COMPLETE DESCRIPTION OF AGOL CYCLES OF PSEUDO-ANOSOV 3-BRAIDS

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Abstract. The equivalence class of an Agol cycle is a conjugacy invariant of a pseudo-Anosov map. Mosher defined train tracks in the torus associated to Farey intervals and investigated the relation between the train tracks and the continued fraction expansions of quadratic irrational numbers. We study Mosher's train tracks and describe Agol cycles of all the pseudo-Anosov 3-braids.

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface with genus g and n punctures. Let $\mathrm{MCG}(\Sigma)$ be the mapping class group of Σ. By the Nielsen-Thurston classification, any homeomorphism $\phi : \Sigma \to \Sigma$ is isotopic to a homeomorphism which is either *periodic*, *reducible* or *pseudo-Anosov* [8, 21]. If ϕ is a pseudo-Anosov map there exist stable and unstable measured laminations $(\mathcal{L}^{\mathrm{s}}, \nu^{\mathrm{s}})$ and $(\mathcal{L}^{\mathrm{u}}, \nu^{\mathrm{u}})$ and the dilatation $\lambda > 1$ such that

$$\phi(\mathcal{L}^{\mathrm{s}}, \nu^{\mathrm{s}}) = (\mathcal{L}^{\mathrm{s}}, \lambda\nu^{\mathrm{s}}) \text{ and } \phi(\mathcal{L}^{\mathrm{u}}, \nu^{\mathrm{u}}) = (\mathcal{L}^{\mathrm{u}}, \lambda^{-1}\nu^{\mathrm{u}}).$$

A *train track* τ is a finite embedded C^1 graph in the surface Σ equipped with a well-defined tangent line at each vertex. It also requires that no component of $\Sigma \setminus \tau$ is an immersed nullgon, monogon, bigon, once-punctured nullgon, or annulus [20]. In this paper following Agol [2] we exclusively study trivalent train tracks, i.e. train tracks all of whose vertices have valence 3. Edges of a train track are called *branches* and vertices are called *switches*. Figure 1(2),(3) illustrates a trivalent train track ω_0 in $\Sigma_{0,4}$ such that each component of $\Sigma_{0,4} \setminus \omega_0$ is a once-punctured monogon.

A branch e is *large at a switch v* of e if in a small neighborhood of v each immersed arc in the train track through v intersects the interior of e. Otherwise e is *small at the switch v*. A branch e is *large* (resp. *small*) if e is large (resp. small) at the both switches. Otherwise e is *mixed*.

A *measured train track* (τ, μ) in Σ is a train track τ together with a transverse measure μ. The transverse measure μ is a function which assigns a weight to each branch. At every

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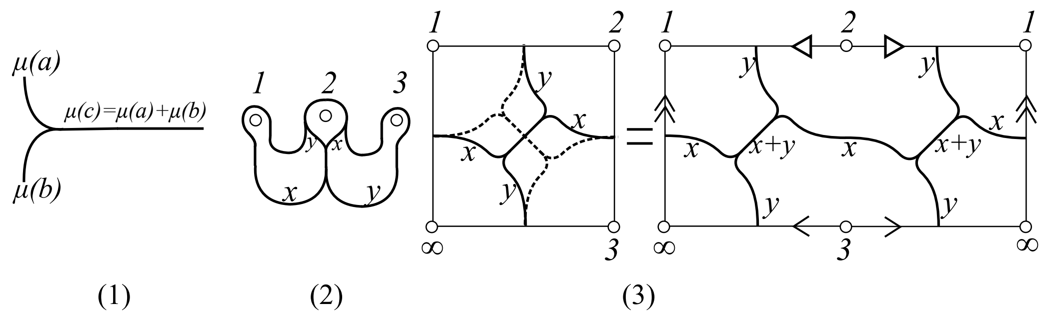

FIGURE 1. (1) Switch condition. (2), (3) The measured train track $(\omega_0, \binom{x}{y})$ in $\Sigma_{0,4}$. In (3) $\Sigma_{0,4}$ is viewed as a square pillowcase with the corners removed. Dotted arcs are branches on the backside.

switch the weights satisfy the *switch condition*: the sums of the weights on each side of the switch are equal to each other. See the equality $\mu(c) = \mu(a) + \mu(b)$ in Figure 1(1).

Left/right splitting, folding and shifting are operations on (τ, μ) that give a new measured train track.

Definition 1.1. (1) Figure 2(1) shows a neighborhood of a large branch of (τ, μ) whose weight is $x + y = z + w$ by the switch condition. If $z > x$ equivalently $y > w$ (resp. $x > z$ equivalently $w > y$) the measured train track (τ, μ) admits a *left* (resp. *right*) *splitting* at the large branch.

(2) A *shifting* (at a mixed branch) is an operation on (τ, μ) as depicted in Figure 2(2).

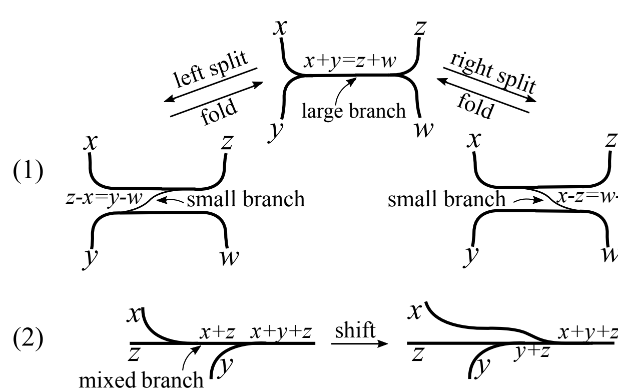

FIGURE 2. (1) Left splitting when $z > x$ ($\Leftrightarrow y > w$), right splitting when $x > z$ ($\Leftrightarrow w > y$) at a large branch, and folding at a small branch. (2) Shifting at a mixed branch.

A *maximal splitting* [2] of a measured train track (τ, μ) is a set of simultaneous splittings along all the large branches of maximal μ-weight. We denote $(\tau, \mu) \rightharpoonup (\tau', \mu')$ if (τ', μ') is the result of the maximal splitting. Consecutive n maximal splittings

$$(\tau, \mu) \rightharpoonup (\tau_1, \mu_1) \rightharpoonup \cdots \rightharpoonup (\tau_n, \mu_n)$$

is denoted by $(\tau, \mu) \rightharpoonup^n (\tau_n, \mu_n)$. If all the splittings in a maximal splitting are of left (resp. right) type then we write $(\tau, \mu) \overset{\mathrm{l}}{\rightharpoonup} (\tau', \mu')$ (resp. $(\tau, \mu) \overset{\mathrm{r}}{\rightharpoonup} (\tau', \mu')$) and say that (τ, μ) admits a *left* (resp. *right*) *maximal splitting.*

Let $\phi : \Sigma \to \Sigma$ be a homeomorphism. For a measured train track (τ, μ) in Σ, we define a new measured train track $\phi(\tau, \mu)$ in Σ by $\phi(\tau, \mu) := (\phi(\tau), \phi_*(\mu))$, where the measure $\phi_*(\mu)$ is defined by $\phi_*(\mu)(e) := \mu(\phi^{-1}(e))$ for every branch e in the train track $\phi(\tau)$.

The following theorem by Agol is a starting point of our study.

Theorem 1.2 (Theorem 3.5 in [2])**.** *Let $\phi : \Sigma \to \Sigma$ be a pseudo-Anosov map with dilatation λ. Let (τ, μ) be a measured train track suited to the stable measured lamination of ϕ. Then there exist $n \geq 0$ and $m > 0$ such that*

$$(\tau, \mu) \rightharpoonup^n (\tau_n, \mu_n) \rightharpoonup^m (\tau_{n+m}, \mu_{n+m}),$$

where $(\tau_{n+m}, \mu_{n+m}) = \phi(\tau_n, \lambda^{-1}\mu_n) = (\phi(\tau_n), \lambda^{-1}\phi_*(\mu_n))$.

See also Agol-Tsang [3]. We note that in [22] Wu worked out orbifold theory and gave more efficient encoding of splitting sequences for braids.

For the terminology *suited to,* see Definition 2.3. Theorem 1.2 states that the maximal splitting sequence is eventually periodic modulo the action of ϕ and a scaling by the dilatation. We call the maximal splitting sequence

$$(\tau_n, \mu_n) \rightharpoonup^m (\tau_{n+m}, \mu_{n+m}) = \phi(\tau_n, \lambda^{-1}\mu_n) \rightharpoonup \cdots$$

a *periodic splitting sequence* of ϕ. We may call the finite subsequence

$$(\tau_n, \mu_n) \rightharpoonup^m (\tau_{n+m}, \mu_{n+m}) = \phi(\tau_n, \lambda^{-1}\mu_n)$$

an *Agol cycle* of ϕ and say that the *length* of the Agol cycle is m. Clearly, the maximal splitting sequence $(\tau_{n+1}, \mu_{n+1}) \rightharpoonup^m (\tau_{n+m+1}, \mu_{n+m+1}) \rightharpoonup \cdots$ starting at (τ_{n+1}, μ_{n+1}) is also a periodic splitting sequence of ϕ, and the finite sequence $(\tau_{n+1}, \mu_{n+1}) \rightharpoonup^m (\tau_{n+m+1}, \mu_{n+m+1})$ is an Agol cycle of ϕ as well.

Definition 1.3. Let $\phi, \phi' : \Sigma \to \Sigma$ be pseudo-Anosov maps with periodic splitting sequences

$$\mathscr{P} : (\tau_n, \mu_n) \rightharpoonup^m (\tau_{n+m}, \mu_{n+m}) = \phi(\tau_n, \lambda^{-1}\mu_n) \rightharpoonup \cdots$$

of ϕ and

$$\mathscr{P}' : (\tau'_{n'}, \mu'_{n'}) \rightharpoonup^{m'} (\tau'_{n'+m'}, \mu'_{n'+m'}) = \phi'(\tau'_{n'}, (\lambda')^{-1}\mu'_{n'}) \rightharpoonup \cdots$$

of ϕ'. We say that $\mathscr{P}$ and $\mathscr{P}'$ are *combinatorially isomorphic* ([13]) if $m = m'$ and there exists an orientation-preserving diffeomorphism $h : \Sigma \to \Sigma$, integers $p, q \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{R}_{>0}$ such that the following conditions (1) and (2) hold.

(1) $\phi' = h \circ \phi \circ h^{-1}$.

(2) $h(\tau_{p+i}, \mu_{p+i}) = (\tau'_{q+i}, c\mu'_{q+i})$ for all $i \in \mathbb{Z}_{\geq 0}$.

We say that Agol cycles $(\tau_n, \mu_n) \rightharpoonup^m (\tau_{n+m}, \mu_{n+m})$ of ϕ and $(\tau'_{n'}, \mu'_{n'}) \rightharpoonup^{m'} (\tau'_{n'+m'}, \mu'_{n'+m'})$ of ϕ' are *equivalent* if $m = m'$ and there exists a diffeomorphism $h : \Sigma \to \Sigma$, non-negative integers p and p', and a number $c > 0$ such that $h(\tau_{n+p}, \mu_{n+p}) = (\tau'_{n'+p'}, c\mu'_{n'+p'})$. This implies the above condition (2), see Lemma 2.2.

By the definition, Agol cycles of ϕ and ϕ' are equivalent if periodic splitting sequences $\mathscr{P}$ of ϕ and $\mathscr{P}'$ of ϕ' are combinatorially isomorphic. Hodgson-Issa-Segerman proved the following.

Theorem 1.4 (Theorem 5.3 in [13]). *The mapping classes of pseudo-Anosov maps $\phi, \phi' : \Sigma \to \Sigma$ are conjugate in $\mathrm{MCG}(\Sigma)$ if and only if $\mathscr{P}$ and $\mathscr{P}'$ are combinatorially isomorphic.*

As a consequence, the equivalence class of an Agol cycle of ϕ is a conjugacy invariant of the pesudo-Anosov map ϕ. The length of an Agol cycle of ϕ is also a conjugacy invariant and we call it the *Agol cycle length* of ϕ.

Each element of the 3-braid group B_3 induces an element of the mapping class group $\mathrm{MCG}(\Sigma_{0,4})$ of a 4-punctured sphere (Section 4.1). By Murasugi's classification of 3-braids [19, Proposition 2.1], a braid $b \in B_3$ is pseudo-Anosov (i.e. a representative of the corresponding mapping class in $\mathrm{MCG}(\Sigma_{0,4})$ is pseudo-Anosov) if and only if b is conjugate to a braid $\Delta^{2j}\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_k}\sigma_2^{-q_k}$ where $\Delta = \sigma_1\sigma_2\sigma_1$, j is an integer, and $p_1, q_1, \cdots, p_k, q_k$ and k are positive integers. Moreover, j and k are unique and the pairs $(p_1, q_1), \cdots, (p_k, q_k)$ are unique up to cyclic permutation.

In this paper we give a complete description of the Agol cycles of pseudo-Anosov 3-braids:

Theorem 1.5 (cf. Theorem 4.1). *Let $\beta = \Delta^{2j}\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_k}\sigma_2^{-q_k}$ be a pseudo-Anosov 3-braid with dilatation λ. Let $\ell = p_1 + q_1 + \cdots + p_k + q_k$ and s be the number admitting a purely periodic continued fraction expansion $[\overline{q_k : p_k, q_{k-1}, p_{k-1}, \cdots, q_1, p_1}]$.*

1. *The measured train track (ω_0, ν_0) with $\nu_0 = \binom{1}{s}$ (Figure 1(3)) is suited to the stable measured lamination of β.*
2. *Let $A = L^{q_k}R^{p_k}\cdots L^{q_1}R^{p_1}$ where $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $A\binom{1}{s} = \lambda\binom{1}{s}$.*
3. *Starting with the measured train track (ω_0, ν_0), the first $\ell + 1$ terms*

$$(\omega_0, \nu_0) \overset{\mathrm{l}}{\rightharpoonup}^{q_k} \overset{\mathrm{r}}{\rightharpoonup}^{p_k} \cdots \overset{\mathrm{l}}{\rightharpoonup}^{q_1} \overset{\mathrm{r}}{\rightharpoonup}^{p_1} (\omega_\ell, \nu_\ell) \tag{1.1}$$

of the maximal splitting sequence form a length ℓ Agol cycle of β.

In other words, (ω_0, ν_0) admits q_k left maximal splittings consecutively. Then the resulting measured train track $(\omega_{q_k}, \nu_{q_k})$ admits p_k right maximal splittings consecutively. After repeating this $k - 1$ more times we obtain the measured train track (ω_ℓ, ν_ℓ).

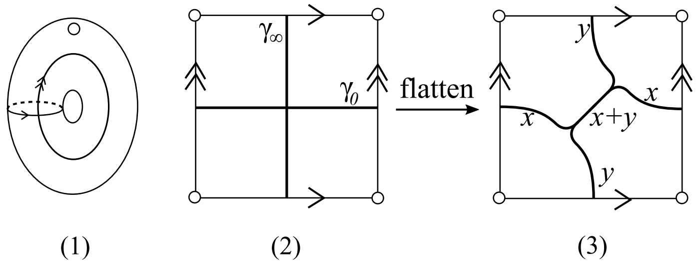

FIGURE 3. (1) Once-punctured torus $\Sigma_{1,1}$. (2) Simple closed curves of slope 0 and ∞. (3) The measured train track $(\tau_0, \binom{x}{y})$ in $\Sigma_{1,1}$.

Here is a geometric description of the above train tracks ω_i $(i = 0, \cdots, \ell)$: Each (ω_i, ν_i) can be written by $(\omega_i, \nu_i) = (\omega_i, K_i{}^{-1}\binom{1}{s})$ where $K_i = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z})$. In the once-punctured torus $\Sigma_{1,1}$ we consider simple closed curves $\gamma_{\frac{b}{a}}$ of slope $\frac{b}{a}$ and $\gamma_{\frac{d}{c}}$ of slope $\frac{d}{c}$ that intersect at a single point. Flattening the obtuse angles at the intersection (Figure 6(2),(3)) gives a train track τ_i in $\Sigma_{1,1}$. Two copies of τ_i nicely placed in $\Sigma_{0,4}$ gives the train track ω_i. For example in the case $i = 0$ we have $(\omega_0, \nu_0) = (\omega_0, K_0{}^{-1}\binom{1}{s})$ with $K_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Figure 3(3) illustrates τ_0 and the two copies of τ_0 give the train track ω_0 as in Figure 1(3).

As a corollary of Theorem 1.5 we have the following.

Corollary 4.6. *Pseudo-Anosov 3-braids β and β' are conjugate in $B_3/Z(B_3)$ where $Z(B_3)$ is the center of B_3 if and only if their Agol cycles are equivalent.*

On the other hand, the equivalence class of an Agol cycle is not a complete conjugacy invariant of a pseudo-Anosov element of $\mathrm{MCG}(\Sigma_{1,1})$ (Proposition 3.14).

Given a pseudo-Anosov map $\phi : \Sigma \to \Sigma$, let $\Sigma^\circ \subset \Sigma$ denote the surface obtained from Σ by removing all the singular points of the stable/unstable foliations for ϕ, and let $\phi^\circ : \Sigma^\circ \to \Sigma^\circ$ be the restriction of ϕ. Agol uses an Agol cycle of ϕ° to give a veering ideal triangulation of the mapping torus of ϕ° [2]. If $\phi : \Sigma_{0,4} \to \Sigma_{0,4}$ is a pseudo-Anosov map induced by a pseudo-Anosov 3-braid then $\Sigma_{0,4}^\circ = \Sigma_{0,4}$ so $\phi^\circ = \phi$. Every maximal splitting in the Agol cycle (1.1) takes place in two large branches for each maximal splitting, which yields two tetrahedra in Agol's construction. Thus, for the 3-braid case, Theorem 1.5 states that twice the Agol cycle length is exactly the number of tetrahedra in the Agol's triangulation.

To prove Theorem 1.5, we give a thorough description of an Agol cycle of every pseudo-Anosov map on $\Sigma_{1,1}$ up to the hyperelliptic involution (Theorem 3.6). It is known that the veering ideal triangulation of the mapping torus obtained from the Agol cycle is the canonical triangulation constructed in Lackenby [15] and Guéritaud [11].

Question 1.6 (Margalit [16])**.** For a fixed surface, what are the possible lengths of Agol cycles? How does the length of the Agol cycle relate to other invariants?

Theorem 1.5 implies that any integer greater than 1 can be realized as the Agol cycle length of some pseudo-Anosov 3-braid. Theorem 1.5 partially answers the second question by Margalit as follows.

Theorem 4.7. *For every pseudo-Anosov* 3*-braid* β*, the Agol cycle length of* β*, the Garside canonical length of any element in the super summit set* $\mathrm{SSS}(\beta)$ *are the same.*

Question 1.7. Is there a pseudo-Anosov map whose Agol cycle length is 1?

We note that a length 1 Agol cycle does not necessarily mean that the induced veering triangulation of the mapping torus consists of one ideal tetrahedron, since a maximal splitting may contain multiple splittings simultaneously.

The paper is organized as follows. In Section 2, we recall basic definitions and facts regarding measured train tracks and laminations. In Section 3, we introduce train tracks in $\Sigma_{1,1}$ associated to Farey intervals following Mosher [17, 18]. Then we study Agol cycles of pseudo-Anosov maps on $\Sigma_{1,1}$. In Section 4, we prove Theorem 1.5. We also discuss a relation between Garside canonical lengths and Agol cycle lengths for 3-braids and prove Theorem 4.7.

2. Preliminaries

The mapping class group $\mathrm{MCG}(\Sigma)$ of a surface $\Sigma = \Sigma_{g,n}$ is the group of isotopy classes of orientation preserving homeomorphisms of Σ which preserve the punctures setwise. For simplicity, we do not distinguish between a homeomorphism $\phi : \Sigma \to \Sigma$ and its mapping class $[\phi] \in \mathrm{MCG}(\Sigma)$.

Measured train tracks are useful tools to encode measured laminations. Measured train tracks (τ, μ), (τ', μ') in Σ are *equal* (and write $(\tau, \mu) = (\tau', \mu')$) if there exists a diffeomorphism $f : \Sigma \to \Sigma$ isotopic to the identity map on Σ such that $f(\tau, \mu) = (\tau', \mu')$.

Measured train tracks (τ, μ), (τ', μ') in Σ are *equivalent* if they are related to each other by a sequence of splittings, foldings, shiftings (Definition 1.1) and isotopies. Equivalence classes of measured train tracks are in one-to-one correspondence with measured laminations [20, Theorem 2.8.5]. For example, all the five measured train tracks in $\Sigma_{0,4}$ in Figure 4 are equivalent for any $s > 0$.

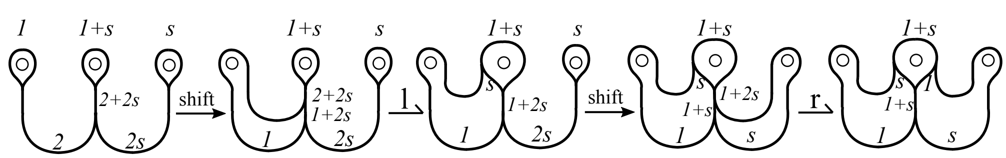

Figure 4. The last measured train track is $(\omega_0, (\begin{smallmatrix}1\\ s\end{smallmatrix}))$ in Figure 1(2).

We adopt the following conventions.

(1) When we regard a maximal splitting $(\tau, \mu) \rightharpoonup (\tau', \mu')$ as an operation on the measured train track we may write $(\tau', \mu') = \rightharpoonup (\tau, \mu)$.
(2) Similarly, using the operator notation we may write n consecutive left (resp. right) maximal splittings $(\tau, \mu) \overset{\mathrm{l}}{\rightharpoonup}^{n} (\tau_n, \mu_n)$ (resp. $(\tau, \mu) \overset{\mathrm{r}}{\rightharpoonup}^{n} (\tau_n, \mu_n)$) as
$$(\tau_n, \mu_n) = \overset{\mathrm{l}}{\rightharpoonup}^{n} (\tau, \mu), \quad \text{(resp. } (\tau_n, \mu_n) = \overset{\mathrm{r}}{\rightharpoonup}^{n} (\tau, \mu)).$$
(3) We may also write a finite sequence $(\tau, \mu) \overset{\mathrm{l}}{\rightharpoonup}^{n} (\tau_n, \mu_n) \overset{\mathrm{r}}{\rightharpoonup}^{m} (\tau_{n+m}, \mu_{n+m})$ as
$$(\tau_{n+m}, \mu_{n+m}) = \overset{\mathrm{r}}{\rightharpoonup}^{m} \circ \overset{\mathrm{l}}{\rightharpoonup}^{n} (\tau, \mu);$$
that is, first apply $\overset{\mathrm{l}}{\rightharpoonup}^{n}$ to (τ, μ) then next apply $\overset{\mathrm{r}}{\rightharpoonup}^{m}$ to obtain (τ_{n+m}, μ_{n+m}).

The next lemma states that the operation $\rightharpoonup$ and a map $\phi : \Sigma \to \Sigma$ commute on measured train tracks in Σ.

Lemma 2.1. *Let (τ, μ) be a measured train track in Σ. Let $\phi : \Sigma \to \Sigma$ be an orientation-preserving diffeomorphism. If (τ, μ) admits consecutive n maximal left splittings, then we have*
$$(\phi \circ \overset{\mathrm{l}}{\rightharpoonup}^{n})(\tau, \mu) = (\overset{\mathrm{l}}{\rightharpoonup}^{n} \circ \phi)(\tau, \mu).$$
A parallel statement holds for right splittings.

Proof. Since a left splitting operation is supported in a small oriented disk neighborhood of the large branch, they commute with any orientation-preserving diffeomorphism $\phi : \Sigma \to \Sigma$ up to isotopy. This gives $(\phi \circ \overset{\mathrm{l}}{\rightharpoonup})(\tau, \mu) = (\overset{\mathrm{l}}{\rightharpoonup} \circ \phi)(\tau, \mu)$. Repeating this for n times, we obtain $(\phi \circ \overset{\mathrm{l}}{\rightharpoonup}^{n})(\tau, \mu) = (\overset{\mathrm{l}}{\rightharpoonup}^{n} \circ \phi)(\tau, \mu)$. ☐

As a corollary of Lemma 2.1 we have the following.

Lemma 2.2. *Let $(\tau_n, \mu_n) \rightharpoonup (\tau_{n+1}, \mu_{n+1}) \rightharpoonup \cdots$ and $(\tau'_{n'}, \mu'_{n'}) \rightharpoonup (\tau'_{n'+1}, \mu'_{n'+1}) \rightharpoonup \cdots$ be maximal splitting sequences. If there exist an orientation-preserving diffeomorphism $h : \Sigma \to \Sigma$, integers $p \geq n$, $q \geq n'$, and a positive number c such that $h(\tau_p, \mu_p) = (\tau'_q, c\mu'_q)$ then $h(\tau_{p+i}, \mu_{p+i}) = (\tau'_{q+i}, c\mu'_{q+i})$ for all $i \geq 0$.*

Definition 2.3. Let $(\mathcal{L}, \nu)$ be a measured lamination in Σ, and let (τ, μ) be a measured train track in Σ. Then $(\mathcal{L}, \nu)$ is *suited* to (τ, μ), and we also say that (τ, μ) is *suited* to $(\mathcal{L}, \nu)$, if there exists a differentiable map $f : \Sigma \to \Sigma$ homotopic to the identity map on Σ with the following conditions:

- $f(\mathcal{L}) = \tau$.
- f is non-singular on the tangent spaces to the leaves of $\mathcal{L}$.
- If p is an interior point of a branch e of τ then $\nu(f^{-1}(p)) = \mu(e)$.

3. Once-punctured torus

3.1. **Mapping class group of once-punctured torus via** $\mathrm{SL}(2;\mathbb{Z})$**.** The quotient space $\mathbb{R}^2/\mathbb{Z}^2$ gives the torus, $\mathbb{T} = S^1 \times S^1$. If no confusion occurs, under the quotient map (or a covering map)

$$Q : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}$$

the image of $(x, y) \in \mathbb{R}^2$ will be denoted by the same (x, y). We may think the torus is obtained by the square $[0, 1] \times [0, 1]$ whose parallel boundary edges are identified.

The special linear group $\mathrm{SL}(2;\mathbb{Z})$ is generated by

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $A \in \mathrm{SL}(2;\mathbb{Z})$, the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$; $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto A \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ induces a well-defined homeomorphism $\bar{f}_A : \mathbb{T} \to \mathbb{T}$. Note that $\bar{f}_A$ fixes the point $(0, 0)$. The restriction of $\bar{f}_A$ to the once-punctured torus $\Sigma_{1,1} = \mathbb{T} \setminus \{(0, 0)\} = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ yields a homeomorphism, denoted by

$$f_A : \Sigma_{1,1} \to \Sigma_{1,1}.$$

We have $Q \circ A \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = f_A \circ Q \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ and the map $\mathrm{SL}(2;\mathbb{Z}) \to \mathrm{MCG}(\Sigma_{1,1})$ which takes A to f_A gives a group isomorphism.

Observe that f_L induced by L is the left-handed Dehn twist about a simple closed curve with the slope ∞ and f_R induced by R is the right-handed Dehn twist about a simple closed curve with the slope 0. (cf. Figure 5 for f_L and f_R.)

Let $\mathrm{tr}(A)$ denote the trace of $A \in \mathrm{SL}(2;\mathbb{Z})$. We have $|\mathrm{tr}(A)| > 2$ if and only if the induced map $\bar{f}_A : \mathbb{T} \to \mathbb{T}$ is Anosov [8, Section 13.1]. A parallel statement for the punctured torus $\Sigma_{1,1}$ is that $|\mathrm{tr}(A)| > 2$ if and only if the induced map $f_A : \Sigma_{1,1} \to \Sigma_{1,1}$ is pseudo-Anosov. The condition $\mathrm{tr}(A) > 2$ is equivalent to that A possesses distinct eigenvalues $\lambda > 1$ and $0 < \lambda^{-1} < 1$. We call λ the *expanding eigenvalue* of A. Since the pseudo-Anosov map f_A is restriction of the Anosov map $\bar{f}_A$, their dilatations are the same and equal to the expanding eigenvalue λ. Let $\left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right)$ be an eigenvector with respect to λ; that is, $A \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right) = \lambda \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right)$. We call s the *slope* of the eigenvectors with respect to λ for A.

The following proposition is well known (cf. [11, Proposition 2.1]).

Proposition 3.1. *Let $A \in \mathrm{SL}(2;\mathbb{Z})$ with $\mathrm{tr}(A) > 2$. Then A is conjugate to $L^{q_k} R^{p_k} \cdots L^{q_1} R^{p_1}$ for some positive integers $p_1, q_1, \cdots, p_k, q_k$ and k. Moreover, the ordered pairs $(p_1, q_1), \cdots, (p_k, q_k)$ are unique up to cyclic permutation.*

3.2. **Mosher's train track.** In the rest of the paper, we assume that a, b, c, d are nonnegative integers with $ad - bc = 1$. In other words, $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{SL}^+(2;\mathbb{Z})$, where $\mathrm{SL}^+(2;\mathbb{Z})$ is the monoid generated by L and R. This condition is equivalent to that $\frac{b}{a}$ and $\frac{d}{c}$ are joined by an arc in the *Farey diagram* (see [12, Section 1.1] for the Farey diagram) and $\frac{0}{1} \leq \frac{b}{a} < \frac{d}{c} \leq \frac{1}{0}$. We call the interval $[\frac{b}{a}, \frac{d}{c}]$ a *Farey interval.* We say that $[\frac{b}{a}, \frac{d}{c}]$ is the *corresponding Farey* *interval* to the matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. We also say that $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ is the *corresponding matrix* to the Farey interval $[\frac{b}{a}, \frac{d}{c}]$.

In [17] and Sections 1.3 and 10.1 of [18] Mosher defined a train track $\tau_{[\frac{b}{a},\frac{d}{c}]}$ in the torus $\mathbb{T}$ and showed an intriguing relation between the train track and a continued fraction expansion. In this paper we study Mosher's train track in the subspace $\Sigma_{1,1} \subset \mathbb{T}$.

We now define a train track $\tau_{[\frac{b}{a},\frac{d}{c}]}$ in $\Sigma_{1,1}$ inductively. Our definition is different from Mosher's original one but in Proposition 3.5 we will see ours coincides with Mosher's.

Let $\tau_{[\frac{0}{1},\frac{1}{0}]} := \tau_0$ be a train track with measure $\mu = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ in $\Sigma_{1,1}$ as shown in Figure 3(3). The component of $\Sigma_{1,1} \setminus \tau_0$ is a once-punctured bigon. The train track τ_0 consists of three branches, one large branch and two small branches. The small branches from South and West merge to form a large branch then it separates into two small branches going North and East. We call τ_0 the *base train track*. The vector $\mu = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ represents the weight x of the horizontal small branch and the weight y of the vertical small branch. From now on, the notation τ_0 is exclusively used for the base train track.

Assume that we have defined a train track $\tau_{[\frac{b}{a},\frac{d}{c}]}$ consisting of one large and two small branches. Note that $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) L = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+c & c \\ b+d & d \end{smallmatrix}\right) \in \mathrm{SL}^+(2;\mathbb{Z})$. In the Farey diagram, the corresponding Farey interval $[\frac{b+d}{a+c}, \frac{d}{c}]$ is the *right* half of the original Farey interval $[\frac{b}{a}, \frac{d}{c}]$. We write

$$\left[\frac{b}{a}, \frac{d}{c}\right] \overset{\text{r-half}}{\supset} \left[\frac{b+d}{a+c}, \frac{d}{c}\right].$$

We define the train track $\tau_{[\frac{b+d}{a+c},\frac{d}{c}]}$ as a result of the left splitting of $\tau_{[\frac{b}{a},\frac{d}{c}]}$ at the unique large branch. Although this is a topological operation (the measure is forgotten) abusing the left maximal splitting symbol, $\overset{\mathrm{l}}{\rightharpoonup}$, on measured train tracks, we may write

$$\tau_{[\frac{b}{a},\frac{d}{c}]} \overset{\mathrm{l}}{\rightharpoonup} \tau_{[\frac{b+d}{a+c},\frac{d}{c}]}. \tag{3.1}$$

Similarly, the corresponding Farey interval of the matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) R = \left(\begin{smallmatrix} a & a+c \\ b & b+d \end{smallmatrix}\right) \in \mathrm{SL}^+(2;\mathbb{Z})$ is $[\frac{b}{a}, \frac{b+d}{a+c}]$ which is the *left* half of the original Farey interval $[\frac{b}{a}, \frac{d}{c}]$. We write

$$\left[\frac{b}{a}, \frac{d}{c}\right] \overset{\text{l-half}}{\supset} \left[\frac{b}{a}, \frac{b+d}{a+c}\right].$$

We define the train track $\tau_{[\frac{b}{a},\frac{b+d}{a+c}]}$ as a result of the right splitting of $\tau_{[\frac{b}{a},\frac{d}{c}]}$. Again, abusing the right maximal splitting symbol, $\overset{\mathrm{r}}{\rightharpoonup}$, we may write

$$\tau_{[\frac{b}{a},\frac{d}{c}]} \overset{\mathrm{r}}{\rightharpoonup} \tau_{[\frac{b}{a},\frac{b+d}{a+c}]}. \tag{3.2}$$

The both new train tracks $\tau_{[\frac{b+d}{a+c},\frac{d}{c}]}$ and $\tau_{[\frac{b}{a},\frac{b+d}{a+c}]}$ consist of three branches, one large and two small branches.

For every Farey interval $[\frac{b}{a}, \frac{d}{c}]$, one can find a unique finite nested sequence of Farey intervals starting from $[\frac{0}{1}, \frac{1}{0}]$ and choosing the left/right half of it. Therefore, the train track $\tau_{[\frac{b}{a}, \frac{d}{c}]}$ is well-defined.

Example 3.2. The Farey interval $[\frac{10}{7}, \frac{3}{2}]$ is uniquely obtained as follows:

$$\left[\frac{0}{1}, \frac{1}{0}\right] \overset{\text{r-half}}{\supset} \left[\frac{1}{1}, \frac{1}{0}\right] \overset{\text{l-half}}{\supset} \left[\frac{1}{1}, \frac{2}{1}\right] \overset{\text{l-half}}{\supset} \left[\frac{1}{1}, \frac{3}{2}\right] \overset{\text{r-half}}{\supset} \left[\frac{4}{3}, \frac{3}{2}\right] \overset{\text{r-half}}{\supset} \left[\frac{7}{5}, \frac{3}{2}\right] \overset{\text{r-half}}{\supset} \left[\frac{10}{7}, \frac{3}{2}\right].$$

In terms of the corresponding matrices, we get

$$\left(\begin{smallmatrix} 7 & 2 \\ 10 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) LRRLLL,$$

where $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the corresponding matrix of the Farey interval $[\frac{0}{1}, \frac{1}{0}]$. The rule is to convert $\overset{\text{l-half}}{\supset} / \overset{\text{r-half}}{\supset}$ into the matrix R/L and multiply it from the right. Here we remark that l-half becomes R, and r-half becomes L. Thus, the train track $\tau_{[\frac{10}{7}, \frac{3}{2}]}$ is defined as a result of the following consecutive splittings:

$$\tau_{[\frac{0}{1}, \frac{1}{0}]} \overset{\mathrm{l}}{\rightharpoonup} \tau_{[\frac{1}{1}, \frac{1}{0}]} \overset{\mathrm{r}}{\rightharpoonup} \tau_{[\frac{1}{1}, \frac{2}{1}]} \overset{\mathrm{r}}{\rightharpoonup} \tau_{[\frac{1}{1}, \frac{3}{2}]} \overset{\mathrm{l}}{\rightharpoonup} \tau_{[\frac{4}{3}, \frac{3}{2}]} \overset{\mathrm{l}}{\rightharpoonup} \tau_{[\frac{7}{5}, \frac{3}{2}]} \overset{\mathrm{l}}{\rightharpoonup} \tau_{[\frac{10}{7}, \frac{3}{2}]},$$

which is also written as $\tau_{[\frac{0}{1}, \frac{1}{0}]} \overset{\mathrm{l}}{\rightharpoonup} \overset{\mathrm{r}}{\rightharpoonup}{}^{2} \overset{\mathrm{l}}{\rightharpoonup}{}^{3} \tau_{[\frac{10}{7}, \frac{3}{2}]}$.

Recall that $f_L, f_R : \Sigma_{1,1} \to \Sigma_{1,1}$ are induced by $L, R \in \mathrm{SL}(2; \mathbb{Z})$ respectively.

Lemma 3.3. *We have* $\tau_0 \overset{\mathrm{l}}{\rightharpoonup} \tau_{[\frac{1}{1}, \frac{1}{0}]} = f_L(\tau_0)$ *and* $\tau_0 \overset{\mathrm{r}}{\rightharpoonup} \tau_{[\frac{0}{1}, \frac{1}{1}]} = f_R(\tau_0)$. *In other words* $f_L(\tau_0) = \overset{\mathrm{l}}{\rightharpoonup} (\tau_0)$ *and* $f_R(\tau_0) = \overset{\mathrm{r}}{\rightharpoonup} (\tau_0)$.

Proof. The left (resp. right) splitting of $\tau_0 = \tau_{[\frac{0}{1}, \frac{1}{0}]}$ yields $\tau_{[\frac{1}{1}, \frac{1}{0}]}$ (resp. $\tau_{[\frac{0}{1}, \frac{1}{1}]}$) by (3.1) (resp. (3.2)) and it is equal to $f_L(\tau_0)$ (resp. $f_R(\tau_0)$) by Figure 5. $\square$

Proposition 3.4. *For the base train track* τ_0 *we have the following.*

1. *For* $p, q \geq 1$ *we have* $\tau_0 \overset{\mathrm{l}}{\rightharpoonup}{}^{q} \tau_{[\frac{q}{1}, \frac{1}{0}]} = f_L^q(\tau_0)$ *and* $\tau_0 \overset{\mathrm{r}}{\rightharpoonup}{}^{p} \tau_{[\frac{0}{1}, \frac{1}{p}]} = f_R^p(\tau_0)$. *In other words* $\overset{\mathrm{l}}{\rightharpoonup}{}^{q} (\tau_0) = f_L^q(\tau_0)$ *and* $\overset{\mathrm{r}}{\rightharpoonup}{}^{p} (\tau_0) = f_R^p(\tau_0)$.
2. *For positive integers* $p_1, q_1, \cdots, p_k, q_k$ *and* k *we have*
$$\tau_0 \overset{\mathrm{l}}{\rightharpoonup}{}^{q_k} \overset{\mathrm{r}}{\rightharpoonup}{}^{p_k} \cdots \overset{\mathrm{l}}{\rightharpoonup}{}^{q_1} \overset{\mathrm{r}}{\rightharpoonup}{}^{p_1} \tau_\ell,$$
where $\ell = p_1 + q_1 + \cdots + p_k + q_k$ *and* $\tau_\ell = (f_L^{q_k} \circ f_R^{p_k} \circ \cdots \circ f_L^{q_1} \circ f_R^{p_1})(\tau_0)$. *In other words*
$$(f_L^{q_k} \circ f_R^{p_k} \circ \cdots \circ f_L^{q_1} \circ f_R^{p_1})(\tau_0) = (\overset{\mathrm{r}}{\rightharpoonup}{}^{p_1} \circ \overset{\mathrm{l}}{\rightharpoonup}{}^{q_1} \circ \cdots \circ \overset{\mathrm{r}}{\rightharpoonup}{}^{p_k} \circ \overset{\mathrm{l}}{\rightharpoonup}{}^{q_k})(\tau_0).$$

Proof. We first prove Statement (1). When $q = 1$ we have $\tau_0 \overset{\mathrm{l}}{\rightharpoonup} f_L(\tau_0)$ by Lemma 3.3. Suppose that $q = 2$. By Lemmas 2.1 and 3.3 it follows that

$$(\overset{\mathrm{l}}{\rightharpoonup} \circ \overset{\mathrm{l}}{\rightharpoonup})(\tau_0) = (\overset{\mathrm{l}}{\rightharpoonup} \circ f_L)(\tau_0) = (f_L \circ \overset{\mathrm{l}}{\rightharpoonup})(\tau_0) = (f_L \circ f_L)(\tau_0).$$

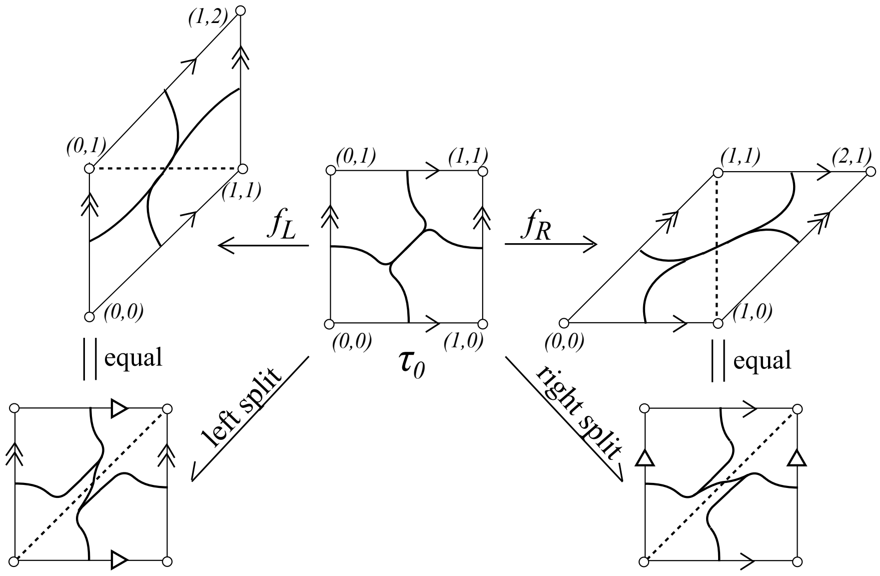

FIGURE 5. Proof of Lemma 3.3: $\tau_0 \overset{\mathrm{l}}{\rightharpoondown} f_L(\tau_0)$ and $\tau_0 \overset{\mathrm{r}}{\rightharpoondown} f_R(\tau_0)$.

Thus $\tau_0 \overset{\mathrm{l}}{\rightharpoondown}{}^2 f_L^2(\tau_0)$. Repeating this argument we have $\tau_0 \overset{\mathrm{l}}{\rightharpoondown}{}^q f_L^q(\tau_0)$. One can similarly prove $\tau_0 \overset{\mathrm{r}}{\rightharpoondown}{}^p f_R^p(\tau_0)$.

For Statement (2), we first consider the case $k = 1$. Statement (1) together with Lemma 2.1 implies that

$$(\overset{\mathrm{r}}{\rightharpoondown}{}^p \circ \overset{\mathrm{l}}{\rightharpoondown}{}^q)(\tau_0) = (\overset{\mathrm{r}}{\rightharpoondown}{}^p \circ f_L^q)(\tau_0) = (f_L^q \circ \overset{\mathrm{r}}{\rightharpoondown}{}^p)(\tau_0) = (f_L^q \circ f_R^p)(\tau_0).$$

The case $k = 1$ is done. We turn to the case $k = 2$. By the above argument it follows that

$$(\overset{\mathrm{l}}{\rightharpoondown}{}^{q_1} \circ \overset{\mathrm{r}}{\rightharpoondown}{}^{p_2} \circ \overset{\mathrm{l}}{\rightharpoondown}{}^{q_2})(\tau_0) = (\overset{\mathrm{l}}{\rightharpoondown}{}^{q_1} \circ (f_L^{q_2} \circ f_R^{p_2}))(\tau_0) = ((f_L^{q_2} \circ f_R^{p_2}) \circ \overset{\mathrm{l}}{\rightharpoondown}{}^{q_1})(\tau_0) = (f_L^{q_2} \circ f_R^{p_2} \circ f_L^{q_1})(\tau_0).$$

Similarly we have

$$(\overset{\mathrm{r}}{\rightharpoondown}{}^{p_1} \circ \overset{\mathrm{l}}{\rightharpoondown}{}^{q_1} \circ \overset{\mathrm{r}}{\rightharpoondown}{}^{p_2} \circ \overset{\mathrm{l}}{\rightharpoondown}{}^{q_2})(\tau_0) = (f_L^{q_2} \circ f_R^{p_2} \circ f_L^{q_1} \circ f_R^{p_1})(\tau_0).$$

Case $k = 2$ is done. The proof for the case $k \geq 3$ is similar. $\square$

Let $\gamma_{\frac{b}{a}}$ be a simple closed curve in $\Sigma_{1,1}$ whose slope $\frac{b}{a}$. We require that $\gamma_{\frac{b}{a}}$ is setwise preserved by the hyperelliptic involution $f_{-\mathrm{I}}$ where $-\mathrm{I} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. See Figure 6(1),(2).

Proposition 3.5. (1) *For* $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{SL}^+(2;\mathbb{Z})$ *we have* $\tau_{[\frac{b}{a},\frac{d}{c}]} = f_A(\tau_0)$.

(2) *Moreover, if* $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *the train track* $\tau_{[\frac{b}{a},\frac{d}{c}]}$ *is formed by flattening the obtuse angles at the intersection of* $\gamma_{\frac{b}{a}}$ *and* $\gamma_{\frac{d}{c}}$ *as in Figure 6(2),(3). In particular,* $\tau_{[\frac{b}{a},\frac{d}{c}]}$ *is preserved by the hyperelliptic involution* $f_{-\mathrm{I}}$.

Statement (2) is how Mosher originally defined $\tau_{[\frac{b}{a},\frac{d}{c}]}$ in [17, 18].

Proof. By definition of the train track $\tau_{[\frac{b}{a},\frac{d}{c}]}$ as in (3.1) and (3.2), the argument in the proof of Proposition 3.4 yields Statement (1). See also Example 3.2.

Recall that the base train track τ_0 is obtained by flattening North West (NW) and South East (SE) right angles at the intersection of the simple closed curves $\gamma_{\frac{0}{1}}$ and $\gamma_{\frac{1}{0}}$ of slope 0 and ∞ as in Figure 3. If $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we have

$$\frac{0}{1} \leq \frac{b}{a} < \frac{d}{c} < \frac{1}{0} \quad \text{or} \quad \frac{0}{1} < \frac{b}{a} < \frac{d}{c} \leq \frac{1}{0}. \tag{3.3}$$

Consider the train tack $f_A(\tau_0)$, the image of τ_0 under $f_A : \Sigma_{1,1} \to \Sigma_{1,1}$. By (3.3) we see that A takes the square $[0,1] \times [0,1] \subset \mathbb{R}^2$ to a parallelogram in the first quadrant of $\mathbb{R}^2$. As a consequence the NW and SE right-angle corners at the intersection of $\gamma_{\frac{0}{1}}$ and $\gamma_{\frac{1}{0}}$ are mapped to the NW and SE obtuse angle corners at the intersection of $f_A(\gamma_{\frac{0}{1}}) = \gamma_{\frac{b}{a}}$ and $f_A(\gamma_{\frac{1}{0}}) = \gamma_{\frac{d}{c}}$. This implies that $f_A(\tau_0)$ is the union of $f_A(\gamma_{\frac{0}{1}}) \cup f_A(\gamma_{\frac{1}{0}})$ with the obtuse angles at the intersection flatten. This proves Statement (2). $\square$

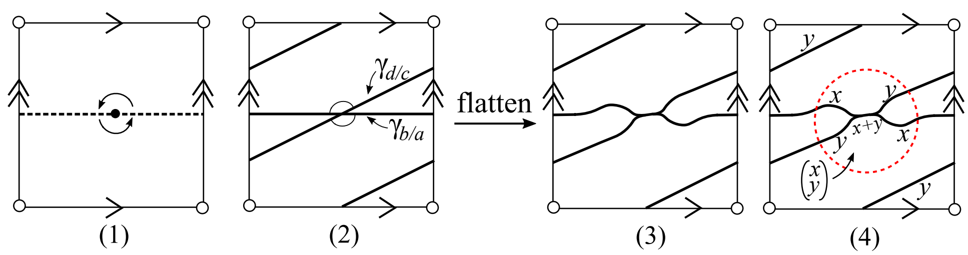

FIGURE 6. (1) The hyperelliptic involution $f_{-\mathrm{I}}$. (2) The obtuse angles at the intersection point $\gamma_{\frac{b}{a}} \cap \gamma_{\frac{d}{c}}$. (3) Flattening gives $\tau_{[\frac{b}{a},\frac{d}{c}]}$. (4) Convention 3.7 for a measure $\mu = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ on $\tau_{[\frac{b}{a},\frac{d}{c}]}$. $\gamma_{\frac{b}{a}} = \gamma_{\frac{0}{1}}$ and $\gamma_{\frac{d}{c}} = \gamma_{\frac{1}{2}}$ in this figure.

3.3. **Agol cycles of pseudo-Anosov maps on $\Sigma_{1,1}$.** The next theorem describes Agol cycles of pseudo-Anosov maps on $\Sigma_{1,1}$ induced by hyperbolic elements $A \in \mathrm{SL}(2;\mathbb{Z})$ with $\mathrm{tr}(A) > 2$.

Theorem 3.6. *Let $A = L^{q_k} R^{p_k} \cdots L^{q_1} R^{p_1} \in \mathrm{SL}(2;\mathbb{Z})$ where $p_1, q_1, \cdots, p_k, q_k$ and k are positive integers. Let $\ell = p_1 + q_1 + \cdots + p_k + q_k$ and s be the slope of the eigenvectors with respect to the expanding eigenvalue $\lambda > 1$ of A. For the pseudo-Anosov map $f_A : \Sigma_{1,1} \to \Sigma_{1,1}$ with dilatation λ, we have the following.*

(1) *The measured train track $(\tau_0, \mu_0) = (\tau_{[\frac{0}{1},\frac{1}{0}]}, \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right))$ (Figure 3(3)) is suited to the stable measured lamination of f_A.*

(2) *Starting with the measured train track* (τ_0, μ_0), *the first* $\ell + 1$ *terms*

$$(\tau_0, \mu_0) \xrightarrow{\text{l}}{}^{q_k} \xrightarrow{\text{r}}{}^{p_k} \ \cdots \ \xrightarrow{\text{l}}{}^{q_1} \xrightarrow{\text{r}}{}^{p_1} (\tau_\ell, \mu_\ell)$$

of the maximal splitting sequence satisfies $(\tau_\ell, \mu_\ell) = f_A(\tau_0, \lambda^{-1}\mu_0)$. *Thus, they form a length* ℓ *Agol cycle of* f_A. *Moreover,* $\tau_\ell = \tau_{[\frac{b}{a}, \frac{d}{c}]}$, *where* $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) = A$.

Each entry of A in Theorem 3.6 is positive; thus, $s > 0$ by the Perron-Frobenius theorem.

Convention 3.7. Let $\tau = \tau_{[\frac{b}{a}, \frac{d}{c}]}$ be the train track in $\Sigma_{1,1}$ defined in Section 3.2. We fix a convention for a measure μ on $\tau_{[\frac{b}{a}, \frac{d}{c}]}$. See Figure 6(4). By Proposition 3.5(2) the train track $\tau_{[\frac{b}{a}, \frac{d}{c}]}$ is formed flattening the obtuse angles at the intersection of $\gamma_{\frac{b}{a}}$ and $\gamma_{\frac{d}{c}}$. Recall that $\frac{0}{1} \leq \frac{b}{a} < \frac{d}{c} \leq \frac{1}{0}$. Let x (resp. y) be the weight of the small branch of $\tau_{[\frac{b}{a}, \frac{d}{c}]}$ which was originally contained in $\gamma_{\frac{b}{a}}$ (resp. $\gamma_{\frac{d}{c}}$) before the flattening. The large branch has weight $x + y$ by the switch condition. The vector $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ represents the measure μ and we write $\mu = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)_\tau$ specifying the train track. When there is no confusion, we simply denote it by $\mu = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Note that $(\tau_{[\frac{0}{1}, \frac{1}{0}]}, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right))$ in Figure 3(3) aligns with this convention.

Remark 3.8. We allow $\mu = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ in Convention 3.7. In these cases, the measured train track $(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu)$ yields the simple closed curves $\gamma_{\frac{b}{a}}$ and $\gamma_{\frac{d}{c}}$ respectively.

Lemma 3.9. (1) *If* $x < y$ *then* $(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)) \xrightarrow{\text{l}} (\tau_{[\frac{b+d}{a+c}, \frac{d}{c}]}, L^{-1}\mu = \left(\begin{smallmatrix} x \\ y-x \end{smallmatrix}\right))$.

(2) *If* $x > y$ *then* $(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)) \xrightarrow{\text{r}} (\tau_{[\frac{b}{a}, \frac{b+d}{a+c}]}, R^{-1}\mu = \left(\begin{smallmatrix} x-y \\ y \end{smallmatrix}\right))$.

Proof. We prove Statement (1). The proof of Statement (2) is similar. Recall that the train track $\tau_{[\frac{b}{a}, \frac{d}{c}]}$ consists of one large and two small branches. If $x < y$ the measured train track $(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu)$ admits a left splitting $(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu) \xrightarrow{\text{l}} (\tau', \mu')$. Then $\tau' = \tau_{[\frac{b+d}{a+c}, \frac{d}{c}]}$ by (3.1). To verify $\mu' = L^{-1}\mu$, we use Figure 2(1) substituting $z = y$ and $w = x$. Then by Convention 3.7 we have $\mu' = L^{-1}\mu = \left(\begin{smallmatrix} x \\ y-x \end{smallmatrix}\right)$. This completes the proof. ☐

Lemma 3.10. *For the measured train track* $(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right))$ *we have the following.*

(1) *Suppose that* $y = qx + r$ *with the quotient* $q \in \mathbb{N}$ *and the remainder* $0 \leq r < x$. *Then* $(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu)$ *admits* q *left splittings consecutively and*

$$(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu) \xrightarrow{\text{l}}{}^{q} (\tau', \mu') := (\tau_{[\frac{b+qd}{a+qc}, \frac{d}{c}]}, L^{-q}\mu = \left(\begin{smallmatrix} x \\ r \end{smallmatrix}\right)).$$

Since $r \not> x$, (τ', μ') *cannot admit any more left splittings. Moreover, if the remainder* $r \neq 0$ *then* (τ', μ') *falls into Case (2) below.*

(2) *Suppose that* $x = py + r$ *with the quotient* $p \in \mathbb{N}$ *and the remainder* $0 \leq r < y$. *Then* $(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu)$ *admits* p *right splittings consecutively and*

$$(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \mu) \xrightarrow{\text{r}}{}^{p} (\tau'', \mu'') := (\tau_{[\frac{b}{a}, \frac{pb+d}{pa+c}]}, R^{-p}\mu = \left(\begin{smallmatrix} r \\ y \end{smallmatrix}\right)).$$

Since $r \not> y$, (τ'', μ'') cannot admit any more right splittings. Moreover, if the remainder $r \neq 0$ then (τ'', μ'') falls into Case (1) above.

Proof. Applying Lemma 3.9 repeatedly, we obtain the desired statements. □

Corollary 3.11. *For the measured train track $(\tau_0, (\begin{smallmatrix} x \\ y \end{smallmatrix}))$ we have the following.*

(1) *If $y > qx$ then $(\tau_0, (\begin{smallmatrix} x \\ y \end{smallmatrix}))$ admits q left splittings consecutively and*

$$(\tau_0, (\begin{smallmatrix} x \\ y \end{smallmatrix})) \overset{\mathrm{l}}{\rightharpoonup}{}^{q} \ f_L^q(\tau_0, L^{-q}(\begin{smallmatrix} x \\ y \end{smallmatrix})).$$

(2) *If $x > py$ then $(\tau_0, (\begin{smallmatrix} x \\ y \end{smallmatrix}))$ admits p right splittings consecutively and*

$$(\tau_0, (\begin{smallmatrix} x \\ y \end{smallmatrix})) \overset{\mathrm{r}}{\rightharpoonup}{}^{p} \ f_R^p(\tau_0, R^{-p}(\begin{smallmatrix} x \\ y \end{smallmatrix})).$$

Proof. We prove Statement (1). Statement (2) follows similarly. Suppose that $y > qx$. By Proposition 3.4-(1) and Lemma 3.10-(1), it follows that $(\tau_0, (\begin{smallmatrix} x \\ y \end{smallmatrix})) \overset{\mathrm{l}}{\rightharpoonup}{}^{q} (f_L^q(\tau_0), L^{-q}(\begin{smallmatrix} x \\ y \end{smallmatrix}))$. By Convention 3.7 it holds $(f_L^q(\tau_0), L^{-q}(\begin{smallmatrix} x \\ y \end{smallmatrix})) = f_L^q(\tau_0, L^{-q}(\begin{smallmatrix} x \\ y \end{smallmatrix}))$ and we obtain $(\tau_0, (\begin{smallmatrix} x \\ y \end{smallmatrix})) \overset{\mathrm{l}}{\rightharpoonup}{}^{q}$ $f_L^q(\tau_0, L^{-q}(\begin{smallmatrix} x \\ y \end{smallmatrix}))$. □

Let $A = L^{q_k} R^{p_k} \cdots L^{q_1} R^{p_1}$ as in Theorem 3.6. Let s be the slope of the eigenvectors with respect to the expanding eigenvalue $\lambda > 1$ of A, that is $A(\begin{smallmatrix} 1 \\ s \end{smallmatrix}) = \lambda(\begin{smallmatrix} 1 \\ s \end{smallmatrix})$. Since $\lambda^2 - \mathrm{tr}(A)\lambda + 1 = 0$ and $\mathrm{tr}(A) > 2$, the eigenvalues of A are quadratic irrationals. This implies that s is also a quadratic irrational.

Let us consider the infinite continued fraction expansion of s.

$$s = n_0 + \cfrac{1}{n_1 + \cfrac{1}{\ddots + \cfrac{1}{n_k + \cdots}}} = [n_0 : n_1, \cdots, n_k, \cdots]$$

with $n_i \in \mathbb{Z}$ and $n_i > 0$ for $i \geq 1$. By Lagrange's theorem the expansion is eventually periodic, i.e. there exists $t \geq 1$ with $n_i = n_{i+t}$ for all $i \gg 1$ [14]. Hence the expansion of s is of the form:

$$s = [n_0 : n_1, \cdots, n_{k-1}, m_0, m_1, \cdots, m_{t-1}, m_0, m_1, \cdots, m_{t-1}, \cdots],$$

which is denoted by $[n_0 : n_1, \cdots, n_{k-1}, \overline{m_0, m_1, \cdots, m_{t-1}}]$. We now claim that the expansion of s is purely periodic.

Proposition 3.12. *The slope s satisfies the following:*

(1) *$s > 1$.*
(2) *s admits a purely periodic continued fraction expansion*

$$s = [\overline{q_k : p_k, q_{k-1}, p_{k-1}, \cdots, q_1, p_1}].$$

Proof. Elementary computation shows that $s > 0$ as follows. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Since $p_1, q_1, \cdots, p_k, q_k$ are positive integers a, b, c, d are also positive integers. We note $s = (\lambda - a)/b$ and $\lambda - a = (-a + d + \sqrt{(a+d)^2 - 4})/2$. We see $-a + d + \sqrt{(a+d)^2 - 4} > 0$ because $\mathrm{tr}(A) = a + d > 2$ and

$$\begin{aligned} (d + \sqrt{(a+d)^2 - 4})^2 - a^2 &= 2d(a+d) - 4 + 2d\sqrt{(a+d)^2 - 4} \\ &\geq 2(a+d) - 4 + 2\sqrt{(a+d)^2 - 4} \\ &\geq 2\sqrt{(a+d)^2 - 4} > 0. \end{aligned}$$

We conclude $s > 0$. (This fact also follows from the Perron-Frobenius theorem.)

Denote

$$\begin{aligned} \left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) &:= \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right), \\ \left(\begin{smallmatrix} x_1 \\ y_0 \end{smallmatrix}\right) &:= R^{p_1}\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_0 + p_1 y_0 \\ y_0 \end{smallmatrix}\right), \\ \left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right) &:= L^{q_1} R^{p_1}\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) = L^{q_1}\left(\begin{smallmatrix} x_1 \\ y_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 \\ q_1 x_1 + y_0 \end{smallmatrix}\right), \\ &\;\vdots \\ \left(\begin{smallmatrix} x_k \\ y_{k-1} \end{smallmatrix}\right) &:= R^{p_k} L^{q_{k-1}} \cdots L^{q_1} R^{p_1}\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) = R^{p_k}\left(\begin{smallmatrix} x_{k-1} \\ y_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_{k-1} + p_k y_{k-1} \\ y_{k-1} \end{smallmatrix}\right), \\ \left(\begin{smallmatrix} x_k \\ y_k \end{smallmatrix}\right) &:= L^{q_k} R^{p_k} \cdots L^{q_1} R^{p_1}\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) = L^{q_k}\left(\begin{smallmatrix} x_k \\ y_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_k \\ q_k x_k + y_{k-1} \end{smallmatrix}\right). \end{aligned}$$

Since exponents $p_1, q_1, \cdots, p_k, q_k$ are all positive integers we have two cases depending on (i) $1 < s$ or (ii) $s < 1$.

(i) $1 = x_0 < s = y_0 < x_1 < y_1 < x_2 < y_2 < \cdots < y_{k-1} < x_k < y_k$.
(ii) $s = y_0 < 1 = x_0 < x_1 < y_1 < x_2 < y_2 < \cdots < y_{k-1} < x_k < y_k$.

Since $A\left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right) = \lambda\left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right)$ we have $\left(\begin{smallmatrix} x_k \\ y_k \end{smallmatrix}\right) = \lambda\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) = \lambda\left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right)$. For (ii) we have $x_k = \lambda > s\lambda = y_k$, which is a contradiction. Thus, we conclude $s > 1$. Statement (1) is proved.

Reading the above computation from backward we obtain $2k$ set of division-with-remainder equations as appear in the Euclidean Algorithm:

$$\begin{aligned} y_k &= q_k x_k + y_{k-1}, \\ x_k &= p_k y_{k-1} + x_{k-1}, \\ &\;\vdots \\ y_1 &= q_1 x_1 + y_0, \\ x_1 &= p_1 y_0 + x_0. \end{aligned}$$

Put $s_i = \frac{1}{\lambda} y_i$ and $r_i = \frac{1}{\lambda} x_i$. Since $\lambda \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_k \\ y_k \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_k \\ q_k x_k + y_{k-1} \end{smallmatrix}\right)$, multiplying $\frac{1}{\lambda}$ to both sides of the above $2k$ equations gives the following new $2k$ division-with-remainder equations:

$$\begin{aligned} s = s_k &= q_k \cdot 1 + s_{k-1}, && (3.4)\\ 1 = r_k &= p_k \cdot s_{k-1} + r_{k-1}, && (3.5)\\ &\vdots \\ s_1 &= q_1 r_1 + s_0, && (3.6)\\ r_1 &= p_1 s_0 + r_0, && (3.7) \end{aligned}$$

where $0 < r_0 < s_0 < \cdots < r_{k-1} < s_{k-1} < 1 < s$. We obtain

$$\begin{aligned} \left(\begin{smallmatrix} 1 \\ s_{k-1} \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1 & 0 \\ -q_k & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right), \\ \left(\begin{smallmatrix} r_{k-1} \\ s_{k-1} \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1 & -p_k \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ s_{k-1} \end{smallmatrix}\right), \\ &\vdots \\ \left(\begin{smallmatrix} r_1 \\ s_0 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1 & 0 \\ -q_1 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} r_1 \\ s_1 \end{smallmatrix}\right), \\ \left(\begin{smallmatrix} r_0 \\ s_0 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1 & -p_1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} r_1 \\ s_0 \end{smallmatrix}\right). \end{aligned}$$

Thus, $\left(\begin{smallmatrix} r_0 \\ s_0 \end{smallmatrix}\right) = R^{-p_1} L^{-q_1} \cdots R^{-p_k} L^{-q_k} \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right) = A^{-1} \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right) = \lambda^{-1} \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right)$. In other words, $\frac{s_0}{r_0} = s$ and $s = [q_k : p_k, q_{k-1}, p_{k-1}, \cdots, q_1, p_1, \frac{s_0}{r_0} = s]$. This shows that s admits the purely periodic continued fraction expansion $s = [\overline{q_k : p_k, q_{k-1}, p_{k-1}, \cdots, q_1, p_1}]$. $\square$

Stable and unstable measured laminations of a pseudo-Anosov map $\phi : \Sigma \to \Sigma$ are unique up to scaling and isotopy. The measured train track (τ, μ) in Theorem 1.2 is assumed to be trivalent and suited to the stable measured lamination. We explain how to find such (τ, μ): The Bestvina-Handel algorithm [5] determines the Nielsen-Thurston type of a homeomorphism $\phi : \Sigma \to \Sigma$. If ϕ is pseudo-Anosov then the algorithm provides us a measured train track (τ', μ') which is suited to the stable measured lamination $(\mathcal{L}, \nu)$ of ϕ. If τ' is trivalent, then we set $(\tau, \mu) := (\tau', \mu')$. Otherwise, by *combing* branches near switches with degree greater than 3 (see [13, Figure 7]) we can obtain a trivalent train track τ with a transverse measure μ so that (τ, μ) is suited to $(\mathcal{L}, \nu)$. See [13, Section 3] for more details.

We are ready to prove Theorem 3.6.

Proof of Theorem 3.6. Let $A = L^{q_k} R^{p_k} \cdots L^{q_1} R^{p_1}$ as in Theorem 3.6. One sees that the measured train track $(\tau_0, \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right))$ (up to scaling) is obtained by applying the Bestvina-Handel algorithm to f_A. Since τ_0 is trivalent Statement (1) holds.

We reuse the computations (3.4)–(3.7). Since $s = q_k \cdot 1 + s_{k-1}$ applying Lemma 3.10-(1) and Corollary 3.11-(1) gives

$$(\tau_0, \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right)) \overset{1}{\rightharpoonup}{}^{q_k} (\tau_{q_k}, \left(\begin{smallmatrix} 1 \\ s_{k-1} \end{smallmatrix}\right)) = (\tau_{q_k}, L^{-q_k} \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right)) = f_L^{q_k}(\tau_0, L^{-q_k} \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right)).$$

Since $1 = p_k \cdot s_{k-1} + r_{k-1}$ we can apply Lemma 3.10-(2) to $f_L^{q_k}(\tau_0, L^{-q_k}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right))$ and we see that $f_L^{q_k}(\tau_0, L^{-q_k}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right))$ admits p_k right splittings consecutively. Then by Lemma 2.1 we obtain

$$(\overset{\mathrm{r}}{\rightharpoonup}^{p_k} \circ f_L^{q_k})(\tau_0, L^{-q_k}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right)) = (f_L^{q_k} \circ \overset{\mathrm{r}}{\rightharpoonup}^{p_k})(\tau_0, L^{-q_k}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right)) = (f_L^{q_k} \circ f_R^{p_k})(\tau_0, R^{-p_k}L^{-q_k}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right)).$$

In other words $f_L^{q_k}(\tau_0, L^{-q_k}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right)) \overset{\mathrm{r}}{\rightharpoonup}^{p_k} f_L^{q_k} \circ f_R^{p_k}(\tau_0, R^{-p_k}L^{-q_k}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right)) = f_L^{q_k} \circ f_R^{p_k}(\tau_0, \left(\begin{smallmatrix}r_{k-1}\\s_{k-1}\end{smallmatrix}\right))$. After repeating this $k-1$ more times, we obtain

$$(\tau_0, \left(\begin{smallmatrix}1\\s\end{smallmatrix}\right)) \overset{\mathrm{l}}{\rightharpoonup}^{q_k} \overset{\mathrm{r}}{\rightharpoonup}^{p_k} \cdots \overset{\mathrm{l}}{\rightharpoonup}^{q_1} \overset{\mathrm{r}}{\rightharpoonup}^{p_1} (\tau_\ell, \mu_\ell),$$

where $(\tau_\ell, \mu_\ell) = (f_L^{q_k} \circ f_R^{p_k} \circ \cdots \circ f_L^{q_1} \circ f_R^{p_1})(\tau_0, R^{-p_1}L^{-q_1} \ldots R^{-p_k}L^{-q_k}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right)) = f_A(\tau_0, A^{-1}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right)) = f_A(\tau_0, \lambda^{-1}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right))$, which gives a length ℓ Agol cycle. By Proposition 3.5-(1) we have $\tau_\ell = f_A(\tau_0) = \tau_{[\frac{b}{a}, \frac{d}{c}]}$, where $\left(\begin{smallmatrix}a & c\\b & d\end{smallmatrix}\right) = A$. This completes the proof. □

Example 3.13. Let $A = LR = \left(\begin{smallmatrix}1 & 1\\1 & 2\end{smallmatrix}\right)$. The mapping torus of $f_A : \Sigma_{1,1} \to \Sigma_{1,1}$ is the figure eight knot complement. See [9] for example. The slope of eigenvectors with respect to the expanding eigenvalue $\lambda = \frac{3+\sqrt{5}}{2}$ of A is the golden ratio $s = \frac{1+\sqrt{5}}{2}$ which admits the continued fractional expansion $[\bar{1}]$. By Theorem 3.6 and Lemma 3.9, $(\tau_0, \left(\begin{smallmatrix}1\\s\end{smallmatrix}\right)) \overset{\mathrm{l}}{\rightharpoonup} (\tau_{[\frac{1}{1}, \frac{1}{0}]}, \left(\begin{smallmatrix}1\\s-1\end{smallmatrix}\right)) \overset{\mathrm{r}}{\rightharpoonup} (\tau_{[\frac{1}{1}, \frac{2}{1}]}, \left(\begin{smallmatrix}2-s\\s-1\end{smallmatrix}\right)) = f_A(\tau_0, \lambda^{-1}\left(\begin{smallmatrix}1\\s\end{smallmatrix}\right))$ forms a length 2 Agol cycle of f_A. See Figure 7.

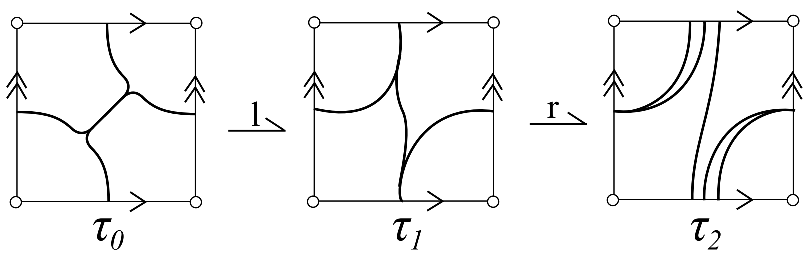

FIGURE 7. Example 3.13: $\tau_0 = \tau_{[\frac{0}{1}, \frac{1}{0}]} \overset{\mathrm{l}}{\rightharpoonup} \tau_1 = \tau_{[\frac{1}{1}, \frac{1}{0}]} \overset{\mathrm{r}}{\rightharpoonup} \tau_2 = \tau_{[\frac{1}{1}, \frac{2}{1}]}$.

Agol cycles of pseudo-Anosov maps induced by hyperbolic elements $A' \in \mathrm{SL}(2;\mathbb{Z})$ with $\mathrm{tr}(A') < -2$ have the following property.

Proposition 3.14. *Let $A \in \mathrm{SL}^+(2;\mathbb{Z})$ with $\mathrm{tr}(A) > 2$. The pseudo-Anosov maps f_A and f_{-A} are not conjugate but they have equivalent Agol cycles.*

Proof. We note that f_{-A} is pseudo-Anosov with dilatation equal to that of f_A. Since $\mathrm{tr}(A) = -\mathrm{tr}(-A) \neq \mathrm{tr}(-A)$, f_{-A} is not conjugate to f_A. Let

$$(\tau_0, \mu_0) \overset{\mathrm{l}}{\rightharpoonup}^{q_k} \overset{\mathrm{r}}{\rightharpoonup}^{p_k} \cdots \overset{\mathrm{l}}{\rightharpoonup}^{q_1} \overset{\mathrm{r}}{\rightharpoonup}^{p_1} (\tau_\ell, \mu_\ell) = f_A(\tau_0, \lambda^{-1}\mu_0) \tag{3.8}$$

be the Agol cycle of f_A as in Theorem 3.6. Since $(\tau_{[\frac{b}{a}, \frac{d}{c}]}, \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right))$ is preserved by $f_{-\mathrm{I}}$ we have

$$(\tau_\ell, \mu_\ell) = f_{-\mathrm{I}}(\tau_\ell, \mu_\ell) = f_{-\mathrm{I}} \circ f_A(\tau_0, \lambda^{-1}\mu_0) = f_{-A}(\tau_0, \lambda^{-1}\mu_0).$$

Thus (3.8) is also an Agol cycle of f_{-A}. □

4. Applications

4.1. **The braid groups and related mapping class groups.** Let B_n be the group of n-braids, which has the well known presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$ and $\sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i$ if $|i-j| = 1$.

Let D_n be a disk with n punctures $a_1, \dots, a_n$. We denote by $\mathrm{MCG}(D_n, \partial D)$, the mapping class group of homeomorphisms which fix the boundary ∂D of the n-punctured disk pointwise. There is a well known isomorphism $B_n \to \mathrm{MCG}(D_n, \partial D)$ sending σ_i to a positive half twist h_i which interchanges the punctures a_i and a_{i+1} counterclockwise. Under the identification $B_n = \mathrm{MCG}(D_n, \partial D)$ a braid word $b_1b_2 \in B_n$ (read from left to right) is identified with the composition $b_2 \circ b_1 \in \mathrm{MCG}(D_n, \partial D)$ (read from right to left) of mapping classes.

Capping the boundary of D_n with a once-punctured disk yields an $n+1$-punctured sphere $\Sigma_{0,n+1}$. The $n+1$ punctures of $\Sigma_{0,n+1}$ come from the punctures $a_1, \dots, a_n$ of D_n and the puncture, say a_∞, of the once-punctured capping disk. Let $\mathrm{MCG}(D_n, \partial D) \overset{Cap}{\to} \mathrm{MCG}(\Sigma_{0,n+1}, \{a_\infty\})$ denote the induced homomorphism where $\mathrm{MCG}(\Sigma_{0,n+1}, \{a_\infty\})$ denotes the subgroup of $\mathrm{MCG}(\Sigma_{0,n+1})$ consisting of elements that fix the puncture a_∞. The following sequence is exact:

$$1 \to \langle \Delta^2 \rangle \to B_n \simeq \mathrm{MCG}(D_n, \partial D) \overset{Cap}{\to} \mathrm{MCG}(\Sigma_{0,n+1}, \{a_\infty\}) \to 1,$$

where $\Delta \in B_n$ is a positive half twist and the infinite cyclic group $\langle \Delta^2 \rangle$ generated by the positive full twist Δ^2 gives the center of B_n. Composing with the injective inclusion $\mathrm{MCG}(\Sigma_{0,n+1}, \{a_\infty\}) \hookrightarrow \mathrm{MCG}(\Sigma_{0,n+1})$ we obtain a homomorphism (strictly speaking anti-homomorphism)

$$\Gamma : B_n \to \mathrm{MCG}(\Sigma_{0,n+1}).$$

A braid $b \in B_n$ is *pseudo-Anosov* if $\Gamma(b)$ is represented by a pseudo-Anosov map. The dilatation $\lambda(b) > 1$ of the pseudo-Anosov braid b is defined to be the dilatation of $\Gamma(b)$.

Abusing the notation, the image of the braid element $\sigma_i \in B_n$ via $\Gamma : B_n \to \mathrm{MCG}(\Sigma_{0,n+1})$ is also denoted by σ_i. When $n = 3$, $\sigma_1 \in \mathrm{MCG}(\Sigma_{0,4})$ fixes the punctures a_∞ and a_3 and interchanges the punctures a_1 and a_2 counterclockwise. Similarly, $\sigma_2^{-1} \in \mathrm{MCG}(\Sigma_{0,4})$ fixes a_∞ and a_1 and interchanges a_2 and a_3 clockwise. See Figure 8.

4.2. **Agol cycles of pseudo-Anosov 3-braids.** To compute Agol cycles of pseudo-Anosov 3-braids, we use the measured train track $(\omega_0, \binom{x}{y})$ in $\Sigma_{0,4}$ defined in Figure 1(3). Notice that $(\omega_0, \binom{x}{y})$ is triply weighted in the sense that the weights of the six branches are either x, y or $x+y$. Importance of triply weighted train tracks in $\Sigma_{0,4}$ in the study of Agol cycles was first pointed by Aceves and Kawamuro [1]. The measure is represented by the column vector $\binom{x}{y}$. There are two large branches with the same weight $x+y$, and others are small branches.

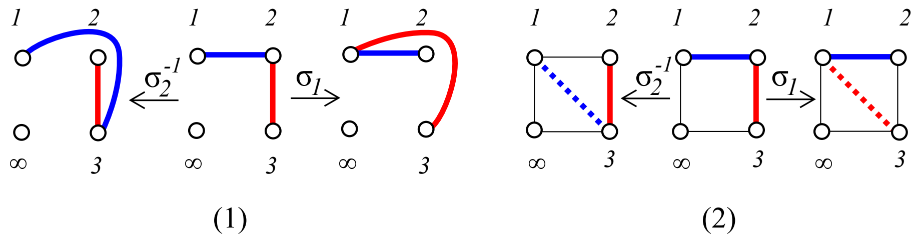

FIGURE 8. (1), (2) $\sigma_1, \sigma_2^{-1} : \Sigma_{0,4} \to \Sigma_{0,4}$. (Labels $1, 2, 3$ and ∞ represent punctures a_1, a_2, a_3 and a_∞.) In (2) we regard $\Sigma_{0,4}$ as a square pillowcase with the corners removed.

For the proof of Theorem 1.5 it is enough to prove the following result.

Theorem 4.1. *Let $A = L^{q_k} R^{p_k} \cdots L^{q_1} R^{p_1} \in \mathrm{SL}(2;\mathbb{Z})$ where $p_1, q_1, \cdots, p_k, q_k$ and k are positive integers. Let $\ell = p_1 + q_1 + \cdots + p_k + q_k$ and s be the slope of the eigenvectors with respect to the expanding eigenvalue $\lambda > 1$ of A. For the pseudo-Anosov map*

$$\phi_A = \Gamma(\sigma_1^{p_1}\sigma_2^{-q_1} \cdots \sigma_1^{p_k}\sigma_2^{-q_k}) = \sigma_2^{-q_k} \circ \sigma_1^{p_k} \circ \cdots \circ \sigma_2^{-q_1} \circ \sigma_1^{p_1} : \Sigma_{0,4} \to \Sigma_{0,4}$$

we have the following.

(1) *The measured train track $(\omega_0, \nu_0) := (\omega_0, \left(\begin{smallmatrix} 1 \\ s \end{smallmatrix}\right))$ (see Figure 1(3)) is suited to the stable measured lamination of ϕ_A.*
(2) *Starting with the measured train track (ω_0, ν_0), the first $\ell + 1$ terms*

$$(\omega_0, \nu_0) \overset{\mathrm{l}}{\rightharpoonup}{}^{q_k} \overset{\mathrm{r}}{\rightharpoonup}{}^{p_k} \cdots \overset{\mathrm{l}}{\rightharpoonup}{}^{q_1} \overset{\mathrm{r}}{\rightharpoonup}{}^{p_1} (\omega_\ell, \nu_\ell)$$

of the maximal splitting sequence satisfies $(\omega_\ell, \nu_\ell) = \phi_A(\omega_0, \lambda^{-1}\nu_0)$. Thus, they form a length ℓ Agol cycle of ϕ_A.

The rest of this section is devoted to prove Theorem 4.1.

We start with taking four points $a_\infty := (0, 0)$, $a_1 := (0, 1/2)$, $a_2 := (1/2, 1/2)$, and $a_3 := (1/2, 0)$ in $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$. The homeomorphism $\bar{f}_A : \mathbb{T} \to \mathbb{T}$ induced by $A \in \mathrm{SL}(2;\mathbb{Z})$ fixes a_∞ and permutes the three points a_1, a_2 and a_3. The involution $\bar{f}_{-\mathrm{I}} : \mathbb{T} \to \mathbb{T}$ induced by $-\mathrm{I} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ fixes the four points a_∞, a_1, a_2, a_3. For $A \in \mathrm{SL}(2;\mathbb{Z})$ let

$$g_A : \Sigma_{1,4} \to \Sigma_{1,4}; \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto A \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$$

be the restriction of $\bar{f}_A$ to the subspace $\Sigma_{1,4} = (\mathbb{R}^2 - (\frac{1}{2}\mathbb{Z})^2)/\mathbb{Z}^2 = \mathbb{T} \setminus \{a_\infty, a_1, a_2, a_3\}$, the 4-punctured torus. In particular, $g_{-\mathrm{I}} : \Sigma_{1,4} \to \Sigma_{1,4}$ is an involution. The action $\mathbb{Z}_2 \ni 1 \mapsto g_{-\mathrm{I}} \in \mathrm{Homeo}_+(\Sigma_{1,4})$ induces a double covering map

$$\pi : \Sigma_{1,4} \to \Sigma_{1,4}/\mathbb{Z}_2 = \Sigma_{0,4}.$$

Alternatively we can regard π as the natural $2:1$ map

$$(\mathbb{R}^2 - (\tfrac{1}{2}\mathbb{Z})^2)/\mathbb{Z}^2 \to (\mathbb{R}^2 - (\tfrac{1}{2}\mathbb{Z})^2)/H,$$

where H is the group generated by π-rotations about points in the lattice $(\frac{1}{2}\mathbb{Z})^2$.

Proposition 4.2. *Let $g_L, g_R : \Sigma_{1,4} \to \Sigma_{1,4}$ be the maps induced by $L, R \in \mathrm{SL}(2;\mathbb{Z})$, respectively. Then $\sigma_1, \sigma_2^{-1} : \Sigma_{0,4} \to \Sigma_{0,4}$ satisfy*

$$\pi \circ g_L = \sigma_2^{-1} \circ \pi,$$
$$\pi \circ g_R = \sigma_1 \circ \pi.$$

Proof. Recall that g_L, g_R are restrictions of $\bar{f}_L, \bar{f}_R : \mathbb{T} \to \mathbb{T}$ to $\Sigma_{1,4} \subset \Sigma_{1,0} = \mathbb{T}$, respectively. The assertion follows from Figure 9. □

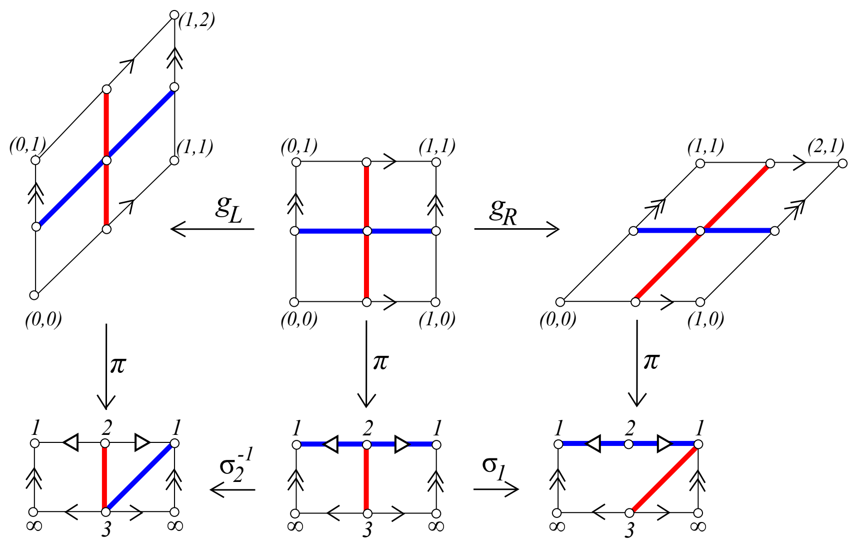

FIGURE 9. Proof of Proposition 4.2. See also Figure 8.

Lemma 4.3. *Let $A = L^{q_k}R^{p_k} \cdots L^{q_1}R^{p_1}$ be as in Theorem 4.1. Consider*

$$\phi_A = \sigma_2^{-q_k} \circ \sigma_1^{p_k} \circ \cdots \circ \sigma_2^{-q_1} \circ \sigma_1^{p_1} : \Sigma_{0,4} \to \Sigma_{0,4}$$

induced by A. Then we have the following.

(1) $\pi \circ g_A = \phi_A \circ \pi$.
(2) *$g_A : \Sigma_{1,4} \to \Sigma_{1,4}$ and $\phi_A : \Sigma_{0,4} \to \Sigma_{0,4}$ are pseudo-Anosov maps with the same dilatation which is equal to the expanding eigenvalue $\lambda > 1$ of A.*

Proof. Statement (1) follows from Proposition 4.2. For Statement (2) we recall the Anosov map $\bar{f}_A : \mathbb{T} \to \mathbb{T}$ induced by A. The dilatation of $\bar{f}_A$ is equal to the expanding eigenvalue λ of A. Since $g_A : \Sigma_{1,4} \to \Sigma_{1,4}$ is the restriction of $\bar{f}_A$ to $\Sigma_{1,4} \subset \Sigma_{1,0} = \mathbb{T}$, one sees that g_A is a pseudo-Anosov map and $\bar{f}_A$ and g_A have the same dilatation. Let $\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$ be the stable and unstable foliations with respect to g_A. Since $g_{-\mathrm{I}} \circ g_A = g_A \circ g_{-\mathrm{I}}$ the invariant foliations $\mathcal{F}^{\mathrm{s}}$ and $\mathcal{F}^{\mathrm{u}}$ are preserved by $g_{-\mathrm{I}}$. This implies that the images $\pi(\mathcal{F}^{\mathrm{s}})$ and $\pi(\mathcal{F}^{\mathrm{u}})$ under π give the stable and unstable foliation with respect to ϕ_A. Hence ϕ_A is also a pseudo-Anosov map with the same dilatation as that of g_A. Thus the dilatation of ϕ_A is also the expanding eigenvalue λ of A. This completes the proof. $\square$

Next we introduce the measured train track $(\mathcal{T}_0, \binom{x}{y})$ in $\Sigma_{1,4}$ defined as the preimage of the measured train track $(\omega_0, \binom{x}{y})$ under the double covering map $\pi : \Sigma_{1,4} \to \Sigma_{0,4}$ (Figure 10);

$$(\mathcal{T}_0, \tbinom{x}{y}) := \pi^{-1}(\omega_0, \tbinom{x}{y}). \tag{4.1}$$

We note that $(\mathcal{T}_0, \binom{x}{y})$ can be obtained from four copies of the measured train track $(\tau_0, \binom{x}{y})$ in $\Sigma_{1,1}$ as in Figure 10. To see this, consider the 4-fold covering $\overline{\rho} : \mathbb{T} \to \mathbb{T}$ corresponding to the subgroup $2\mathbb{Z} \oplus 2\mathbb{Z}$ of the fundamental group $\pi_1(\mathbb{T}) = \mathbb{Z} \oplus \mathbb{Z}$; that is, the subgroup $\overline{\rho}_*(\pi_1(\mathbb{T})) < \pi_1(\mathbb{T})$ is isomorphic to $2\mathbb{Z} \oplus 2\mathbb{Z}$ and its deck transformation group is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The covering map $\overline{\rho}$ induces a 4-fold covering map

$$\rho : \Sigma_{1,4} \to \Sigma_{1,1}.$$

Alternatively, we can view ρ as the natural $4:1$ map

$$(\mathbb{R}^2 - (\tfrac{1}{2}\mathbb{Z})^2)/\mathbb{Z}^2 \to (\mathbb{R}^2 - (\tfrac{1}{2}\mathbb{Z})^2)/(\tfrac{1}{2}\mathbb{Z})^2.$$

It satisfies $\rho \circ g_A = f_A \circ \rho$ for any $A \in SL(2;\mathbb{Z})$. Hence the preimage of $(\tau_0, \binom{x}{y})$ is

$$(\mathcal{T}_0, \tbinom{x}{y}) = \rho^{-1}(\tau_0, \tbinom{x}{y}), \tag{4.2}$$

and $g_A : \Sigma_{1,4} \to \Sigma_{1,4}$ is a pull-back of $f_A : \Sigma_{1,1} \to \Sigma_{1,1}$ under ρ.

Proposition 4.4. *For the measured train tracks $(\mathcal{T}_0, \binom{x}{y})$ in $\Sigma_{1,4}$ and $(\omega_0, \binom{x}{y})$ in $\Sigma_{0,4}$ we have the following.*

(1) *If $y > qx$ then both $(\mathcal{T}_0, \binom{x}{y})$ and $(\omega_0, \binom{x}{y})$ admit q left splittings consecutively:*

$$(\mathcal{T}_0, \tbinom{x}{y}) \overset{\mathrm{l}}{\rightharpoonup}{}^{q}\ g_L^q(\mathcal{T}_0, L^{-q}\tbinom{x}{y}) \text{ and } (\omega_0, \tbinom{x}{y}) \overset{\mathrm{l}}{\rightharpoonup}{}^{q}\ \sigma_2^{-q}(\omega_0, L^{-q}\tbinom{x}{y}).$$

Moreover $\pi \circ \overset{\mathrm{l}}{\rightharpoonup}{}^{q} (\mathcal{T}_0, \binom{x}{y}) = \overset{\mathrm{l}}{\rightharpoonup}{}^{q} \circ \pi(\mathcal{T}_0, \binom{x}{y}) = \sigma_2^{-q}(\omega_0, L^{-q}\binom{x}{y})$ as in the left commutative diagram below.

(2) *If $x > py$ then both $(\mathcal{T}_0, \binom{x}{y})$ and $(\omega_0, \binom{x}{y})$ admits p right splittings consecutively:*

$$(\mathcal{T}_0, \tbinom{x}{y}) \overset{\mathrm{r}}{\rightharpoonup}{}^{p}\ g_R^p(\mathcal{T}_0, R^{-p}\tbinom{x}{y}) \text{ and } (\omega_0, \tbinom{x}{y}) \overset{\mathrm{r}}{\rightharpoonup}{}^{p}\ \sigma_1^{p}(\omega_0, R^{-p}\tbinom{x}{y}).$$

Moreover $\pi \circ \overset{\mathrm{r}}{\rightharpoonup}{}^{p} (\mathcal{T}_0, \binom{x}{y}) = \overset{\mathrm{r}}{\rightharpoonup}{}^{p} \circ \pi(\mathcal{T}_0, \binom{x}{y}) = \sigma_1^{p}(\omega_0, R^{-p}\binom{x}{y})$ as in the right commutative diagram below.

$$\begin{array}{ccc} (\mathcal{T}_0, \tbinom{x}{y}) & \overset{\mathrm{l}}{\rightharpoonup}{}^{q} & g_L^q(\mathcal{T}_0, L^{-q}\tbinom{x}{y}) \\ \pi\downarrow & & \pi\downarrow \\ (\omega_0, \tbinom{x}{y}) & \overset{\mathrm{l}}{\rightharpoonup}{}^{q} & \sigma_2^{-q}(\omega_0, L^{-q}\tbinom{x}{y}) \end{array} \qquad \begin{array}{ccc} (\mathcal{T}_0, \tbinom{x}{y}) & \overset{\mathrm{r}}{\rightharpoonup}{}^{p} & g_R^p(\mathcal{T}_0, R^{-p}\tbinom{x}{y}) \\ \pi\downarrow & & \pi\downarrow \\ (\omega_0, \tbinom{x}{y}) & \overset{\mathrm{r}}{\rightharpoonup}{}^{p} & \sigma_2^{p}(\omega_0, R^{-p}\tbinom{x}{y}) \end{array}$$

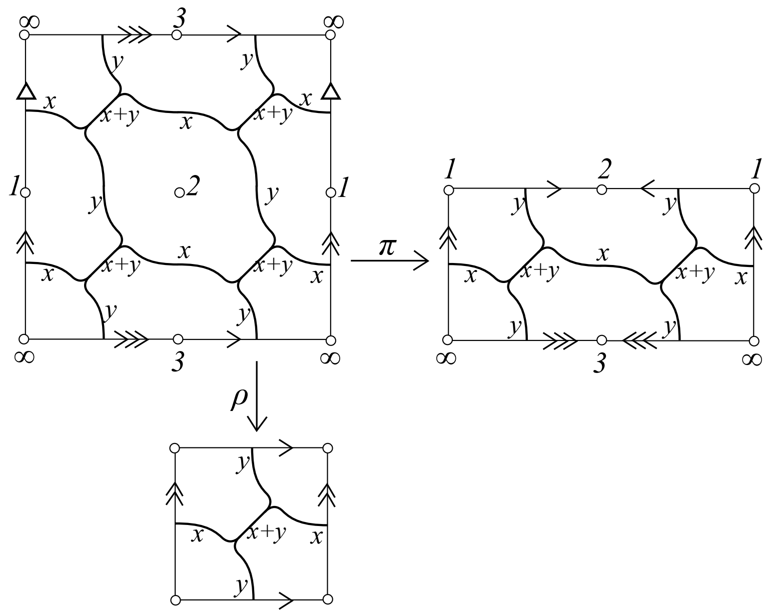

FIGURE 10. The double covering map $\pi : \Sigma_{1,4} \to \Sigma_{0,4}$ with $\pi(\mathcal{T}_0) = \omega_0$ and the 4-fold covering map $\rho : \Sigma_{1,4} \to \Sigma_{1,1}$ with $\rho(\mathcal{T}_0) = \tau_0$.

Proof. Suppose that $y > qx$. By Corollary 3.11-(1) we have $(\tau_0, \binom{x}{y}) \overset{\mathrm{l}}{\rightharpoonup}^{q} f_L^q(\tau_0, L^{-q}\binom{x}{y})$. Using the fact that $g_A : \Sigma_{1,4} \to \Sigma_{1,4}$ is a restriction of $f_A : \Sigma_{1,1} \to \Sigma_{1,1}$ and the description of $(\mathcal{T}_0, \binom{x}{y})$ in (4.2), one can verify that $(\mathcal{T}_0, \binom{x}{y}) \overset{\mathrm{l}}{\rightharpoonup}^{q} g_L^q(\mathcal{T}_0, L^{-q}\binom{x}{y})$.

We turn to the proof of $(\omega_0, \binom{x}{y}) \overset{\mathrm{l}}{\rightharpoonup}^{q} \sigma_2^{-q}(\omega_0, L^{-q}\binom{x}{y})$. We forget the measure for a moment. Figure 11 explains $\omega_0 \overset{\mathrm{l}}{\rightharpoonup} \sigma_2^{-1}(\omega_0)$ and $\omega_0 \overset{\mathrm{r}}{\rightharpoonup} \sigma_1(\omega_0)$. As in the proof of Proposition 3.4-(1) one can prove that $\omega_0 \overset{\mathrm{l}}{\rightharpoonup}^{q} \sigma_2^{-q}(\omega_0)$ and $\omega_0 \overset{\mathrm{r}}{\rightharpoonup}^{p} \sigma_1^p(\omega_0)$. Now we consider the measured train track $(\omega_0, \binom{x}{y})$ under the assumption $y > qx$. It is not hard to see that $(\omega_0, \binom{x}{y}) \overset{\mathrm{l}}{\rightharpoonup} \sigma_2^{-1}(\omega_0, L^{-1}\binom{x}{y})$. Using Lemma 2.1 repeatedly one can prove that $(\omega_0, \binom{x}{y})$ admits q left splittings consecutively and $(\omega_0, \binom{x}{y}) \overset{\mathrm{l}}{\rightharpoonup}^{q} \sigma_2^{-q}(\omega_0, L^{-q}\binom{x}{y})$. Then by Proposition 4.2

$$\begin{aligned} \pi \circ \overset{\mathrm{l}}{\rightharpoonup}^{q} (\mathcal{T}_0, \tbinom{x}{y}) &= \pi \circ g_L^q(\mathcal{T}_0, L^{-q}\tbinom{x}{y}) = \sigma_2^{-q}(\omega_0, L^{-q}\tbinom{x}{y}), \text{ and} \\ \overset{\mathrm{l}}{\rightharpoonup}^{q} \circ \pi(\mathcal{T}_0, \tbinom{x}{y}) &= \overset{\mathrm{l}}{\rightharpoonup}^{q} (\omega_0, \tbinom{x}{y}) = \sigma_2^{-q}(\omega_0, L^{-q}\tbinom{x}{y}). \end{aligned}$$

The proof of (1) is done. By a similar argument we can prove Statement (2). $\square$

We now prove Theorem 4.1.

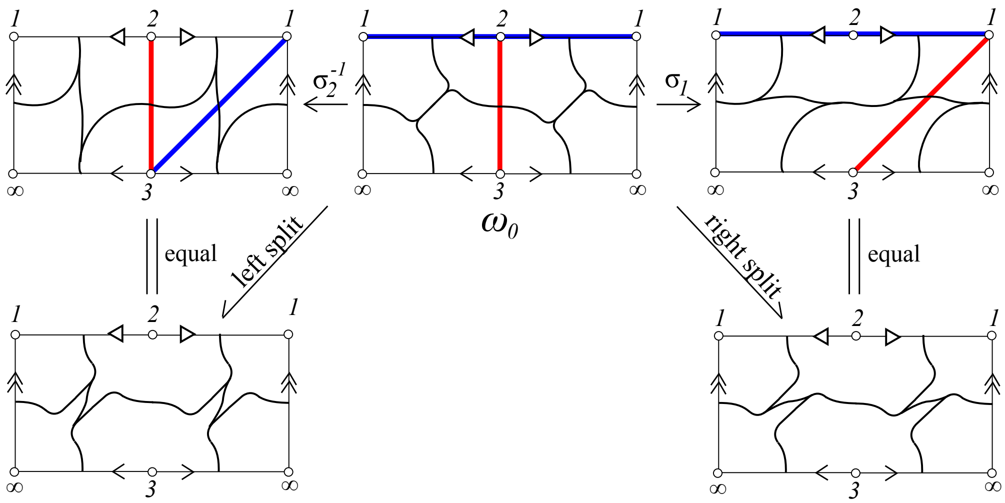

FIGURE 11. Proof of Proposition 4.4: $\omega_0 \overset{\mathrm{l}}{\rightharpoondown} \sigma_2^{-1}(\omega_0)$ and $\omega_0 \overset{\mathrm{r}}{\rightharpoondown} \sigma_1(\omega_0)$.

Proof of Theorem 4.1. By Theorem 3.6-(1) $(\tau_0, (\begin{smallmatrix}1\\ s\end{smallmatrix}))$ is suited to the stable measured lamination of f_A. The pseudo-Anosov map $g_A : \Sigma_{1,4} \to \Sigma_{1,4}$ is a pull-back of $f_A : \Sigma_{1,1} \to \Sigma_{1,1}$ under the covering map $\rho : \Sigma_{1,4} \to \Sigma_{1,1}$, and g_A is also a pull-back of $\phi_A : \Sigma_{0,4} \to \Sigma_{0,4}$ under the covering map $\pi : \Sigma_{1,4} \to \Sigma_{0,4}$. Recall the measured train track $(\omega_0, (\begin{smallmatrix}1\\ s\end{smallmatrix}))$ is defined by Figure 1(3). Figure 10 gives $(\omega_0, (\begin{smallmatrix}1\\ s\end{smallmatrix})) = \pi \circ \rho^{-1}(\tau_0, (\begin{smallmatrix}1\\ s\end{smallmatrix}))$. Thus $(\omega_0, (\begin{smallmatrix}1\\ s\end{smallmatrix}))$ is suited to the stable measured lamination of ϕ_A.

Theorem 3.6 states that $(\tau_0, \mu_0 = (\begin{smallmatrix}1\\ s\end{smallmatrix})) \overset{\mathrm{l}}{\rightharpoondown}^{q_k} \overset{\mathrm{r}}{\rightharpoondown}^{p_k} \cdots \overset{\mathrm{l}}{\rightharpoondown}^{q_1} \overset{\mathrm{r}}{\rightharpoondown}^{p_1} (\tau_\ell, \mu_\ell)$ forms a length ℓ Agol cycle of f_A. Note that similar commutative diagrams as in Proposition 4.4 hold for the pair of measured train tracks $(\mathcal{T}_0, (\begin{smallmatrix}x\\ y\end{smallmatrix}))$ and $(\tau_0, (\begin{smallmatrix}x\\ y\end{smallmatrix}))$. Moreover by Lemma 4.3-(2) the dilatations of f_A, g_A and ϕ_A are the same which is the expanding eigenvalue of A. These facts together with the property $\rho \circ g_{A'} = f_{A'} \circ \rho$ for each $A' \in \mathrm{SL}(2; \mathrm{Z})$ imply that

$$(\mathcal{T}_0, (\begin{smallmatrix}1\\ s\end{smallmatrix})) = \rho^{-1}(\tau_0, \mu_0) \overset{\mathrm{l}}{\rightharpoondown}^{q_k} \overset{\mathrm{r}}{\rightharpoondown}^{p_k} \cdots \overset{\mathrm{l}}{\rightharpoondown}^{q_1} \overset{\mathrm{r}}{\rightharpoondown}^{p_1} \rho^{-1}(\tau_\ell, \mu_\ell)$$

gives a length ℓ Agol cycle of g_A. Each measured train track of the above Agol cycle of g_A is of the form $\rho^{-1}(\tau_i, \mu_i)$. By Proposition 3.5-(2) τ_i is preserved by the involution $f_{-\mathrm{I}} : \Sigma_{1,1} \to \Sigma_{1,1}$. Thus $\rho^{-1}(\tau_i)$ is also preserved by the involution $g_{-\mathrm{I}} : \Sigma_{1,4} \to \Sigma_{1,4}$. This means that $\pi \circ \rho^{-1}(\tau_i)$ gives a train track in the quotient space $\Sigma_{0,4}$.

On the other hand, we have the commutative diagrams in Proposition 4.4. The above Agol cycle of g_A together with Proposition 4.2 tells us that

$$(\omega_0, \nu_0) \overset{\mathrm{l}}{\rightharpoondown}^{q_k} \overset{\mathrm{r}}{\rightharpoondown}^{p_k} \cdots \overset{\mathrm{l}}{\rightharpoondown}^{q_1} \overset{\mathrm{r}}{\rightharpoondown}^{p_1} (\omega_\ell, \nu_\ell),$$

where $(\omega_i, \nu_i) = \pi \circ \rho^{-1}(\tau_i, \mu_i)$ where $i = 1, \ldots, \ell$ gives a length ℓ Agol cycle of ϕ_A. This completes the proof. ☐

Example 4.5 (The pseudo-Anosov 3-braid $\sigma_1\sigma_2^{-1}$)**.** Let $A = LR$ and $s = \frac{1+\sqrt{5}}{2}$ be as in Example 3.13. Then $(\omega_0, \nu_0 = (\begin{smallmatrix}1\\ s\end{smallmatrix})) \overset{\mathrm{l}}{\rightharpoondown} (\omega_1, \nu_1) \overset{\mathrm{r}}{\rightharpoondown} (\omega_2, \nu_2)$ forms a length 2 Agol cycle of $\phi_A = \sigma_2^{-1} \circ \sigma_1 : \Sigma_{0,4} \to \Sigma_{0,4}$. See Figure 12.

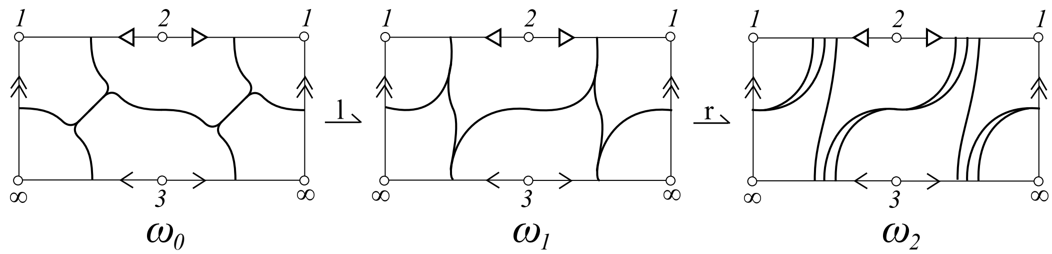

FIGURE 12. Example 4.5: $\omega_0 \overset{\mathrm{l}}{\rightharpoondown} \omega_1 \overset{\mathrm{r}}{\rightharpoondown} \omega_2$. See also Figure 7.

Corollary 4.6. *Pseudo-Anosov 3-braids β and β' are conjugate in $B_3/Z(B_3)$ where $Z(B_3)$ is the center of B_3 if and only if their Agol cycles are equivalent.*

Proof. The only-if-part follows from Theorem 1.4. For the if-part we prove the contrapositive statement. Assume that pseudo-Anosov 3-braids β and β' are not conjugate in $B_3/Z(B_3)$. Due to Murasugi's classification we may assume that β is conjugate to $\Delta^{2j}\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_k}\sigma_2^{-q_k}$ and β' is conjugate to $\Delta^{2j'}\sigma_1^{p'_1}\sigma_2^{-q'_1}\cdots\sigma_1^{p'_l}\sigma_2^{-q'_l}$ for some $j, k, p_1, q_1, \dots, p_k, q_k$ and $j', l, p'_1, q'_1, \dots,\ p'_l,\ q'_l$. If their Agol cycles were equivalent then by Theorem 1.5 and Lemma 2.1 the cyclically-ordered sets $\{(p_1, q_1), \cdots, (p_k, q_k)\}$ and $\{(p'_1, q'_1), \cdots, (p'_l, q'_l)\}$ had to be equal, which means that β and β' are conjugate in $B_3/Z(B_3)$. This is a contradiction. ☐

4.3. **Garside canonical lengths v.s. Agol cycle lengths.** The *Garside (left) normal form* of an n-braid is used to improve Garside's solution to the conjugacy problem for B_n [6, 7]. Garside introduced the fundamental braid Δ which is a positive half-twist on all of the n strands. The *simple* elements in B_n are the positive braids in which every pair of strands cross at most once. There are $n!$ simple elements in B_n. Given a braid $b \in B_n$, its Garside normal form is a special representation by a braid word of the form $\Delta^r P_1 P_2 \cdots P_s$ for some integer r and simple elements $P_1, \cdots, P_s$ satisfying the condition that each $P_i P_{i+1}$ is *left-weighted* introduced by Elrifai and Morton [6]. The number of simple elements in the Garside normal form is called the *canonical length* of b. The super summit set SSS(b) is a finite set of n-braids with minimal canonical length in the conjugacy class of b. The conjugacy problem in B_n can be solved by computing the super summit set SSS(b).

For 3-braids, the fundamental element is $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. The Garside normal form of the pseudo-Anosov 3-braid $\sigma_1^p\sigma_2^{-q}$ for positive integers p and q is given as follows.

$$\begin{array}{ll} \Delta^{-q}\,\sigma_2^p \cdot (\sigma_2\sigma_1)\cdot(\sigma_1\sigma_2)\cdots(\sigma_2\sigma_1) & \text{if } q \text{ is odd,} \\ \Delta^{-q}\,\sigma_1^p \cdot (\sigma_1\sigma_2)\cdot(\sigma_2\sigma_1)\cdots(\sigma_1\sigma_2)\cdot(\sigma_2\sigma_1) & \text{if } q \text{ is even,} \end{array}$$

where the simple elements $\sigma_2\sigma_1$ and $\sigma_1\sigma_2$ alternate in the tail and the number of the simple elements $\sigma_2\sigma_1$ or $\sigma_1\sigma_2$ is q. This can be obtained by replacing σ_2^{-1} with $\Delta^{-1}\sigma_2\sigma_1$, and shift Δ^{-1} to the left. For example when $q = 1$ and 2, we have $\sigma_1^p\sigma_2^{-1} = \Delta^{-1}\sigma_2^p \cdot (\sigma_2\sigma_1)$ and $\sigma_1^p\sigma_2^{-2} = \Delta^{-2}\sigma_1^p \cdot (\sigma_1\sigma_2) \cdot (\sigma_2\sigma_1)$. Thus, the canonical length of $\sigma_1^p\sigma_2^{-q}$ is $p + q$.

Similarly, a direct computation shows that the Garside canonical length of the pseudo-Anosov 3-braid $\beta = \Delta^{2m}\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_k}\sigma_2^{-q_k}$ is $p_1+q_1+\cdots+p_k+q_k$ which is by Theorem 4.1 exactly the Agol cycle length of β. It is known that β belongs to its super summit set, see Aguilera [4] for example. Therefore, we obtain the following result.

Theorem 4.7. *For every pseudo-Anosov* 3*-braid* β*, the Agol cycle length of* β*, the Garside canonical length of any element in the super summit set* $\mathrm{SSS}(\beta)$ *are the same.*

The same result as Theorem 4.7 does not hold for higher braid index. For example consider the 5-braid $\alpha := \sigma_3\sigma_2\sigma_3\sigma_4\sigma_1^{-1}$. By [2, Section 5] the Agol cycle length of α is 6 but the Garside canonical length of any element in $\mathrm{SSS}(\alpha)$ is 2. (One can use a computer program *Braiding* by González-Meneses to compute the super summit set [10].) It would be interesting to explore the relation between Agol cycle lengths and Garside canonical lengths for general pseudo-Anosov braids.

For the dual-version of the Garside normal form, also known as the Birman-Ko-Lee normal form, we confirm the same result as Theorem 4.7. For the above 5-braid α, it's dual Garside canonical length of any super summit element is 3.

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