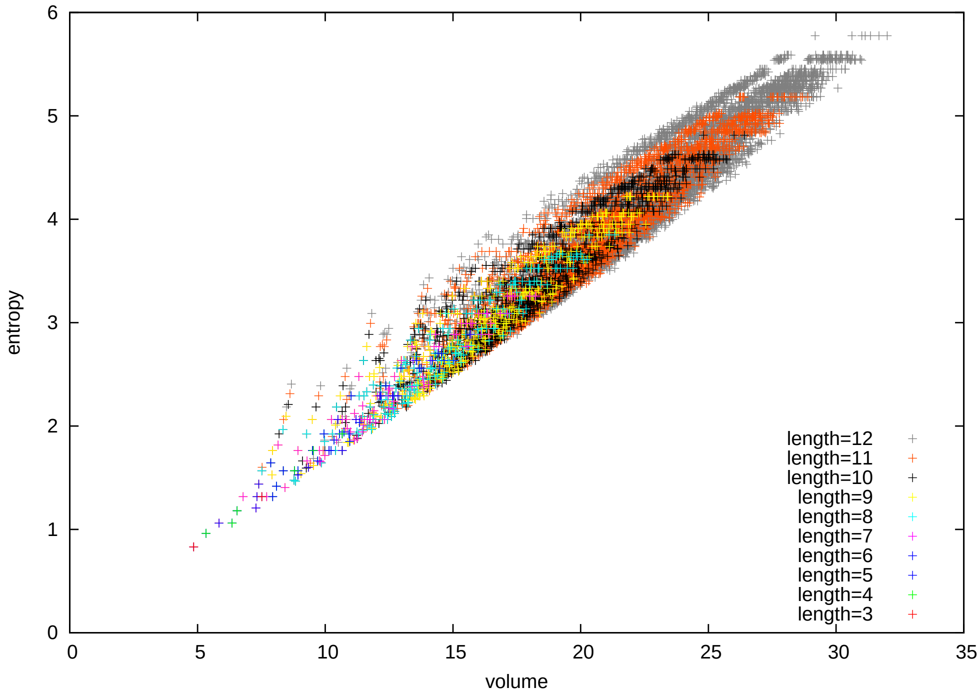

FIGURE 7. $\mathcal{E}_{12}(D_4)$.

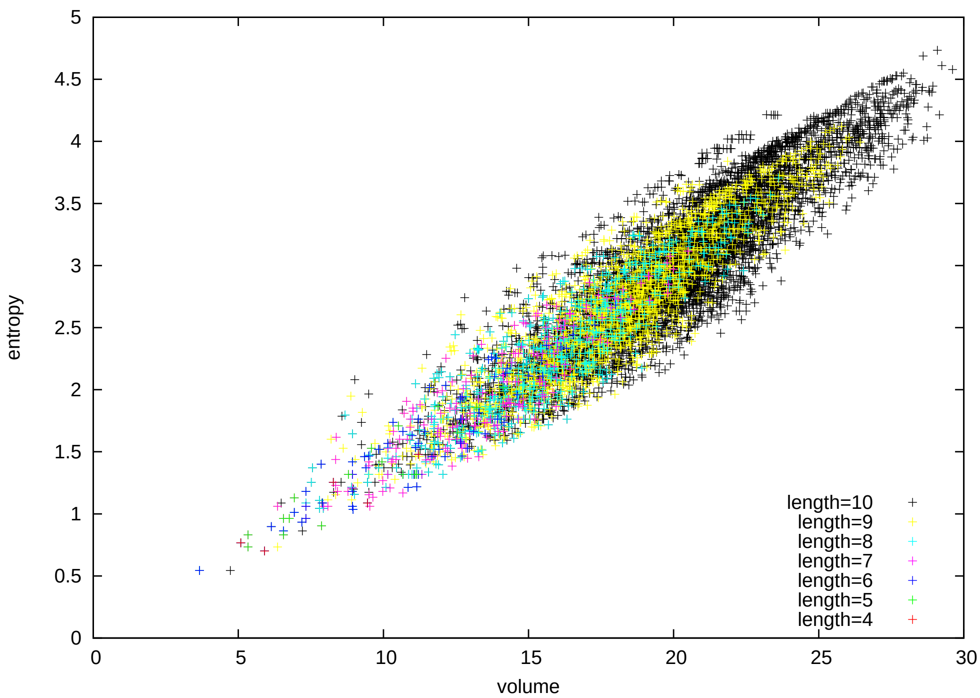

FIGURE 8. $\mathcal{E}_{10}(D_5)$.

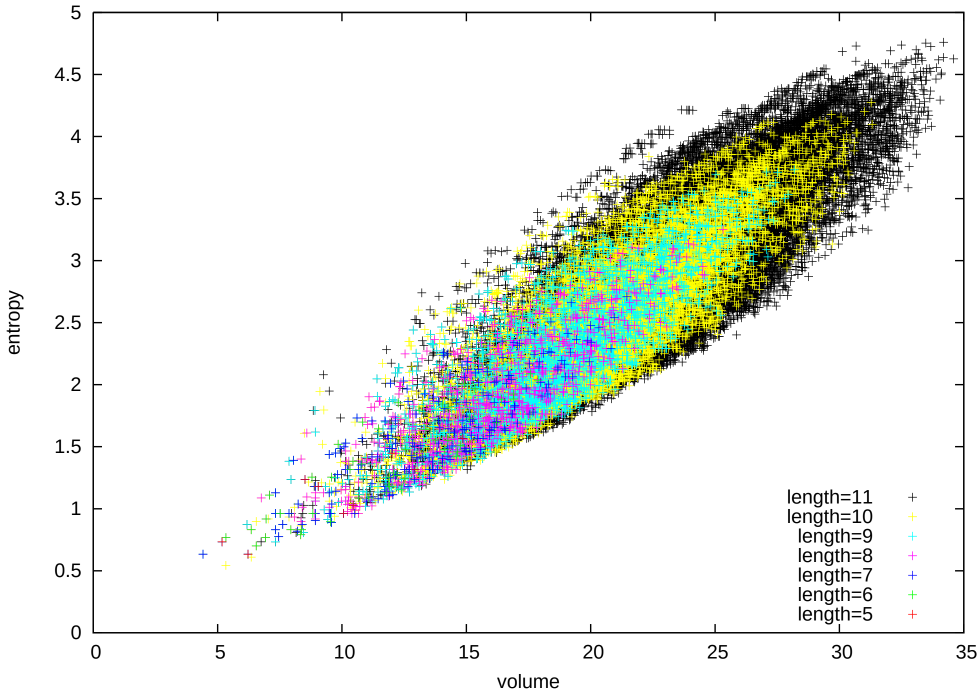

FIGURE 9. $\mathcal{E}_{11}(D_6)$.

5.2. **Observations.** Recall that $\lambda(\Sigma)$ represents the minimal dilatation among $\lambda(\phi)$ for $\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)$. In addition to $\lambda(\Sigma)$, we introduce the following further notations for the exposition:

$$\begin{aligned}\lambda_k(\Sigma) &= \min\{\lambda(\phi) \mid \phi \in \mathcal{M}^{\mathrm{p}A}(\Sigma) \text{ of word length} \leq k\},\\ \lambda(\Sigma; c) &= \min\{\lambda(\phi) \mid \phi \in \mathcal{M}^{\mathrm{p}A}(\Sigma),\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\},\\ \lambda_k(\Sigma; c) &= \min\{\lambda(\phi) \mid \phi \in \mathcal{M}^{\mathrm{p}A}(\Sigma) \text{ of word length} \leq k,\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}.\end{aligned}$$

When $\Sigma = D_n$, the number of cusps of $\mathbb{T}(b)$ for $b \in \mathcal{M}^{\mathrm{p}A}(D_n)$ equals the number of the components of the link $\bar{b}$, since $\mathbb{T}(b) = S^3 \setminus \bar{b}$.

The minimal dilatation $\lambda(\Sigma)$ and the minimal entropy $\mathrm{ent}(\Sigma) = \log \lambda(\Sigma)$ are known for the surfaces in Table 1.

TABLE 1. Minimal dilatations.

Σ	$\lambda(\Sigma)$	$\mathrm{ent}(\Sigma)$	mapping class realizing $\lambda(\Sigma)$	reference
$\Sigma_{1,1}$	≈ 2.61803	≈ 0.96242	$\tau_1\tau_2^{-1}$	folklore
D_3	≈ 2.61803	≈ 0.96242	$\beta_3 := \sigma_1\sigma_2^{-1}$	Matsuoka [17]
D_4	≈ 2.29663	≈ 0.83144	$\beta_4 := \sigma_1\sigma_2\sigma_3^{-1}$	Ko-Los-Song [14]
D_5	≈ 1.72208	≈ 0.54353	$\beta_5 := \sigma_1^3\sigma_2\sigma_3\sigma_4$	Ham-Song [11]
$\Sigma_{2,0}$	≈ 1.72208	≈ 0.54353	$\tau_1^3\tau_2\tau_3\tau_4$	Cho-Ham [5]

We now turn to the volume. The set of volumes of hyperbolic 3-manifolds, called the *volume spectrum*, is known to be a well-ordered closed subset in $\mathbb{R}$ of order type

ω^ω, see [22]. In particular, any subset of the volume spectrum achieves its infimum. We set

$$\begin{aligned}
\mathrm{vol}(\Sigma) &= \min\{\mathrm{vol}(\phi) \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)\}, \\
\mathrm{vol}_k(\Sigma) &= \min\{\mathrm{vol}(\phi) \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma) \text{ of word length} \leq k\}, \\
\mathrm{vol}(\Sigma; c) &= \min\{\mathrm{vol}(\phi) \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma),\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}, \text{ and} \\
\mathrm{vol}_k(\Sigma; c) &= \min\{\mathrm{vol}(\phi) \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma) \text{ of word length} \leq k,\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\},
\end{aligned}$$

again for the exposition.

To discuss which mapping class reaches $\lambda(\Sigma)$ and which one reaches $\mathrm{vol}(\Sigma)$, we first confirm that there exists a mapping class simultaneously reaching both $\lambda(\Sigma)$ and $\mathrm{vol}(\Sigma)$ when $\Sigma = D_3$ or D_5. Guéritaud and Futer [9, Theorem B.1] show that the 3-braid $\beta_3 = \sigma_1\sigma_2^{-1}$ with the minimal dilatation appeared in Table 1 realizes $\mathrm{vol}(D_3) \approx 4.05976$. The braided link $\overline{\beta_5}$ of the 5-braid $\beta_5 = \sigma_1^3\sigma_2\sigma_3\sigma_4$ with minimal dilatation appeared in Table 1 equals the $(-2, 3, 8)$-pretzel link, see Figure 10. On the other hand, Agol [2] shows that the $(-2, 3, 8)$-pretzel link complement and the Whitehead link complement have the minimal volume among orientable 2-cusped hyperbolic 3-manifolds. Hence β_5 also realizes $\mathrm{vol}(D_5) \approx 3.66386$.

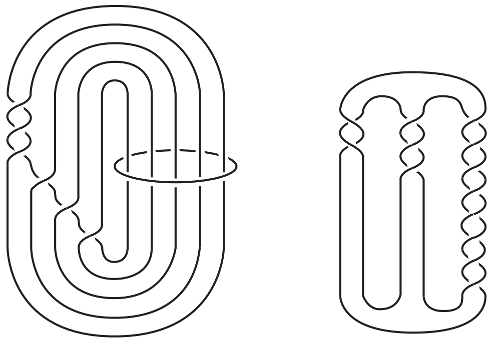

FIGURE 10. Link $\overline{\beta_5}$ on the left is equal to $(-2, 3, 8)$-pretzel link on the right.

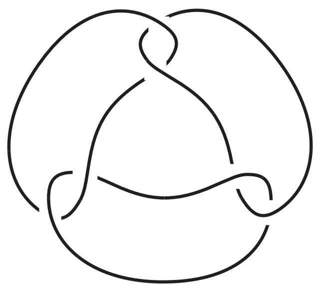

FIGURE 11. Chain-link with 3 components.

Thus one may ask whether there exists a mapping class simultaneously reaching both $\lambda(\Sigma)$ and $\mathrm{vol}(\Sigma)$. However it seems to be false in general. Within our experiments, $\mathrm{vol}_{11}(D_6)$ and $\lambda_{11}(D_6)$ are not reached by the same mapping class. It may be caused by the fact that the mapping torus reaching $\mathrm{vol}_{11}(D_6)$ and the one reaching $\lambda_{11}(D_6)$ have different number of cusps. Thus, we propose a refined problem by taking the number of cusps into account.

Problem 5.1. *Does there exist a mapping class in* $\mathcal{M}^{\mathrm{pA}}(\Sigma)$ *reaching both* $\lambda(\Sigma; c)$ *and* $\mathrm{vol}(\Sigma; c)$ *simultaneously?*

To see our experimental data more carefully for approaching this problem, we exhibit the plots of $\mathcal{E}_{15}(D_3)$, $\mathcal{E}_{12}(D_4)$, $\mathcal{E}_{10}(D_5)$ and $\mathcal{E}_{11}(D_6)$ restricted to the range of the volume < 5.334 in Figure 12. We can observe the following supporting evidences (1-a), $\cdots$, (4-b) for Problem 5.1 to be likely.

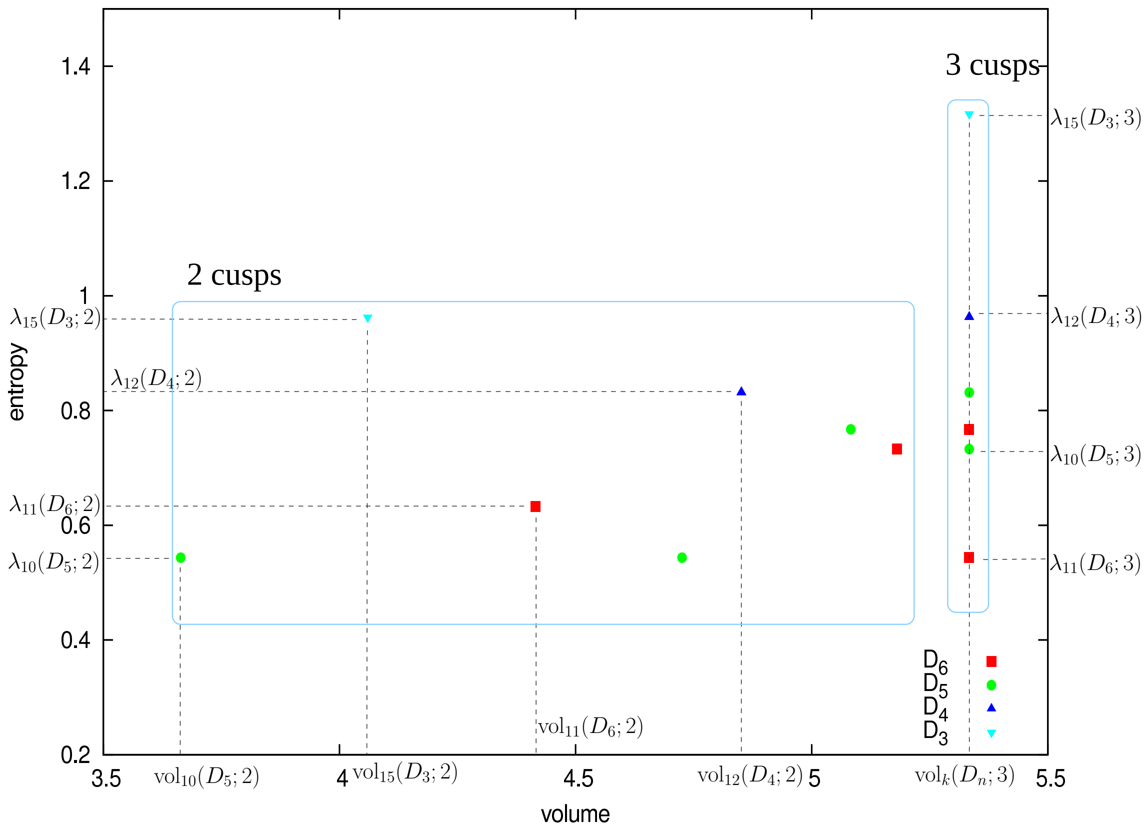

FIGURE 12. Samples with small entropy and small volume.

1-a. The 3-braid $\beta_3 = \sigma_1\sigma_2^{-1}$ reaches both $\lambda(D_3) \approx 2.61803$ and $\mathrm{vol}(D_3) \approx 4.05976$. Thus $\lambda(D_3) = \lambda(D_3;2)$ and $\mathrm{vol}(D_3) = \mathrm{vol}(D_3;2)$.

1-b. It is easy to verify that the 3-braid $\sigma_1^2\sigma_2^{-1}$ reaches $\lambda(D_3;3)$. This 3-braid also reaches $\mathrm{vol}_{15}(D_3;3) = \mathrm{vol}(S^3 \setminus C_3) \approx 5.33348$, where C_3 is the chain-link with 3 components, see Figure 11. Among orientable 3-cusped hyperbolic 3-manifolds, $S^3 \setminus C_3$, which is called the magic manifold, is the one with the smallest known volume.

2-a. The 4-braid $\beta_4 = \sigma_1\sigma_2\sigma_3^{-1}$ reaches both $\lambda(D_4) = \lambda(D_4;2) \approx 2.29663$ and $\mathrm{vol}_{12}(D_4;2) \approx 4.85117$.

2-b. The 4-braid $\sigma_1^2\sigma_2\sigma_3^{-1}$ reaches both $\lambda_{12}(D_4;3) \approx 2.61803$ and $\mathrm{vol}_{12}(D_4;3) = \mathrm{vol}(S^3 \setminus C_3)$.

3-a. The 5-braid $\beta_5 = \sigma_1^3\sigma_2\sigma_3\sigma_4$ reaches both $\lambda(D_5) = \lambda(D_5;2) \approx 1.72208$ and $\mathrm{vol}(D_5) = \mathrm{vol}(D_5;2) \approx 3.66386$.

3-b. The 5-braid $\sigma_1\sigma_2^2\sigma_3\sigma_4$ reaches both $\lambda_{10}(D_5;3) \approx 2.08102$ and $\mathrm{vol}_{10}(D_5;3) = \mathrm{vol}(S^3 \setminus C_3)$.

4-a. The 6-braid $\sigma_1^3\sigma_2\sigma_3\sigma_4\sigma_5$ reaches both $\lambda_{11}(D_6;2) \approx 1.8832$ and $\mathrm{vol}_{11}(D_6;2) \approx 4.41533$.

4-b. The 6-braid $\sigma_1^3\sigma_2\sigma_1^2\sigma_3\sigma_2\sigma_4\sigma_5$ reaches both $\lambda_{11}(D_6;3) = \lambda(\beta_5)$ and $\mathrm{vol}_{11}(D_6;3) = \mathrm{vol}(S^3 \setminus C_3)$.

Remark 5.2. There are braids plotted in Figure 12 other than the ones we identified in the observations (1-a), $\cdots$, (4-b). We have experimentally verified by SnapPea that all the 3-cusped mapping tori in the plots are homeomorphic to either $S^3 \setminus C_3$ or its mirror image, and the other mapping tori having 2 cusps are results of some Dehn surgeries on $S^3 \setminus C_3$. On the other hand, the mapping tori listed in (1-b), (2-b), (3-b) and (4-b) are in fact shown to be homeomorphic to $S^3 \setminus C_3$ rigorously in [13].

Remark 5.3. The 5-braids β_5 and $\beta_5' = \sigma_1^4\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_4$ both realize the minimal dilatation $\lambda(D_5)$, but $\mathrm{vol}(D_5) = \mathrm{vol}(\beta_5) < \mathrm{vol}(\beta_5')$. This example says that the mapping class with minimal dilatation does not always realize the minimal volume.

6. Lower bounds for ent/vol

6.1. **More problems.** The first half of Theorem 3.2 shows that there exists a constant $B = B(\Sigma)$ such that the inequality $B \leq \mathrm{ent}(\phi)/\mathrm{vol}(\phi)$ holds for any $\phi \in \mathcal{M}^{pA}(\Sigma)$. However it is not quite obvious to find accurate value of B. In this subsection, we formulate a few problems concerning with this constant. Let us set

$$\mathcal{I}(\Sigma) = \inf\{\mathrm{ent}(\phi)/\mathrm{vol}(\phi) \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)\},$$
$$\mathcal{I}_k(\Sigma) = \min\{\mathrm{ent}(\phi)/\mathrm{vol}(\phi) \mid \phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma) \text{ of word length} \leq k\},$$

where we fix preferred generating sets as in the previous section.

When the complexity of Σ goes higher, the minimal entropy $\mathrm{ent}(\Sigma)$ approaches zero, while the minimal volume $\mathrm{vol}(\Sigma)$ stays bounded away from zero. Thus the bounding constant B in Theorem 3.2 for D_n (or $\Sigma_{g,0}$) necessarily tends to zero when n (or g) goes to ∞. We simultaneously plot the data of $\{(\mathrm{ent}(\phi), \mathrm{vol}(\phi))\}$ for D_n $(3 \leq n \leq 6)$ in Figure 13. It is natural to ask monotonicity.

Problem 6.1. *Is it true that $\mathcal{I}(D_n) > \mathcal{I}(D_{n+1})$ for all $n \geq 3$? Is it true that $\mathcal{I}(\Sigma_{g,0}) > \mathcal{I}(\Sigma_{g+1,0})$ for all $g \geq 2$?*

Normalizing the entropy by multiplying $\mathrm{area}(D_n) = 2\pi(n-1)$, we again simultaneously plot the data up to some word lengths in Figure 14. Looking at this plot, one may ask

Problem 6.2. *Does the minimal normalized ratio $2\pi(n-1)\,\mathcal{I}(D_n)$ converge to some positive constant as n goes to ∞?*

Also one may ask

Problem 6.3. *Does there exist a mapping class $\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma)$ which attains $\mathcal{I}(\Sigma)$?*

To study Problems 6.1, 6.2 and 6.3, we computed $\mathcal{I}_k(\Sigma)$ for $\Sigma = D_n$ $(3 \leq n \leq 6)$ in Figure 15, and their normalized value in Figure 16. We observe that $\mathcal{I}_k(D_3)$ is achieved by the mapping class $\sigma_1\sigma_2^{-1}$ up to $k = 15$. On the other hand, $\mathcal{I}_k(\Sigma)$ decreases as k increases for the other surfaces. We thus propose

Conjecture 6.4. (1) $\mathcal{I}(D_3) = \dfrac{\mathrm{ent}(\sigma_1\sigma_2^{-1})}{\mathrm{vol}(\sigma_1\sigma_2^{-1})} \approx 0.2370.$

(2) *There are no mapping classes which attain $\mathcal{I}(D_n)$ for $n \geq 4$.*

In contrast with Problem 6.1, the graph in Figure 16 suggests that the normalized ratio may not be monotone as n increases.

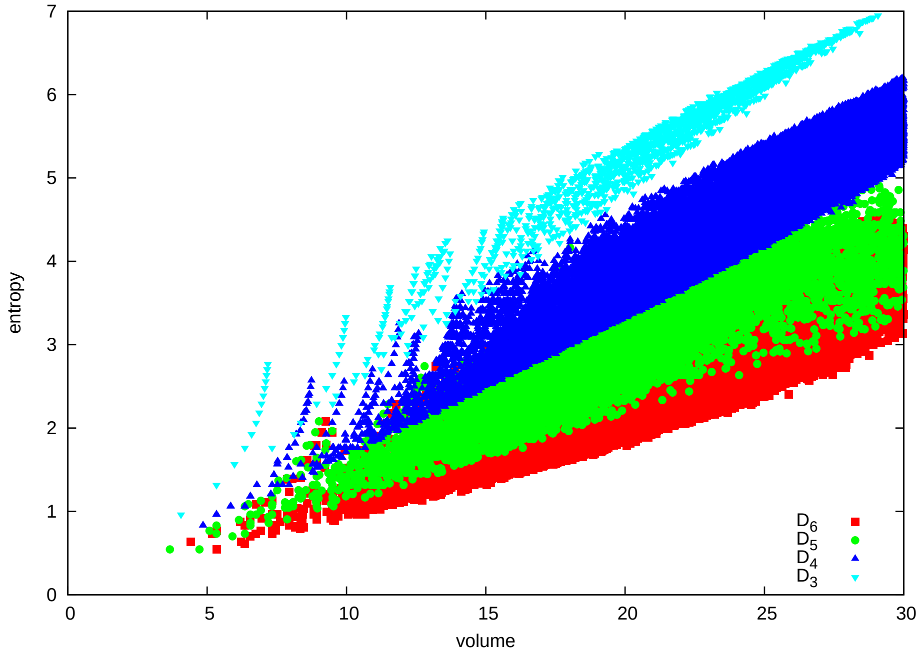

FIGURE 13. Entropy vs. volume for D_3, D_4, D_5 and D_6.

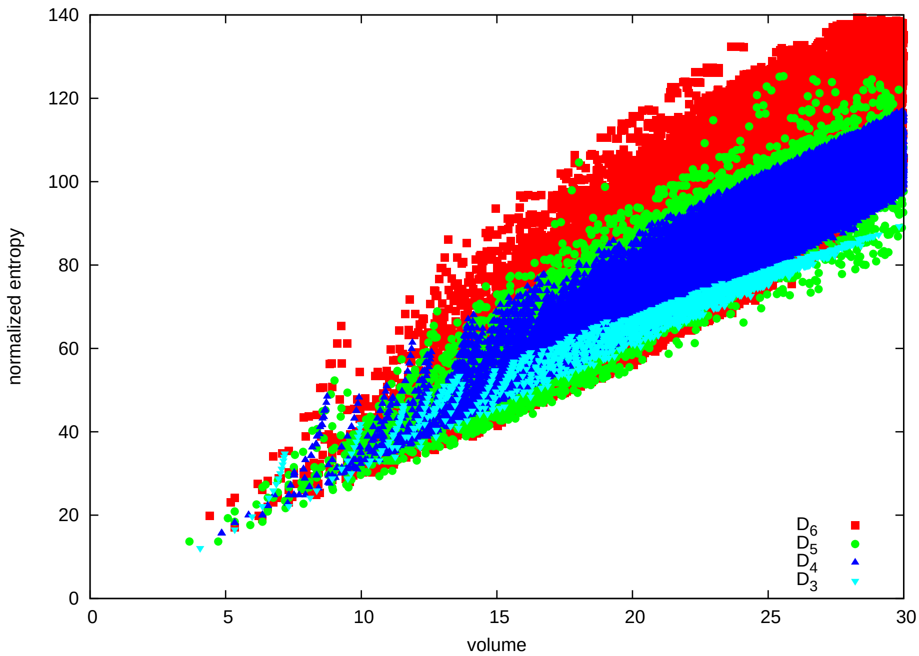

FIGURE 14. Normalized entropy vs. volume for D_3, D_4, D_5 and D_6.

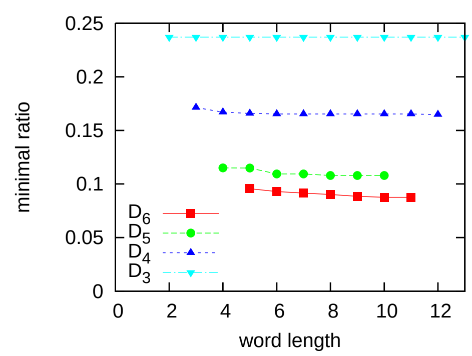

FIGURE 15. Minimal ratio.

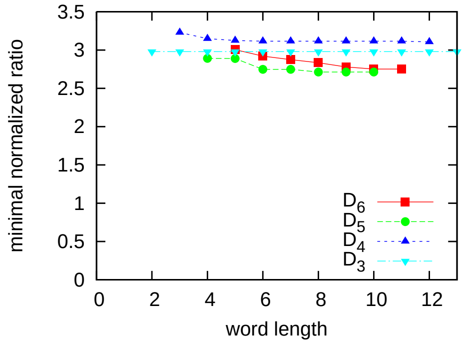

FIGURE 16. Minimal normalized ratio.

Remark 6.5. A related normalized quantity $2(n-1)\pi\,\mathrm{ent}(D_n)$ is not monotone. In fact, we see from Table 1 that if $3 \le n \le 5$, this quantity attains the largest value at $n = 4$.

6.2. **A lower bound for $\mathcal{I}(\Sigma_{1,1})$.** Recall that we choose a preferred generating set $\{\tau_1, \tau_2\}$ for $\mathcal{M}(\Sigma_{1,1})$ in section 4. The following result is well-known.

Lemma 6.6. *Any pseudo-Anosov $\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma_{1,1})$ is conjugate to a mapping class*

$$\tau_1^{m_1}\tau_2^{-n_1}\cdots\tau_1^{m_\ell}\tau_2^{-n_\ell} \tag{6.1}$$

where ℓ, m_i and n_i are some positive integers. A presentation of the mapping class $\tau_1^{m_1}\tau_2^{-n_1}\cdots\tau_1^{m_\ell}\tau_2^{-n_\ell}$ in this form is unique up to cyclic permutations. Conversely, every mapping class of the form (6.1) *is pseudo-Anosov.*

The integer ℓ in the above form is called the *block length* of ϕ.

Theorem 6.7. *For each $\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma_{1,1})$, we have*

$$\frac{\mathrm{ent}(\phi)}{\mathrm{vol}(\phi)} > \frac{\log(\frac{3+\sqrt{5}}{2})}{2v_8} \approx 0.1313,$$

where $v_8 \approx 3.6638$ is the volume of a regular ideal octahedron.

Proof. It is well-known that the mapping class group $\mathcal{M}(\Sigma_{1,1})$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$ and the dilatation of a pseudo-Anosov map corresponds to the largest real eigenvalue of the matrix representative. We set

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

These are the matrix representatives for τ_1 and τ_2^{-1} respectively. Suppose that $\phi = \tau_1^{m_1}\tau_2^{-n_1}\cdots\tau_1^{m_\ell}\tau_2^{-n_\ell}$ is a pseudo-Anosov map of block length ℓ. Then

$$M = M_1^{m_1}M_2^{n_1}M_1^{m_2}M_2^{n_2}\cdots M_1^{m_\ell}M_2^{n_\ell}$$

is the matrix representative for ϕ. Since $M \ge (M_1M_2)^\ell$, the largest eigenvalue of M is greater than that of $(M_1M_2)^\ell$. We thus have

$$\lambda(\phi) \ge \lambda((\tau_1\tau_2^{-1})^\ell) = \left(\frac{3+\sqrt{5}}{2}\right)^\ell.$$

On the other hand, using a result of Agol [1, Corollary 2.4], we have

$$\mathrm{vol}(\phi) < 2\ell v_8. \tag{6.2}$$

Hence

$$\frac{\mathrm{ent}(\phi)}{\mathrm{vol}(\phi)} > \frac{\ell \cdot \log(\frac{3+\sqrt{5}}{2})}{2\ell v_8} = \frac{\log(\frac{3+\sqrt{5}}{2})}{2v_8} \approx 0.1313.$$

□

With the aid of SnapPea, one can have a more accurate estimate for some special case.

Proposition 6.8. *For each $\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma_{1,1})$ of block length* 1*, we have*

$$\frac{\mathrm{ent}(\phi)}{\mathrm{vol}(\phi)} \geq \frac{\mathrm{ent}(\tau_1\tau_2^{-1})}{\mathrm{vol}(\tau_1\tau_2^{-1})} = \frac{\log(\frac{3+\sqrt{5}}{2})}{2v_3} \approx 0.4741. \tag{6.3}$$

Proof. Let d be the constant on the right hand side of (6.3). If $\mathrm{ent}(\phi) \geq d \cdot 2v_8$, then $\mathrm{ent}(\phi)/\mathrm{vol}(\phi) > d$ since $\mathrm{vol}(\phi) < 2v_8$ (see (6.2)). Set

$$Y = \{\phi \in \mathcal{M}^{\mathrm{pA}}(\Sigma_{1,1}) \text{ of block length } 1 \mid \mathrm{ent}(\phi) < d \cdot 2v_8 < 3.4748\}.$$

This is a finite set.

When $\phi = \tau_1^m\tau_2^{-n}$, $\lambda(\phi)$ is the largest eigenvalue of $\begin{pmatrix} 1+mn & m \\ n & 1 \end{pmatrix}$, that is,

$$\lambda(\phi) = \frac{2+mn+\sqrt{4mn+(mn)^2}}{2}.$$

If $\tau_1^m\tau_2^{-n} \in Y$, then $\lambda(\phi) < e^{3.4748} < 33$. Hence we have $mn \leq 31$. Computing $\mathrm{ent}(\phi)/\mathrm{vol}(\phi)$ for each $\phi = \tau_1^m\tau_2^{-n}$ with $mn \leq 31$ by SnapPea, we see that it is greater than or equal to d. □

Remark 6.9. The same strategy in the proof of Proposition 6.8 works for the case with a few more longer block lengths. However, the computational cost becomes larger and larger.

We thus propose

Conjecture 6.10.

$$\mathcal{I}(\Sigma_{1,1}) = \frac{\mathrm{ent}(\tau_1\tau_2^{-1})}{\mathrm{vol}(\tau_1\tau_2^{-1})} \approx 0.4741.$$

We conclude this paper by the following remark.

Remark 6.11. It is well-known that a mapping class ϕ of $\mathcal{M}(D_3)$ can be lifted to $\tilde{\phi}$ of $\mathcal{M}(\Sigma_{1,1})$ and following identities

$$\mathrm{ent}(\phi) = \mathrm{ent}(\tilde{\phi}) \quad \text{and} \quad \mathrm{vol}(\phi) = 2\mathrm{vol}(\tilde{\phi})$$

hold, see for instance [9]. Hence Conjecture 6.10 is equivalent to Conjecture 6.4 (1).

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