A CONSTRUCTION OF PSEUDO-ANOSOV BRAIDS WITH SMALL NORMALIZED ENTROPIES

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Abstract. Let $b$ be a pseudo-Anosov braid whose permutation has a fixed point and let $M_b$ be the mapping torus by the pseudo-Anosov homeomorphism defined on the genus 0 fiber $F_b$ associated with $b$. We prove that there is a 2-dimensional subcone $C_0$ contained in the fibered cone $C$ of $F_b$ such that the fiber $F_a$ for each primitive integral class $a \in C_0$ has genus 0. We also give a constructive description of the monodromy $\phi_a : F_a \to F_a$ of the fibration on $M_b$ over the circle, and consequently provide a construction of many sequences of pseudo-Anosov braids with small normalized entropies. As an application we prove that the smallest entropy among skew-palindromic braids with $n$ strands is comparable to $1/n$, and the smallest entropy among elements of the odd/even spin mapping class groups of genus $g$ is comparable to $1/g$.

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus $g$ with $n$ punctures for $n \geq 0$. We set $\Sigma_g = \Sigma_{g,0}$. By mapping class group $\text{Mod}(\Sigma_{g,n})$, we mean the group of isotopy classes of orientation preserving self-homeomorphisms on $\Sigma_{g,n}$ preserving punctures setwise. By Nielsen-Thurston classification, elements in $\text{Mod}(\Sigma)$ are classified into three types: periodic, reducible, pseudo-Anosov [30, 9]. For $\phi \in \text{Mod}(\Sigma)$ we choose a representative $\Phi \in \phi$ and consider the mapping torus $M_\phi = \Sigma \times \mathbb{R}/\sim$, where $\sim$ identifies $(x, t + 1)$ with $(\Phi(x), t)$ for $x \in \Sigma$ and $t \in \mathbb{R}$. Then $\Sigma$ is a fiber of a

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
\hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm} \hspace{1cm}\hspace{1cm}\hspace{1cm}\hspace{1cm} \\
1 & i & i+1 & \ldots & \ldots & 1 & 2 & 3 \\
\end{tabular}
\end{figure}

Figure 1. (1) $\sigma_i$. (2) $\sigma_i^{-1}\sigma_2$ with the permutation $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$. (3) $\sigma_2^{-1}\sigma_2^{-1}$ whose permutation has a fixed point.

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fibration on $M_\phi$ over the circle $S^1$ and $\phi$ is called the \textit{monodromy}. A theorem by Thurston [31] asserts that $M_\phi$ admits a hyperbolic structure of finite volume if and only if $\phi$ is pseudo-Anosov.

For a pseudo-Anosov element $\phi \in \text{Mod}(\Sigma)$ there is a representative $\Phi : \Sigma \rightarrow \Sigma$ of $\phi$ called a \textit{pseudo-Anosov homeomorphism} with the following property: $\Phi$ admits a pair of transverse measured foliations $(F^u, \mu^u)$ and $(F^s, \mu^s)$ and a constant $\lambda = \lambda(\phi) > 1$ depending on $\phi$ such that $F^u$ and $F^s$ are invariant under $\Phi$, and $\mu^u$ and $\mu^s$ are uniformly multiplied by $\lambda$ and $\lambda^{-1}$ under $\Phi$. The constant $\lambda(\phi)$ is called the \textit{dilatation} and $F^u$ and $F^s$ are called the \textit{unstable} and \textit{stable foliation}. We call the logarithm $\log(\lambda(\phi))$ the \textit{entropy}, and call

$$\text{Ent}(\phi) = |\chi(\Sigma)| \log(\lambda(\phi))$$

the \textit{normalized entropy} of $\phi$, where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$. Such normalization of the entropy is suited for the context of 3-manifolds [8, 21].

Penner [27] proved that if $\phi \in \text{Mod}(\Sigma_{g,n})$ is pseudo-Anosov, then

$$\frac{\log 2}{12g - 12 + 4n} \leq \log(\lambda(\phi)).$$

See also [21, Corollary 2]. For a fixed surface $\Sigma$, the set

$$\{ \log \lambda(\phi) \mid \phi \in \text{Mod}(\Sigma) \text{ is pseudo-Anosov} \}$$

is a closed, discrete subset of $\mathbb{R}$ ([1]). For any subgroup or subset $G \subset \text{Mod}(\Sigma)$ let $\delta(G)$ denote the minimum of $\lambda(\phi)$ over all pseudo-Anosov elements $\phi \in G$. Then $\delta(G) \geq \delta(\text{Mod}(\Sigma))$. We write $f \asymp h$ if there is a universal constant $P > 0$ such that $1/P \leq f/h \leq P$. It is proved by Penner [27] that the minimal entropy among pseudo-Anosov elements in $\text{Mod}(\Sigma_{g,n})$ on the closed surface of genus $g$ satisfies

$$\log \delta(\text{Mod}(\Sigma_g)) \asymp \frac{1}{g}.$$ 

See also [16, 32, 33] for other sequences of mapping class groups.

For any $P > 0$, consider the set $\Psi_P$ consisting of all pseudo-Anosov homeomorphisms $\Phi : \Sigma \rightarrow \Sigma$ defined on any surface $\Sigma$ with the normalized entropy $|\chi(\Sigma)| \log \lambda(\Phi) \leq P$. This is an infinite set in general (take $P > 2 \log(2 + \sqrt{3})$ for example) and is well-understood in the context of hyperbolic fibered 3-manifolds. The universal finiteness theorem by Farb-Leininger-Margalit [8] states that the set of homeomorphism classes of mapping tori of pseudo-Anosov homeomorphisms $\Phi^\circ : \Sigma^\circ \rightarrow \Sigma^\circ$ is finite, where $\Phi^\circ : \Sigma^\circ \rightarrow \Sigma^\circ$ is the fully punctured pseudo-Anosov
There is a sequence of pseudo-Anosov braids. Suppose that $\Phi \in \Psi_P$. (Clearly $\lambda(\Phi^o) = \lambda(\Phi)$.) In other words such $\Phi^o : \Sigma^o \to \Sigma^o$ is a monodromy of a fiber in some fibered cone for a hyperbolic fibered 3-manifold in the finite list determined by $P$. Thus 3-manifolds in the finite list govern all pseudo-Anosov elements in $\Psi_P$. It is natural to ask the dynamics and a constructive description of elements in $\Psi_P$. There are some results about this question by several authors [4, 15, 20, 22, 33], but it is not completely understood. In this paper we restrict our attention to the pseudo-Anosov elements in $\Psi_P$ defined on the genus 0 surfaces, and provide an approach for a concrete description of those elements.

Let $B_n$ be the braid group with $n$ strands. The group $B_n$ is generated by the braids $\sigma_1, \cdots, \sigma_{n-1}$ as in Figure 1. Let $S_n$ be the symmetric group, the group of bijections of $\{1, \ldots, n\}$ to itself. A permutation $P \in S_n$ has a fixed point if $P(i) = i$ for some $i$. We have a surjective homomorphism $\pi : B_n \to S_n$ which sends each $\sigma_j$ to the transposition $(j, j + 1)$.

The closure $cl(b)$ of a braid $b \in B_n$ is a knot or link in the 3-sphere $S^3$. The braided link

$$\text{br}(b) = cl(b) \cup A$$

is a link in $S^3$ obtained from $cl(b)$ with its braid axis $A$ (Figure 2). Let $M_b$ denote the exterior of $\text{br}(b)$ which is a 3-manifold with boundary. It is easy to find an $(n + 1)$-holed sphere $F_b$ in $M_b$ (Figure 2(3)). Clearly $F_b$ is a fiber of a fibration on $M_b \to S^1$ and its monodromy $\phi_b : F_b \to F_b$ is determined by $b$. We call $F_b$ the $F$-surface for $b$.

A braid $b \in B_n$ is periodic (resp. reducible, pseudo-Anosov) if the associated mapping class $f_b \in \text{Mod}(\Sigma_{0, n+1})$ is of the corresponding type (Section 2.3). If $b$ is pseudo-Anosov, then the dilatation $\lambda(b)$ is defined by $\lambda(f_b)$ and the normalized entropy $\text{Ent}(b)$ is defined by $\text{Ent}(f_b)$. The following theorem is due to Hironaka-Kin [16, Proposition 3.36] together with the observation by Kin-Takasawa [22, Section 4.1].

**Theorem 1.1.** There is a sequence of pseudo-Anosov braids $z_n \in B_n$ such that $\text{Ent}(z_n) \neq 2\log(2 + \sqrt{3})$, $M_{z_n} \simeq M_{\sigma_1^z \sigma_2^{-1}}$ for each $n \geq 3$ and $\text{Ent}(z_n) \to 2\log(2 + \sqrt{3})$ as $n \to \infty$.

Here $\simeq$ means they are homeomorphic to each other. The limit point $2\log(2 + \sqrt{3})$ is equal to $\text{Ent}(\sigma_1^z \sigma_2^{-1})$. By the lower bound (1.1), Theorem 1.1 implies that

$$\log \delta(\text{Mod}(\Sigma_{0,n})) \simeq \frac{1}{n}.$$ 

In particular, the hyperbolic fibered 3-manifold $M_{\sigma_1^z \sigma_2^{-1}}$ admits an infinitely family of genus 0 fibers of fibrations over $S^1$.

Let $z_n$ be a pseudo-Anosov braid with $d_n$ strands. We say that a sequence $\{z_n\}$ has a small normalized entropy if $d_n \asymp n$ and there is a constant $P > 0$ which does not depend on $n$ such that $\text{Ent}(z_n) \leq P$. By (1.1) a sequence $\{z_n\}$ having a small normalized entropy means $\log(\lambda(z_n)) \asymp 1/n$. One of the aims in this paper is to give a construction of many sequences of pseudo-Anosov braids with small normalized entropies. The following result generalizes Thereom 1.1.

**Theorem A.** Suppose that $b$ is a pseudo-Anosov braid whose permutation has a fixed point. There is a sequence of pseudo-Anosov braids $\{z_n\}$ with small normalized entropy such that $\text{Ent}(z_n) \to \text{Ent}(b)$ as $n \to \infty$ and $M_{z_n} \simeq M_b$ for $n \geq 1$. 
The proof of Theorem A is constructive. In fact one can describe braids $z_n$ explicitly. For a more general result see Theorems 5.1, 5.2. Let $C \subset H_2(M_b, \partial M_b)$ be the fibered cone containing $[F_b]$. A theorem by Thurston [29] states that for each primitive integral class $a \in C$ there is a connected fiber $F_n$ with the pseudo-Anosov monodromy $\phi_n : F_n \rightarrow F_n$ of a fibration on the hyperbolic 3-manifold $M_b$ over $S^1$. The following theorem states a structure of $C$.

**Theorem B.** Suppose that $b$ is a pseudo-Anosov braid whose permutation has a fixed point. Then there are a 2-dimensional subcone $C_0 \subset C$ and an integer $u \geq 1$ with the following properties.

1. The fiber $F_n$ for each primitive integral class $a \in C_0$ has genus 0.
2. The monodromy $\phi_n : F_n \rightarrow F_n$ for each primitive integral class $a \in C_0$ is conjugate to
   $$\omega_1 \psi \cdots \omega_{u-1} \psi \omega_u \psi^{m-1} : F_n \rightarrow F_n,$$
   where $m \geq 1$ depends on the class $a$, $\psi$ is periodic and each $\omega_j$ is reducible. Moreover there are homeomorphisms $\tilde{\omega}_j : S_0 \rightarrow S_0$ on a surface $S_0$ for $j = 1, \ldots, u$ determined by $b$ and an embedding $h : S_0 \rightarrow F_n$ such that $h(S_0)$ is the support of each $w_j$ and
   $$w_j|_{h(S_0)} = h \circ \tilde{\omega}_j \circ h^{-1}.$$

Theorem B gives a constructive description of $\phi_n$. Also it states that each $w_j : F_n \rightarrow F_n$ is reducible supported on a uniformly bounded subsurface $h(S_0) \subset F_n$. It turns out from the proof that the type of the periodic homeomorphism $\psi : F_n \rightarrow F_n$ does not depend on $a \in C_0$ (Remark 3.3), see Figure 3(1). Theorem B reminds us of the symmetry conjecture in [23] by Farb-Leininger-Margalit.

Clearly the permutation of each pure braid has a fixed point. For any pseudo-Anosov braid $b$, a suitable power $b^k$ becomes a pure braid and one can apply Theorems A, B for $b^k$.

We have a remark about Theorem A. Theorem 10.2 in [25] by McMullen also tells us the existence of a sequence $(F_n, \phi_n)$ of fibers and monodromies in $C$ such that $\text{Ent}(\phi_n) \rightarrow \text{Ent}(b)$ as $n \rightarrow \infty$ and $|\chi(F_n)| \sim n$. However one can not appeal his theorem for the genera of fibers $F_n$. Theorem A says that $F_n$ has genus 0 in fact.

As an application we will determine asymptotic behaviors of the minimal dilatations of a subset of $B_n$ consisting of braids with a symmetry. A braid $b \in B_n$ is palindromic if $\text{rev}(b) = b$, where $\text{rev} : B_n \rightarrow B_n$ is a map such that if $w$ is a word of letters $\sigma_j^{\pm 1}$ representing $b$, then $\text{rev}(b)$ is the braid obtained from $b$ reversing the order of letters in $w$. A braid $b \in B_n$ is skew-palindromic if $\text{skew}(b) = b$, where $\text{skew}(b) = \Delta \text{rev}(b) \Delta^{-1}$ and $\Delta$ is a half twist (Section 2.2). See Figure 4. We will prove that dilatations of palindromic braids have the following lower bound.

**Theorem C.** If $b \in B_n$ is palindromic and pseudo-Anosov for $n \geq 3$, then
$$\lambda(b) \geq \sqrt{2 + \sqrt{5}}.$$ 

In contrast with palindromic braids we have the following result.
Figure 3. Dynamics of $\psi$ and $\omega_j$ in Theorem B. (1) Periodic $\psi : F_a \to F_a$. (2) Reducible $\omega_j : F_a \to F_a$. Subsurface $h(S_0)$ is shaded.

Figure 4. Illustration of braids (1) $b$, (2) $\text{rev}(b)$, (3) $\text{skew}(b)$.

Figure 5. (1) $\mathcal{I}: \Sigma_g \to \Sigma_g$. (2) A basis $\{x_1, y_1, \ldots, x_g, y_g\}$ of $H_1(\Sigma_g; \mathbb{Z}_2)$.

**Theorem D.** Let $PA_n$ be the set of skew-palindromic elements in $B_n$. We have

$$\log \delta(PA_n) \asymp \frac{1}{n}.$$

The hyperelliptic mapping class group $\mathcal{H}(\Sigma_g)$ is the subgroup of $\text{Mod}(\Sigma_g)$ consisting of elements with representative homeomorphisms that commute with some fixed hyperelliptic involution $\mathcal{I}: \Sigma_g \to \Sigma_g$ as in Figure 5(1). It is shown in [16] that $\log \delta(\mathcal{H}(\Sigma_g)) \asymp 1/g$. See also [7, 15, 19] for other subgroups of $\text{Mod}(\Sigma_g)$. As an application we will determine the asymptotic behavior of the minimal dilatations of the odd/even spin mapping class groups of genus $g$. To define these subgroups let $(\cdot, \cdot)_2$ be the mod-2 intersection form on $H_1(\Sigma_g; \mathbb{Z}_2)$. A map $q : H_1(\Sigma_g; \mathbb{Z}_2) \to \mathbb{Z}_2$ is a quadratic form if $q(v + w) = q(v) + q(w) + (v, w)_2$ for $v, w \in H_1(\Sigma_g; \mathbb{Z}_2)$. For a quadratic form $q$, the spin mapping class group $\text{Mod}_g[q]$ is the subgroup of $\text{Mod}(\Sigma_g)$
consisting of elements $\phi$ such that $q \circ \phi = q$. To define the two quadratic forms $q_0$ and $q_1$, we choose a basis $\{x_1, y_1, \ldots, x_g, y_g\}$ of $H_1(\Sigma_g; \mathbb{Z}_2)$ as in Figure 5(2). Let $q_0$ be the quadratic form such that $q_0(x_i) = q_0(y_i) = 0$ for $1 \leq i \leq g$. Let $q_1$ be the quadratic form such that $q_1(x_1) = q_1(y_1) = 1$ and $q_1(x_i) = q_1(y_i) = 0$ for $2 \leq i \leq g$. A result of Dye [5] tells us that $\text{Mod}_g[q_0]$ and $\text{Mod}_g[q_1]$ in $\text{Mod}(\Sigma_g)$. We call $\text{Mod}_g[q_0]$ and $\text{Mod}_g[q_1]$ the even spin and odd spin mapping class group respectively. It is known that $\text{Mod}_g[q_1]$ attains the minimum index for a proper subgroup of $\text{Mod}(\Sigma_g)$ and $\text{Mod}_g[q_0]$ attains the secondary minimum, see Berrick-Gebhardt-Paris [2].

**Theorem E.** We have

1. $\log(\delta(\text{Mod}_g[q_1] \cap \mathcal{H}(\Sigma_g))) > \frac{1}{g}$ and

2. $\log(\delta(\text{Mod}_g[q_0] \cap \mathcal{H}(\Sigma_g))) > \frac{1}{g}$.

In particular $\log(\delta(\text{Mod}_g[q])) \approx 1/g$ for each quadratic form $q$.

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2. **Preliminaries**

2.1. **Links.** Let $L$ be a link in the 3-sphere $S^3$. Let $\mathcal{N}(L)$ denote a tubular neighborhood of $L$ and let $\mathcal{E}(L)$ denote the exterior of $L$, i.e., $\mathcal{E}(L) = S^3 \setminus \text{int}(\mathcal{N}(L))$.

Oriented links $L$ and $L'$ in $S^3$ are equivalent, denoted by $L \sim L'$ if there is an orientation preserving homeomorphism $f : S^3 \to S^3$ such that $f(L) = L'$ with respect to the orientations of the links. Furthermore for components $K_i$ of $L$ and $K'_i$ of $L'$ with $i = 1, \ldots, m$ if $f$ satisfies $f(K_i) = K'_i$ for each $i$, then $(L, K_1, \ldots, K_m)$ and $(L', K'_1, \ldots, K'_m)$ are equivalent and we write

$$(L, K_1, \ldots, K_m) \sim (L', K'_1, \ldots, K'_m).$$

2.2. **Braid groups $B_n$ and spherical braid groups $SB_n$.** Let $\delta_j = \sigma_1 \sigma_2 \cdots \sigma_{j-1}$ and $\rho_j = \sigma_1 \sigma_2 \cdots \sigma_{j-2} \sigma_j^2$. The half twist $\Delta_j$ is given by $\Delta_j = \delta_j \rho_j \cdots \delta_2$. We often omit the subscript $n$ in $\Delta_n$, $\delta_n$ and $\rho_n$ when they are precisely $n$-braids.

We put indices $1, 2, \ldots, n$ from left to right on the bottoms of strands, and give an orientation of strands from the bottom to the top (Figure 1). The closure $\text{cl}(b)$ is oriented by the strands. We think of $\text{br}(b) = \text{cl}(b) \cup A$ as an oriented link in $S^3$ choosing an orientation of $A = A_b$ arbitrarily. (In Section 3 we assign an orientation of the braid axis for $i$-monotonic braids).

If two braids are conjugate to each other, then their braided links are equivalent. Morton proved that the converse holds if their axises are preserved.

**Theorem 2.1.** (Morton [26]). If $(\text{br}(b), A_b)$ is equivalent to $(\text{br}(c), A_c)$ for braids $b, c \in B_n$, then $b$ and $c$ are conjugate in $B_n$.

Let us turn to the spherical braid group $SB_n$ with $n$ strands. We also denote by $\sigma_i$, the element of $SB_n$ as shown in Figure 1(1). The group $SB_n$ is generated
by \(\sigma_1, \ldots, \sigma_{n-1}\). For a braid \(b \in B_n\) represented by a word of letters \(\sigma_j^\pm 1\), let \(S(b)\) denote the element in \(SB_n\) represented by the same word as \(b\).

For a braid \(b\) in \(B_n\) or \(SB_n\) the degree of \(b\) means the number \(n\) of the strands, denoted by \(d(b)\).

2.3. Mapping classes and mapping tori from braids. Let \(D_n\) be the \(n\)-punctured disk. Consider the mapping class group \(\text{Mod}(D_n)\), the group of isotopy classes of orientation preserving self-homeomorphisms on \(D_n\) preserving the boundary \(\partial D\) of the disk setwise. We have a surjective homomorphism

\[
\Gamma : B_n \rightarrow \text{Mod}(D_n)
\]

which sends each generator \(\sigma_i\) to the right-handed half twist \(t_i\) between the \(i\)th and \((i+1)\)st punctures. The kernel of \(\Gamma\) is an infinite cyclic group generated by the full twist \(\Delta^2\).

Collapsing \(\partial D\) to a puncture in the sphere we have a homomorphism

\[
\varsigma : \text{Mod}(D_n) \rightarrow \text{Mod}(\Sigma_{0,n+1}).
\]

We say that \(b \in B_n\) is periodic (resp. reducible, pseudo-Anosov) if \(f_b := \varsigma(\Gamma(b))\) is of the corresponding Nielsen-Thurston type. The braids \(\delta, \rho \in B_n\) are periodic since some power of each braid is the full twist: \(\Delta^n = \delta^n = \rho^{n-1} \in B_n\).

We also have a surjective homomorphism

\[
\hat{\Gamma} : SB_n \rightarrow \text{Mod}(\Sigma_{0,n})
\]

sending each generator \(\sigma_i\) to the right-handed half twist \(t_i\). We say that \(\eta \in SB_n\) is pseudo-Anosov if \(\hat{\Gamma}(\eta) \in \text{Mod}(\Sigma_{0,n})\) is pseudo-Anosov. In this case \(\lambda(\eta)\) is defined by the dilatation of \(\hat{\Gamma}(\eta)\).

2.4. Stable foliations \(F_b\) for pseudo-Anosov braids \(b\).

Recall the surjective homomorphism \(\pi : B_n \rightarrow S_n\). We write \(\pi_b = \pi(b)\) for \(b \in B_n\). Consider a pseudo-Anosov braid \(b \in B_n\) with \(\pi_b(i) = i\). Removing the \(i\)th strand \(b(i)\) from \(b\), we get a braid \(b - b(i) \in B_{n-1}\). Taking its spherical element, we have \(S(b - b(i)) \in SB_{n-1}\). Note that \(b - b(i)\) and \(S(b - b(i))\) are not necessarily pseudo-Anosov.

A well-known criterion uses the stable foliation \(F_b\) for the monodromy \(\phi_b : F_b \rightarrow F_b\) of a fibration on \(M_b \rightarrow S^1\) as we recall now. Such a fibration on \(M_b\) extends naturally to a fibration on the manifold obtained from \(M_b\) by Dehn filling a cusp along the boundary slope of the fiber \(F_b\) which lies on the torus \(\partial\mathcal{N}(\text{cl}(b(i)))\). Also \(\phi_b\) extends to the monodromy defined on \(F_b^*\) of the extended fibration, where \(F_b^*\) is obtained from \(F_b\) by filling in the boundary component of \(F_b\) which lies on \(\partial\mathcal{N}(\text{cl}(b(i)))\) with
a disk. Then \( b - b(i) \) is the corresponding braid for the extended monodromy defined on \( F^*_i \). Suppose that \( F_b \) is not 1-pronged at the boundary component in question. (See Figure 6 in the case where \( F_b \) is 1-pronged at a boundary component.) Then \( F_b \) extends to the stable foliation for \( b - b(i) \), and hence \( b - b(i) \) is pseudo-Anosov with the same dilatation as \( b \). Furthermore if \( F_b \) is not 1-pronged at the boundary component of \( F_b \) which lies on \( \partial N(A) \), then \( S(b - b(i)) \) is still pseudo-Anosov with the same dilatation as \( b \).

2.5. Thurston norm. Let \( M \) be a 3-manifold with boundary (possibly \( \partial M = \emptyset \)). If \( M \) is hyperbolic, i.e. the interior of \( M \) possess a complete hyperbolic structure of finite volume, then there is a norm \( \| \cdot \| \) on \( H_2(M, \partial M; \mathbb{R}) \), now called the Thurston norm [29]. The norm \( \| \cdot \| \) has the property such that for any integral class \( a \in H_2(M, \partial M; \mathbb{R}) \), \( \|a\| = \min_{S} \{-\chi(S)\} \), where the minimum is taken over all oriented surface \( S \) embedded in \( M \) with \( a = [S] \) and with no components of non-negative Euler characteristic. The surface \( S \) realizing this minimum is called a norm-minimizing surface of \( a \).

**Theorem 2.2** (Thurston [29]). The norm \( \| \cdot \| \) on \( H_2(M, \partial M; \mathbb{R}) \) has the following properties.

1. There are a set of maximal open cones \( C_1, \ldots, C_k \) in \( H_2(M, \partial M; \mathbb{R}) \) and a bijection between the set of isotopy classes of connected fibers of fibrations \( M \to S^1 \) and the set of primitive integral classes in the union \( C_1 \cup \cdots \cup C_k \).
2. The restriction of \( \| \cdot \| \) to \( C_j \) is linear for each \( j \).
3. If we let \( F_a \) be a fiber of a fibration \( M \to S^1 \) associated with a primitive integral class \( a \) in each \( C_j \), then \( \|a\| = -\chi(F_a) \).

We call the open cones \( C_j \) fibered cones and call integral classes in \( C_j \) fibered classes.

**Theorem 2.3** (Fried [11]). For a fibered cone \( C \) of a hyperbolic 3-manifold \( M \), there is a continuous function \( \text{ent} : C \to \mathbb{R} \) with the following properties.

1. For the monodromy \( \phi_a : F_a \to F_a \) of a fibration \( M \to S^1 \) associated with a primitive integral class \( a \in C \), we have \( \text{ent}(a) = \log(\lambda(\phi_a)) \).
2. \( \text{Ent} = \| \cdot \| \circ \text{ent} : C \to \mathbb{R} \) is a continuous function which becomes constant on each ray through the origin.
3. If a sequence \( \{a_n\} \subset C \) tends to a point \( \neq 0 \) in the boundary \( \partial C \) as \( n \) tends to \( \infty \), then \( \text{ent}(a_n) \to \infty \). In particular \( \text{Ent}(a_n) = \|a_n\| \text{ent}(a_n) \to \infty \).

We call \( \text{ent}(a) \) and \( \text{Ent}(a) \) the entropy and normalized entropy of the class \( a \in C \).

For a pseudo-Anosov element \( \phi \in \text{Mod} (\Sigma) \) we consider the mapping torus \( M_\phi \). The vector field \( \frac{\partial}{\partial t} \) on \( \Sigma \times \mathbb{R} \) induces a flow \( \phi^t \) on \( M_\phi \) called the suspension flow.

**Theorem 2.4** (Fried [10]). Let \( \phi \) be a pseudo-Anosov mapping class defined on \( \Sigma \) with stable and unstable foliations \( F^s \) and \( F^u \). Let \( \tilde{F}^s \) and \( \tilde{F}^u \) denote the suspensions of \( F^s \) and \( F^u \) by \( \phi \). If \( C \) is a fibered cone containing the fibered class \( [\Sigma] \), then we can modify a norm-minimizing surface \( F_a \) associated with each primitive integral class \( a \in C \) by an isotopy on \( M_\phi \) with the following properties.

1. \( F_a \) is transverse to the suspension flow \( \phi^t \), and the first return map \( \phi_a : F_a \to F_a \) is precisely the pseudo-Anosov monodromy of the fibration on \( M_\phi \to S^1 \) associated with \( a \). Moreover \( F_a \) is unique up to isotopy along flow lines.
The stable and unstable foliations for $\phi_a$ are given by $\mathcal{F}^s \cap F_a$ and $\mathcal{F}^u \cap F_a$.

2.6. Disk twist. Let $L$ be a link in $S^3$. Suppose an unknot $K$ is a component of $L$. Then the exterior $\mathcal{E}(K)$ (resp. $\partial \mathcal{E}(K)$) is a solid torus (resp. torus). We take a disk $D$ bounded by the longitude of a tubular neighborhood $N(K)$ of $K$. We define a mapping class $T_D$ defined on $\mathcal{E}(K)$ as follows. We cut $\mathcal{E}(K)$ along $D$. We have resulting two sides obtained from $D$, and reglue two sides by twisting either of the sides 360 degrees so that the mapping class defined on $\partial \mathcal{E}(K)$ is the right-handed Dehn twist about $\partial D$. Such a mapping class on $\mathcal{E}(K)$ is called the disk twist about $D$. For simplicity we also call a self-homeomorphism representing the mapping class $T_D$ the disk twist about $D$, and denote it by the same notation

$$T_D : \mathcal{E}(K) \to \mathcal{E}(K).$$

Clearly $T_D$ equals the identity map outside a neighborhood of $D$ in $\mathcal{E}(K)$. We observe that if $u + 1$ segments of $L - K$ pass through $D$ for $u \geq 1$, then $T_D(L - K)$ is obtained from $L - K'$ by adding the full twist near $D$. In the case $u = 1$, see Figure 7. We may assume that $T_D$ fixes one of these segments, since any point in $D$ becomes the center of the twisting about $D$.

For any integer $\ell$, consider a homeomorphism

$$T_D^\ell : \mathcal{E}(K) \to \mathcal{E}(K).$$

Observe that $T_D^\ell$ converts $L$ into a link $K \cup T_D^\ell(L - K)$ such that $S^3 \setminus L$ is homeomorphic to $S^3 \setminus (K \cup T_D^\ell(L - K))$. Then $T_D^\ell$ induces a homeomorphism between the exteriors of links

$$h_{D,\ell} : \mathcal{E}(L) \to \mathcal{E}(K \cup T_D^\ell(L - K)).$$

We use the homeomorphism in (2.1) in later section.

3. $i$-increasing braids and Theorem 3.2

Definitions of $i$-increasing braids, signs and intersection numbers. Let $L$ be an oriented link in $S^3$ with a trivial component $K$. We take an oriented disk $D$ bounded by the longitude of $N(K)$ so that the orientation of $D$ agrees with the orientation of $K$. For each component $K'$ of $L - K$ such that $D$ and $K'$ intersect transversally with $D \cap K' \neq \emptyset$, we assign each point of intersection +1 or −1 as shown in Figure 8.
Let $b$ be a braid with $\pi_b(i) = i$. We consider an oriented disk $D = D_{b(i)}$ bounded by the longitude \( \ell_i \) of \( N(\text{cl}(b(i))) \). Such a disk $D$ is unique up to isotopy on $\mathcal{E}(\text{cl}(b(i)))$. We say that a braid $b \in B_n$ with $\pi_b(i) = i$ is $i$-increasing (resp. $i$-decreasing) if there is a disk $D = D_{b(i)}$ as above with the following conditions.

(D1) There is at least one component $K'$ of $\text{cl}(b - b(i))$ such that $D \cap K' \neq \emptyset$.

(D2) Each component of $\text{cl}(b - b(i))$ and $D$ intersect with each other transversally, and every point of intersection has the sign +1 (resp. -1).

We set $\epsilon(b, i) = 1$ (resp. $\epsilon(b, i) = -1$), and call it the sign of the pair $(b, i)$. We also call $D$ the associated disk of the pair $(b, i)$. We say that $b$ is $i$-monotonic if $b$ is $i$-increasing or $i$-decreasing. Then we set

$$I(b, i) = D \cap \text{cl}(b - b(i))$$

and let $u(b, i) \geq 1$ be the cardinality of $I(b, i)$. We call $u(b, i)$ the intersection number of the pair $(b, i)$. If the pair $(b, i)$ is specified, then we simply denote $\epsilon(b, i)$ and $u(b, i)$ by $\epsilon$ and $u$ respectively. For example $\sigma_i^2\sigma_i^{-1}$ is 1-increasing with $u(\sigma_i^2\sigma_i^{-1}, 1) = 1$.

A braid $b$ is positive if $b$ is represented by a word in letters $\sigma_j$, but not $\sigma_j^{-1}$. A braid $b$ is irreducible if the Nielsen-Thurston type of $b$ is not reducible.

**Lemma 3.1.** Let $b$ be a positive braid with $\pi_b(i) = i$. Then $b$ is $i$-increasing if $b$ is irreducible.
Proof. Suppose that a positive braid $b$ with $\pi_b(i) = i$ is irreducible. Since $b$ is positive, there is a disk $D = D(b,i)$ with the condition (D2). Assume that $D$ fails in (D1). Let $\partial D_n$ be the boundary of the disk $D_n$ containing $n$ punctures. Consider a neighborhood of $\partial D_n \cup (D_n \cap D)$ in $D_n$ which is an annulus. One of the boundary components of this annulus is an essential simple closed curve in $\Gamma(b) \in \text{Mod}(D_n)$. This means that $b$ is reducible, a contradiction. Thus $D$ satisfies (D1), and $b$ is $i$-increasing.

Orientation of the axis $A$. Let $b$ be $i$-monotonic with $\epsilon(b,i) = \epsilon$ and $u(b,i) = u$. Consider the braided link $\text{br}(b) = \text{cl}(b) \cup A$. The associated disk $D$ has a unique point of intersection with $A$, and the cardinality of $I(b,i) \cup (D \cap A)$ is $u(b,i) + 1$. To deal with $\text{br}(b) = \text{cl}(b) \cup A$ as an oriented link, we consider an orientation of $\text{cl}(b)$ as we described before, and assign an orientation of $A$ so that the sign of the intersection between $D$ and $A$ coincides with $\epsilon(b,i)$. See Figure 2(2).

Recall that $M_b = \mathcal{E}(\text{br}(b))$ is the exterior of $\text{br}(b)$ which is a surface bundle over $S^1$. We consider an orientation of the $F$-surface $F_b$ which agrees with the orientation of $A$.

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dot

$E$-surface. We now define an oriented surface $E_{(b,i)}$ of genus 0 embedded in $M_b$. Consider small $u(b,i) + 1$ disks in the oriented disk $D = D(b,i)$ whose centers are points of $I(b,i) \cup (D \cap A)$. Then $E_{(b,i)}$ is a sphere with $u(b,i) + 2$ boundary components obtained from $D$ by removing the interiors of those small disks. We choose the orientation of $E_{(b,i)}$ so that it agrees with the orientation of $D$. We call $E_{(b,i)}$ the $E$-surface for $b$. For example, the $1$-increasing braid $\sigma_1^2 \sigma_2^{-1}$ has the $E$-surface $E_{(\sigma_1^2 \sigma_2^{-1},1)}$ homeomorphic to a 3-holed sphere.

Subcone $C_{(b,i)}$. Let us consider the $2$-dimensional subcone $C_{(b,i)}$ of $H_2(M_b, \partial M_b; \mathbb{R})$ spanned by $[F_b]$ and $[E_{(b,i)}]$ (Figure 9):

$$C_{(b,i)} = \{ x[F_b] + y[E_{(b,i)}] \mid x > 0, \ y > 0 \}.$$ 

Let $\overline{C_{(b,i)}}$ denote the closure of $C_{(b,i)}$. We write $(x,y) = x[F_b] + y[E_{(b,i)}]$. We prove the following theorem in Section 4.

Theorem 3.2. For a pseudo-Anosov, $i$-increasing braid $b$ with $u(b,i) = u$, let $C$ be the fibered cone containing $[F_b]$. We have the following.

1. $C_{(b,i)} \subset C$.
2. The fiber $F_{(x,y)}$ for each primitive integral class $(x,y) \in C_{(b,i)}$ has genus 0.
3. The monodromy $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ for each primitive integral class $(x,y) \in C_{(b,i)}$ is conjugate to

$$(\omega_1^1 \psi) \cdots (\omega_{u-1} \psi)(\omega_u \psi)^{-m-1} : F_{(x,y)} \to F_{(x,y)},$$

where $m \geq 1$ depends on $(x,y)$, $\psi$ is periodic and each $\omega_j$ is reducible. Moreover there are homeomorphisms $\hat{\omega}_j : S_0 \to S_0$ for $j = 1, \ldots, u$ on a surface $S_0$ determined by $b$ and an embedding $h : S_0 \to F_{(x,y)}$ such that the subsurface $h(S_0)$ of $F_{(x,y)}$ is the support of each $w_j$ and $w_j \circ h \circ \hat{\omega}_j = h^{-1}$.

The conclusion of Theorem 3.2 holds for $i$-decreasing braids as well. We now claim that Theorem 3.2 implies Theorem B.
Proof of Theorem B. Suppose that Theorem 3.2 holds. Let \( b \in B_n \) be a pseudo-Anosov braid such that \( \tau_b(i) = i \). We consider the braid \( b\Delta^{2k} \in B_n \) for \( k \geq 1 \). The full twist \( \Delta^2 \) is an element in the center \( Z(B_n) \) and \( \Delta^2 = \sigma_j P_j \) holds for each \( 1 \leq j \leq n - 1 \), where \( P_j \) is positive. Such properties imply that \( b\Delta^{2k} \) is positive for \( k \) large. We fix such large \( k \).

We consider the braid \( b\Delta_k \) for \( k \geq 1 \). The full twist \( \Delta^2 \) is an element in the center \( Z(B_n) \) and \( \Delta^2 = \sigma_j P_j \) holds for each \( 1 \leq j \leq n - 1 \), where \( P_j \) is positive. Such properties imply that \( b\Delta^{2k} \) is positive for \( k \) large. We fix such large \( k \).

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Consider the \( k \)th power of the disk twist about the disk \( D_A \) bounded by the longitude of \( N(A) \):

\[
T^k_{D_A} : \mathcal{E}(A) \to \mathcal{E}(A).
\]

Since \( A \cup T^k_{D_A}(\text{cl}(b)) = A \cup \text{cl}(b\Delta^{2k}) = \text{br}(b\Delta^{2k}) \), we have \( S^3 \setminus \text{br}(b) \simeq S^3 \setminus \text{br}(b\Delta^{2k}) \).

Let us set
\[
f_k := h_{D_A,k} : M_b \to M_{b\Delta^{2k}},
\]
where \( h_{D_A,k} \) is the homeomorphism in (2.1). The isomorphism
\[
f_{k*} : H_2(M_b, \partial M_b) \to H_2(M_{b\Delta^{2k}}, \partial M_{b\Delta^{2k}})
\]
obeys \([F_b] \to [F_b\Delta^{2k}]\). (Here we note that the above \( k \) is suppose to be large, but the homeomorphism \( f_k \) makes sense for all integer \( k \).) The pullback of the subcone \( C_{(b\Delta^{2k},i)} \) into \( H_2(M_b, \partial M_b) \) is a desired subcone contained in \( \mathcal{C} \).

Remark 3.3. If \( F_{(x,y)} \) is a \((d+1)\)-holed sphere, then the periodic homeomorphism \( \psi : F_{(x,y)} \to F_{(x,y)} \) in Theorem 3.2 is determined by the periodic braid \( \rho = \sigma_1 \sigma_2 \ldots \sigma_{d-2} \sigma_{d-1} \in B_d \). See the proof of Theorem 3.2(3) in Section 4.3.

4. PROOF OF THEOREM 3.2

We fix integers \( n \geq 3 \) and \( 1 \leq i \leq n \). Throughout Section 4, we assume that \( b \in B_n \) is pseudo-Anosov and \( i \)-increasing with \( u(b,i) = u \). We now choose an associated disk about the pair \((b,i)\) suitably. Let \( \mathbb{D} \) denote the unit disk with the center \((0,0)\) in the plane \( \mathbb{R}^2 \). Let \( J = (-1,1) \times \{0\} \subset \mathbb{D} \) be the interval and let \( A_0 = (-2,0) \) be a point in \( \mathbb{R}^2 \). We denote by \( \mathbb{D}_n \), the disk \( \mathbb{D} \) with equally spaced \( n \) points in \( J \). Let us denote these \( n \) points by \( A_1, \ldots, A_n \) from left to right. We take a point \( Q_i \neq A_i \in J \) between \( A_{i-1} \) and \( A_i \) so that the Euclidean distance
Figure 11. Case: \( b \) is \( i \)-increasing. (1) Associated disk \( D \) with conditions \( \diamondsuit 1, 2, 3 \). (2) \( \text{br}(b_1) \). Circles \( \circ \) indicate points of intersection between \( D \) and components of \( \text{br}(b - b(i)) \). See also Figure 12.

\( d(Q_i, A_i) \) is sufficiently small (e.g. \( d(Q_i, A_i) < \frac{1}{n+1} \)). Let \( r_i \) denote the closed interval in \([-2, 1] \times \{0\} \) with endpoints \( A_0 \) and \( Q_i \). (Figure 10(1).) We regard \( b \) as a braid contained in the cylinder \( \mathbb{D}^2 \times [0, 1] \subset \mathbb{R}^3 \) and \( b \) is based at \( n \) points \( A_1 \times \{0\}, \ldots, A_n \times \{0\} \). Since \( \pi_b(i) = i \), one can take a representative of \( b \) such that \( b(i) \) is an interval in the cylinder:

\( \diamondsuit 1. \) \( b(i) = \bigcup_{0 \leq t \leq 1} A_t \times \{t\} \).

Furthermore we may assume that \( \partial D(= \ell_i) \) of an associated disk \( D \) of \( (b, i) \) is a union of the following four segments as a set (Figure 10):

\( \diamondsuit 2. \) \( \bigcup_{-1 \leq t \leq 2} A_0 \times \{t\} \cup (r_i \times \{-1\}) \cup \bigcup_{-1 \leq t \leq 2} Q_i \times \{t\} \cup (r_i \times \{2\}) \).

Preserving \( \diamondsuit 1, 2 \) we may further assume the following (Figures 10(2), 11(1)):

\( \diamondsuit 3. \) For a regular neighborhood \( U_i \) of \( \ell_i \) in \( D \), we have \( I(b, i) \subset U_i \).

This is because every point \( x \in D \cap K' \), where \( K' \) is a component of \( \text{cl}(b - b(i)) \), one can slide \( x \) along \( K' \) so that the resulting point on \( K' \) is in \( U_i \). Said differently, preserving \( \partial D \) pointwise, we can modify a small neighborhood of \( D \) near \( K' \) so that the resulting associated disk satisfies \( \diamondsuit 3 \).

Under the conditions \( \diamondsuit 1, 2, 3 \) we have the following. For each \( x \in D \cap K' \subset U_i \), there is a segment \( s' \subset K' \) through \( x \) such that \( s' \) passes over \( b(i) \) since \( b \) is \( i \)-increasing. See Figure 11(1). Such a local picture of \( \text{cl}(b) \) is used in the the next section. Hereafter we assume that associated disks possess conditions \( \diamondsuit 1, 2, 3 \).

4.1. **Proof of Theorem 3.2(1).** Let \( s \) be the open segment in \( H_2(M_b, \partial M_b; \mathbb{R}) \) with the endpoints \( \frac{n-1}{u}[E_{(b,i)}] = (0, \frac{n-1}{u}) \) and \( [F_0] = (1, 0) \):

\[
\text{(4.1)} \quad s = \{(x, y) \in C_{(b,i)} \mid y = -\frac{n-1}{u}x + \frac{n-1}{u}, 0 < x < 1\}.
\]

The ray of each point in \( C_{(b,i)} \) through the origin intersects with \( s \). Thus for the proof of (1), it suffices to prove that \( s \subset C \).
We now introduce a sequence of braided links \( \{ \text{br}(b_p) \}_{p=1}^{\infty} \) from an \( i \)-increasing braid \( b \in B_n \) such that \( M_{b_{n+p}} \simeq M_b \) for each \( p \geq 1 \). (We use the 1-increasing braid \( \sigma_1^2 \sigma_2^{-1} \in B_3 \) to illustrate the idea.) Let \( D \) be an associated disk of the pair \((b,i)\). We take a disk twist

\[
T_D : \mathcal{E}(\text{cl}(b(i))) \to \mathcal{E}(\text{cl}(b(i)))
\]

so that the point of intersection \( D \cap A \) becomes the center of the twisting about \( D \), i.e. \( T_D(D \cap A) = D \cap A \). We may assume that \( T_D(A) = A \) as a set. Figure 11 illustrates the image of the segment \( s' \) under \( T_D \). The condition \( \triangleright 3 \) ensures that \( T_D \) equals the identity map outside a neighborhood of \( U_i \) in \( \mathcal{E}(\text{cl}(b(i))) \). Then by \( \triangleright 1, 2 \), it follows that

\[
T_D(\text{br}(b - b(i))) \cup \text{cl}(b(i))
\]

is a braided link of some \((i + u)\)-increasing braid with \((n + u)\) strands. We define \( b_1 \in B_{n+u} \) to be such a braid. The trivial knot \( T_D(A)(= A) \) becomes a braid axis of \( b_1 \). By definition of the disk twist, we have \( M_{b_1} \simeq M_b \). See Figure 12 for \( \text{br}(\sigma_1^2 \sigma_2^{-1}) \).

As discussed below, there is some ambiguity in defining \( b_1 \). As we will see, the ambiguity is irrelevant for the study of pseudo-Anosov monodromies defined on fibers of fibrations on the mapping torus. Suppose that both \( D \) and \( D' \) are the associated disks of the pair \((b,i)\) with conditions \( \triangleleft 1, 2, 3 \). We consider the disk twists \( T_D \) and \( T_{D'} \) with the above condition, i.e. both \( D \cap A \) and \( D' \cap A \) become the center of the twisting about \( D \) and \( D' \) respectively. Observe that the resulting two links obtained from \( D \) and \( D' \) are equivalent:

\[
T_D(\text{br}(b - b(i))) \cup \text{cl}(b(i)) \sim T_{D'}(\text{br}(b - b(i))) \cup \text{cl}(b(i)).
\]

They are braided links, say \( \text{br}(b_1) \) and \( \text{br}(b_1') \) of some braids \( b_1, b_1' \in B_{n+u} \) respectively with the same axis \( T_D(A) = A = T_{D'}(A) \). This means that a more stronger claim holds:

\[
(\text{br}(b_1), A) \sim (\text{br}(b_1'), A).
\]

Thus \( b_1 \) and \( b_1' \) are conjugate in \( B_{n+u} \) by Theorem 2.1. In particular both \( b_1 \) and \( b_1' \) are pseudo-Anosov (since the initial braid \( b \) is pseudo-Anosov and \( M_b \) is hyperbolic) and they have the same dilatation.
For each integer $C$ we have $C C$ where $h F$ and $D$p

**Remark 4.2.**

Proof of Theorem 3.2(1).

**Lemma 4.1.** For each integer $p \geq 1$, $g_p$ sends $(0, 1) \in \overline{C_{(b, i)}}$ to $(0, 1) \in \overline{C_{(b, i+p-u)}}$, and sends $(1, p) \in \overline{C_{(b, i)}}$ to $(1, 0) \in \overline{C_{(b, i+p-u)}}$. In particular for integers $x, y \geq 1$ with $y = xp + r$ for $0 \leq r < p$, $g_p$ sends $(x, y) \in \overline{C_{(b, i)}}$ to $(x, r) \in \overline{C_{(b, i+p-u)}}$.

**Proof.** We consider the oriented sum $F(x, y) := xF_b + yE_{(b, i)}$. This is an oriented surface embedded in $M_b$, and is obtained from the cut and past construction of parallel $x$ copies of $F_b$ and parallel $y$ copies of $E_{(b, i)}$. The orientation of $F(x, y)$ agrees with those of $F_b$ and $E_{(b, i)}$. We have $[F(x, y)] = (x, y) \in \overline{C_{(b, i)}}$. Then $g_p$ sends $E_{(b, i)}$ to $E_{(b, i+p-u)}$, and sends $F(1, p)$ to $F_{b, p}$. Thus $g_p$ sends $(0, 1)$ to $(0, 1)$, and sends $(1, p)$ to $(1, 0)$. This completes the proof.

By the proof of Lemma 4.1, $g_1$ sends $F(1, 1) = F_b + E_{(b, i)}$ to the fiber $F_{b, 1}$ of a fibration on $M_b$ associated with $(1, 1) \in \overline{C_{(b, i)}}$. Since the fibers $F(1, 1)$ and $F_b$ are norm-minimizing, $E_{(b, i)}$ is also norm-minimizing.

**Proof of Theorem 3.2(1).** We have $||F_b|| = n - 1$ and $||F_{b, p}|| = n + pu - 1$ since $F_b$ and $F_{b, p}$ are fibers, and $||E_{(b, i)}|| = u$ since $E_{(b, i)}$ is norm-minimizing. By Lemma 4.1, $[F_{b, p}] = (1, p) \in \overline{C_{(b, i)}}$. Consider the rational class

$$c_p := \frac{n - 1}{n + pu - 1} [F_{b, p}] = \left( \frac{n - 1}{n + pu - 1}, \frac{p(n - 1)}{n + pu - 1} \right).$$

Then $||c_p|| = n - 1$ for $p \geq 1$. The ray of $[F_{b, p}]$ through the origin is contained in some fibered cone for each $p \geq 1$. We easily check that $c_p$ lies on $s$ in (4.1). This means that three classes $[F_b]$, $c_p$ and $c_{p+1}$ with the same Thurston norm are contained in $C$. Observe that the small segment $s'$ in $s$ connecting $[F_b]$ and $c_{p+1}$ contains $c_p$, and $s' \subset C$ since $||\cdot||$ is linear on each fibered cone. Moreover $c_p \to (0, 0) \in \partial s \subset \partial C_{(b, i)}$ as $p \to \infty$. Putting all things together, we conclude that $s \subset C$. This completes the proof.

**Remark 4.2.** From the proof of Theorem 3.2(1), one sees the following: If $[E_{(b, i)}] \in \overline{C_{(b, i)}}$ is a fibered class, then $[E_{(b, i)}] \in C$. Otherwise $[E_{(b, i)}] \in \partial C$. See Figure 9(2)(3).
4.2. **Proof of Theorem 3.2(2).** We start with a simple observation: \( \Delta^2 \in B_n \) is \( j \)-increasing for each \( 1 \leq j \leq n \), and \( u(\Delta^2, j) = n - 1 \) holds. The following lemma is immediate.

**Lemma 4.3.** If \( b \in B_n \) is \( i \)-increasing, then \( b\Delta^2 \in B_n \) is \( i \)-increasing with \( u(b\Delta^2, i) = u(b, i) + n - 1 \).

We explain the idea of Theorem 3.2(2). Let \( D \) be the associated disk of the pair \((b, i)\). We have two types of the disk twist. One is \( T^k_{DA} : \mathcal{E}(A) \to \mathcal{E}(A) \) which appears in the proof of Theorem B in Section 3 and the other is \( T^p_D : \mathcal{E}(\text{cl}(b(i))) \to \mathcal{E}(\text{cl}(b(i))) \). If \( k \) and \( p \) are positive, then we obtain the \( i \)-increasing \( b\Delta^{2k} \) from the former type \( T^k_{DA} \), and another increasing braid \( b_p \) from the latter type \( T^p_D \). Since both resulting braids are increasing, we can further apply two types of the disk twist for the resulting braid. This is a key of the proof. Choosing two types of the disk twist alternatively, we get a sequence of increasing and pseudo-Anosov braids (since the initial braid \( b \) is pseudo-Anosov). We shall see that the desired monodromies associated with primitive classes in \( C(b, i) \) are given by these braids.

Let \( p_1, \ldots, p_j \) be integers such that \( p_1 \geq 0 \) and \( p_2, \ldots, p_j \geq 1 \). Given an \( i \)-increasing braid \( b \in B_n \) with \( u(b, i) = u \), we define an integer \( i[p_1, \ldots, p_j] \geq 1 \) and an \( i[p_1, \ldots, p_j] \)-increasing braid \( b[p_1, \ldots, p_j] \) inductively as follows.

- If \( j = 1 \) and \( p_1 = 0 \), then \( i[0] = i \) and \( b[0] = b \). If \( j = 1 \) and \( p_1 = p \geq 1 \), then \( i[p] = i + pu \) and \( b[p] = b_p \).
- If \( j > 1 \) is even, then
  \[
  \begin{align*}
  i[p_1, \ldots, p_{j-1}, p_j] &= i[p_1, \ldots, p_{j-1}], \\
  b[p_1, \ldots, p_{j-1}, p_j] &= (b[p_1, \ldots, p_{j-1}])^2 p_j.
  \end{align*}
  \]

The right-hand side is \( i[p_1, \ldots, p_{j-1}] \)-increasing by Lemma 4.3.

- If \( j > 1 \) is odd, then
  \[
  \begin{align*}
  i[p_1, \ldots, p_{j-1}, p_j] &= i[p_1, \ldots, p_{j-1}] + p_j u(b[p_1, \ldots, p_{j-1}], i[p_1, \ldots, p_{j-1}]), \\
  b[p_1, \ldots, p_{j-1}, p_j] &= (b[p_1, \ldots, p_{j-1}])^2 p_j.
  \end{align*}
  \]

We say that \( b[p_1, \ldots, p_j] \) has length \( j \).

**Example 4.4.**

1. \( b[p] = b_p \) by definition.
2. Let \( \beta = b\Delta^2 \). Then \( b[0, 1] = \beta \) and \( b[0, 1, p] = \beta_p \).
3. We have \( b[0, p] = b\Delta^{2p} \) and \( b[0, p, 1] = (b\Delta^{2p})_1 \), where \( (b\Delta^{2p})_1 \) is obtained from \( i \)-increasing \( b\Delta^{2p} \) by the disk twist.

For each \( k \geq 1 \), let \( f_k : M_b \to M_{b\Delta^{2k}} \) be the homeomorphism which in the proof of Theorem B. Consider the isomorphism \( f_{k*} : H_2(M_b, \partial M_b) \to H_2(M_{b\Delta^{2k}}, \partial M_{b\Delta^{2k}}) \). We have the following property.

**Lemma 4.5.** For each integer \( k \geq 1 \), \( f_{k*} \) sends \((1, 0) \in C(b, i) \) to \((1, 0) \in C(b\Delta^{2k}, i) \), and sends \((k, 1) \in C(b, i) \) to \((0, 1) \in C(b\Delta^{2k}, i) \). In particular for integers \( x, y \geq 1 \) with \( x = yk + r \) for \( 0 \leq r < k \), then \( f_{k*} \) sends \((x, y) \in C(b, i) \) to \((r, y) \in C(b\Delta^{2k}, i) \).

**Proof.** The homeomorphism \( f_k \) sends \( F_0 \) to \( F_{b\Delta^{2k}} \), and sends \( F_{(k, 1)} = kF_b + E_{(b, i)} \) to \( E_{(b\Delta^{2k}, i)} \). This implies that the claim holds. \( \square \)
Proof of Theorem 3.2(2). Let \((x, y) \in C(b,i)\) be a primitive integral class. (Hence \(x, y\) are positive integers with \(\gcd(x, y) = 1\).) We consider the continued fraction of \(y/x\) by the Euclidean algorithm

\[
\frac{y}{x} = p_1 + \frac{1}{p_2 + \frac{1}{p_3 + \cdots + \frac{1}{p_j} + \frac{1}{p_{j+1} + \frac{1}{p_{j+2} + \cdots + \frac{1}{p_l}}}}}
\]

with length \(j\) and \(p_j \geq 2\) and \(p_1 = 0\) if \(0 < y < x\). There is another expression

\[
\frac{y}{x} = p_1 + \frac{1}{p_2 + \frac{1}{p_3 + \cdots + \frac{1}{p_{j-1} + \frac{1}{p_j}} + \frac{1}{p_{j+1} + \frac{1}{p_{j+2} + \cdots + \frac{1}{p_l}}}}}
\]

with length \(j + 1\). We choose one of the two expressions with odd length \(\ell\):

\[
\frac{y}{x} = p_1 + \frac{1}{p_2 + \frac{1}{p_3 + \cdots + \frac{1}{p_{\ell-1} + \frac{1}{p_{\ell}}}}}
\]

This encodes the fiber \(F_{(x,y)}\) and its monodromy \(\phi_{(x,y)}\). In fact Lemmas 4.1, 4.5 ensure that

\[
(g_{p_1}f_{p_{\ell-2}} \cdots f_{p_3}g_{p_1}),_* : H_2(M_b, \partial M_b) \to H_2(M_{b[p_1, \ldots, p_{\ell}]}, \partial M_{b[p_1, \ldots, p_{\ell}]})
\]

sends \((x, y) = [xF_b + yE_{(b,i)}]\) to \((1, 0)\) which is the integral class of the \(F\)-surface of \(b[p_1, \ldots, p_{\ell}]\). \((g_{p_1} = id : M_b \to M_b\) if \(p_1 = 0\). Thus \(F_{(x,y)}\) has genus 0. Moreover this means that one can take \(F_{b[p_1, \ldots, p_{\ell}]\) as a representative of \((x, y) \in C(b,i)\) and the monodromy \(\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}\) is determined by \(b[p_1, \ldots, p_{\ell}]\). This completes the proof.

We denote by \(b_{(x,y)}\) the braid \(b[p_1, \ldots, p_{\ell}]\) which determines \(\phi_{(x,y)}\). Here is an example: If \((x, y) = (5, 14)\), then \(\frac{14}{5} = 2 + \frac{4}{1 + \frac{1}{4}}\) and \(\phi_{(5,14)}\) is determined by \(b_{(5,14)} = b[2,1,4]\). If \((x, y) = (14, 5)\), then \(\frac{5}{14} = 0 + \frac{1}{2 + \frac{1}{3 + \frac{1}{1 + 1}}}\) and \(\phi_{(14,5)}\) is determined by \(b_{(14,5)} = b[0,2,1,3,1]\).

4.3. Proof of Theorem 3.2(3). We begin with the following lemma.

Lemma 4.6 (Standard form). If \(b \in B_n\) is \(i\)-increasing with \(u(b, i) = u\), then \(b\) is conjugate to an \(n\)-increasing braid \(b'\) of the form

\[
b' = (w_1\sigma_{n-1}^2) \cdots (w_n\sigma_{n-1}^2),
\]

where each \(w_k\) is a word of \(\sigma_1^{\pm 1}, \ldots, \sigma_{n-2}^{\pm 1}\), but not \(\sigma_{n-1}^{\pm 1}\), possibly \(w_k = \emptyset\) for some \(k\).

Figure 13(1) shows the form of \(b'\) in Lemma 4.6 in case \(u = 2\).

Proof. We regard \(b\) as a braid in \(\mathbb{D} \times [0, 1]\). By \(\Diamond 1\), \(b(i)\) is an interval in \(\mathbb{D} \times [0, 1]\). If \(i = n\), then \(b\) is \(n\)-increasing and it is not hard to see that a representative of \(b\) is of the desired form in Lemma 4.6. Suppose that \(b\) is \(i\)-increasing for \(1 \leq i < n\). We set \(\sigma = \sigma_{n-1}\sigma_{n-2} \cdots \sigma_1\) if \(1 \leq i < n-1\) and \(\sigma = \sigma_{n-1}\) if \(i = n-1\). We consider the \(n\)-braid \(b' = \sigma b \sigma^{-1}\) which is \(n\)-increasing with \(u(b', n) = u\). We pull \(b'(n)\) tight in \(\mathbb{D} \times [0, 1]\) and make it straight. Then a representative of \(b'\) is of the desired form. \(\square\)
Since each $b$ generates \{b_p\}. (1) $b = w_1 \sigma_1^2 w_2 \sigma_2^2 = (\nu_1 \rho) (\nu_2 \rho) \in B_4$, where $\nu_j = w_j (\sigma_1 \sigma_2)^{-1}$. 
(2) $b_1 = (\nu_1 \rho) (\nu_2 \rho) \in B_6$. (3) $b_2 = (\nu_1 \rho) (\nu_2 \rho) \in B_8$.

Proof of Theorem 3.2(3). Since each $i$-increasing braid is conjugate to an $n$-increasing braid of a standard form in Lemma 4.6, we may assume that $b \in B_n$ is an $i$-increasing braid of the form $b = (w_1 \sigma_1^2 \cdots w_n \sigma_n^2)$. Since $\rho \in B_n$ is the periodic braid such that each $\rho_j$ is a reducible braid and $\rho = \sigma_1 \sigma_2 \cdots \sigma_{n-2} \sigma_{n-1}$, we have $\sigma_{n-1} = (\sigma_1 \cdots \sigma_{n-2})^{-1} \rho$. Then $b$ is expressed as follows.

$$b = (\nu_1 \rho) \cdots (\nu_n \rho),$$

where $\nu_i = w_i (\sigma_1 \cdots \sigma_{n-2})^{-1}$ is written by a word of $\sigma_1^\pm, \cdots, \sigma_{n-2}^\pm$, but not $\sigma_{n-1}^\pm$. Each $\nu_j$ in $b$ is a reducible braid and $\rho$ in $b$ is the periodic braid. Let $\omega_j : F_b \to F_b$ denote a reducible representative whose mapping class is determined by $\nu_j$, and let $\psi : F_b \to F_b$ denote a periodic representative whose mapping class is determined by $\rho$. The monodromy $\phi_b$ defined on $F_b$ is written by $\phi_b = (\omega_1 \psi) \cdots (\omega_n \psi)$.

Recall that $\mathbb{D}_{n-1}$ is the disk $\mathbb{D}$ with marked points $A_1, \cdots, A_{n-1}$. Let $S_0$ be an $n$-holed sphere obtained from $\mathbb{D}_{n-1}$ by removing the interiors of small $(n-1)$ disks with centers $A_1, \cdots, A_{n-1}$. Each $\nu_j$ as an $(n-1)$-braid determines a homeomorphism $\hat{\omega}_j : S_0 \to S_0$. We may assume that $\hat{\omega}_j$ fixes one of the boundary components corresponding to $\partial \mathbb{D}$ pointwise. It is clear that we have an embedding $h : S_0 \to F_b$ such that each $\omega_j$ in $\phi_b$ is reducible supported on the subsurface $h(S_0)$ and the restriction of $\omega_j$ to $h(S_0)$ is given by $h \circ \omega_j \circ h^{-1}$.

By the proof of Theorem 3.2(2), $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ associated with each primitive class $(x, y) \in C_{(b, i)}$ is determined by the braid of the form $b[p_1, \ldots, p_\ell]$. We now prove by the induction on length $\ell$ that

$$b[p_1, \ldots, p_\ell] = (\nu_1 \rho) \cdots (\nu_{\ell-1} \rho) (\nu \rho) \rho^{m-1} = (\nu_1 \rho) \cdots (\nu_{\ell-1} \rho) (\nu \rho^m)$$

for some $m \geq 1$ depending on $(x, y)$. Here each $\nu_j$ in $b[p_1, \ldots, p_\ell]$ is a reducible braid which is an extension of $\nu_j$ in $b$ and $\rho$ is the periodic braid with the degree of $b[p_1, \ldots, p_\ell]$. If this holds, then $\phi_{(x,y)}$ has a desired property as in Theorem 3.2(3). Suppose that $\ell = 1$. If $p_1 = 0$, then $b[0] = b$ and we are done. If $p_1 \geq 1$, then $b[p_1] = b_{p_1}$. Using the above expression of $b$ we observe that $b_{p_1}$ is written by

$$b_{p_1} = (\nu_1 \rho) \cdots (\nu \rho) \in B_{n+p_1}$$

(see Figure 13). We are done.
For $\ell \geq 2$, suppose that $b[p_1, \ldots, p_{\ell-1}] = (v_1\rho_d) \cdots (v_{a-1}\rho_d)(v_u\rho_d^m)$ for some $m$, where $d$ is the degree of $b[p_1, \ldots, p_{\ell-1}]$. Consider $b[p_1, \ldots, p_{\ell}]$ with length $\ell$. If $\ell$ is even, then by induction hypothesis
\begin{equation*}
\Delta_d^a = (b[p_1, \ldots, p_{\ell-1}])\Delta_d^{2p} = (v_1\rho_d) \cdots (v_{a-1}\rho_d)(v_u\rho_d^m)\Delta_d^{2p}.
\end{equation*}
Since $\Delta_d^2 = \rho_d^{d-1}$ we have $(v_u\rho_d^m)\Delta_d^{2p} = \rho_d^{m+p(d-1)}$. Thus $b[p_1, \ldots, p_{\ell}]$ has a desired expression and we are done. If $\ell$ is odd, then by induction hypothesis again
\begin{equation*}
b[p_1, \ldots, p_{\ell}] = (b[p_1, \ldots, p_{\ell-1}])_{p_{\ell}} = ((v_1\rho_d) \cdots (v_{a-1}\rho_d)(v_u\rho_d^m))_{p_{\ell}}.
\end{equation*}
As in the case of $\ell = 1$, the braid in the right-hand side is expressed as
\begin{equation*}
((v_1\rho_d) \cdots (v_{a-1}\rho_d)(v_u\rho_d^m))_{p_{\ell}} = (v_1\rho_d) \cdots (v_{a-1}\rho_d)(v_u\rho_d^m),
\end{equation*}
where $\dagger$ is the degree of $b[p_1, \ldots, p_{\ell}]$. This completes the proof. \hfill \square

5. Sequences of pseudo-Anosov braids with small normalized entropies

In this section we prove Theorem A. We begin with an observation. Let $\Omega \subset \{ a \in C \mid \|a\| = 1 \}$ be a compact set in $H_2(M_b, \partial M_b; \mathbb{R})$ and let $C_{\Omega} \subset C$ denote the cone over $\Omega$ through the origin. By Theorem 2.3(2) there is a constant $P = P(\Omega) > 0$ depending on $\Omega$ such that $\operatorname{Ent}(a) < P$ for any $a \in C_{\Omega}$. This observation provides us many sequences of pseudo-Anosov braids with small normalized entropies from a single pseudo-Anosov braid $b$.

**Theorem 5.1.** Suppose that $b$ is a pseudo-Anosov braid whose permutation has a fixed point. We fix any $0 < \ell < \infty$. Let $\{(x_p, y_p)\}$ be a sequence of primitive integral classes in $C_{(b, i)}$ such that $y_p/x_p < \ell$ and $\| (x_p, y_p) \| \approx p$. Then the sequence of pseudo-Anosov braids $\{b(x_p, y_p)\}$ has a small normalized entropy.

**Proof.** If $\{(x_p, y_p)\}$ is the sequence under the assumption, then we have $d(b(x_p, y_p)) \approx \| (x_p, y_p) \| \approx p$. Since $(1, 0) \in C_{(b, i)} \subset C$ and the slope of $y_p/x_p$ is bounded by $\ell$ from above, the set of projective classes $(x_p, y_p)$ is contained in some compact set in $\{ a \in C \mid \|a\| = 1 \}$ (Figure 9). Thus there is a constant $P = P(\ell) > 1$ such that $\operatorname{Ent}(b(x_p, y_p)) < P$ for any $p$. This completes the proof. \hfill \square

Let us discuss three sequences coming from Example 4.4. They are $\{b_p\}$, $\{\beta_p\}$ and $\{(b\Delta^{2p})_1\}$ varying $p$. It is not hard to see that $d(b_p)$, $d(\beta_p)$, $d((b\Delta^{2p})_1) \approx p$.

**Theorem 5.2.** For an $i$-increasing and pseudo-Anosov $b \in B_n$, we have the following on the sequences of pseudo-Anosov braids.

1. $\{b_p\}$ has a small normalized entropy if and only if $[E_{(b, i)}]$ is a fibered class.
2. For $\beta = b\Delta^2 \in B_n$, $\{\beta_p\}$ has a small normalized entropy and $\operatorname{Ent}(\beta_p) \approx \operatorname{Ent}(1, 1)$ as $p \rightarrow \infty$.
3. $\{(b\Delta^{2p})_1\}$ has a small normalized entropy and $\operatorname{Ent}((b\Delta^{2p})_1) \approx \operatorname{Ent}(b)$ as $p \rightarrow \infty$.

**Proof of Theorem 5.2.** For $a = (x, y) \in C_{(b, i)}$, let $a = (x, y)$ denote its projective class. We have $[F_{(b, i)}] = (1, p) \rightarrow [E_{(b, i)}] = (0, 1)$ as $p \rightarrow \infty$. If $[E_{(b, i)}]$ is a fibered class, then $[E_{(b, i)}] \in C$ by Remark 4.2 and $\operatorname{Ent}(b_p) \rightarrow \operatorname{Ent}([E_{(b, i)}])$ as $p \rightarrow \infty$ by Theorem 2.3(2). If $[E_{(b, i)}]$ is a non-fibered class, then $[E_{(b, i)}] \in \partial C$ by Remark 4.2,
and $\text{Ent}(b_p) \to \infty$ as $p \to \infty$ by Theorem 2.3(3). We finish the proof of (1). We turn to (2). Since $[F_{\beta_p}] = [(p+1, p)] \in C(b;i)$, its projective class goes to $(1, 1)$ as $p \to \infty$. Since $(1, 1) \in C(b;i) \subset C$ by Theorem 3.2(1), $\text{Ent}(\beta_p) \to \text{Ent}((1, 1))$ as $p \to \infty$ by Theorem 2.3(2). This completes the proof of (2). Finally we prove (3). The fibered class of $F$-surface of $(b \Delta^2)_1$ is given by $(p+1, 1)$ in $C(b;i)$. Its projective class goes to $[F_b] = (1, 0)$ as $p \to \infty$. Thus $\text{Ent}((b \Delta^2)_1) \to \text{Ent}(b)$ as $p \to \infty$. This completes the proof. \hfill \square

We use Theorem 5.2(1)(2) in Section 8. For an application using (3), see [19].

Proof of Theorem A. Suppose that $b \in B_n$ is pseudo-Anosov with $\pi_b(i) = i$. Let $\beta(k)$ denote $b \Delta^{2k} \in B_n$ for $k \geq 1$. Clearly $\beta(k)$ is pseudo-Anosov with the same dilatation as $b$ (for any $k$) and $\beta(k)$ is positive for $k$ large. We fix such large $k$. By Lemma 3.1 $\beta(k)$ is $i$-increasing. If we let $z_p = (\beta(k) \Delta^2)_1$, then $M_{z_p} \simeq M_{\beta(k)} \simeq M_b$ holds for $p \geq 1$. By Theorem 5.2(3), $\{z_p\}$ has a small normalized entropy and $\text{Ent}(z_p) \to \text{Ent}(\beta(k)) = \text{Ent}(b)$ as $p \to \infty$. \hfill \square

Let $b_p^\star$ denote the braid obtained from $(i + pu)$-increasing $b_p$ by removing the strand of the index $i + pu$. Taking its spherical element we have $S(b_p^\star)$. A mild generalization of the sequence $\{b_p\}$ is the ones $\{b_p^\star\}$ and $\{S(b_p^\star)\}$ varying $p$. Although $b_p^\star$, $S(b_p^\star)$ may not be pseudo-Anosov, they are frequently pseudo-Anosov. To be more precise, we need to consider the number of prongs of singularities in the stable foliation $F_{b_p}$ for $b_p$ as we explained in Section 2.3. This is the motivation of the study in Section 6

6. Stable foliation for the monodromy

Let $b$ be pseudo-Anosov and $i$-monotonic with the sign $\epsilon(b,i) = \epsilon$. For any primitive integral class $(x, y) \in C(b,i)$, the oriented sum $F_{(x,y)} = xF_b + yE(b,i)$ is connected. Let $\partial \mathcal{N}(A)$ and $\partial \mathcal{N}(\text{cl}(b(i)))$ respectively.
Let us set
\[ \partial_{(b,A)} F_{(x,y)} = \partial F_{(x,y)} \cap T_{(b,A)} \quad \text{and} \quad \partial_{(b,i)} F_{(x,y)} = \partial F_{(x,y)} \cap T_{(b,i)}, \]
each of which is a single simple closed curve on the torus (since \(\gcd(x, y) = 1\)). Recall that we chose the orientation of the axis for the \(i\)-monotonic \(b\) in Section 3. We use the meridian and longitude basis \(\{m_i, \ell_i\}\) for \(T_{(b,A)}\) to represent a homology class of a disjoint union of simple closed curves on \(T_{(b,A)}\). We also use the meridian and the longitude basis \(\{m_i, \ell_1\}\) for \(T_{(b,i)}\). Observe that the homology classes \([\partial_{(b,A)} F_{(x,y)}]\) and \([\partial_{(b,i)} F_{(x,y)}]\) are given by the pairs of integers
\[
(6.1) \quad \quad [\partial_{(b,A)} F_{(x,y)}] = (-\epsilon y, x) \quad \text{and} \quad [\partial_{(b,i)} F_{(x,y)}] = (-\epsilon x, y).
\]
They are called boundary slopes of \(F_{(x,y)}\). See Figure 14.

Let \(\phi_b : F_b \to F_b\) be the pseudo-Anosov monodromy of a fiber \(F_b\) of the fibration on \(M_b \to S^1\). The stable foliation \(F_b\) of \(\phi_b\) has singularities on each boundary component of \(F_b\). Now we consider the suspension flow \(\phi_t^\iota \ (t \in \mathbb{R})\) on the mapping torus \(M_b\). We obtain a disjoint union of simple closed curves \(c_A = c_{(b,A)}\) on \(T_{(b,A)}\) (possibly a single simple closed curve) which is a union of closed orbits for singularities in \(\partial_{(b,A)} F_b\) under the flow. Similarly we have a disjoint union of simple closed curves \(c_i = c_{(b,i)}\) on \(T_{(b,i)}\) (possibly a single simple closed curve again) which is a union of closed orbits for singularities in \(\partial_{(b,i)} F_b\). (Figure 17 depicts these closed curves for some pseudo-Anosov 3-braid.) A useful tool is train track maps which encode those data \(\phi_b, F_b\). They also enable us to compute homology classes \([c_A]\) and \([c_i]\).

The following lemma is a consequence of Theorem 2.4(2) by Fried.

**Lemma 6.1.** Let \(\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}\) be the monodromy of a fibration on \(M_b \to S^1\) associated with a primitive integral class \((x, y) \in C_{(b,i)}\). Then the stable foliation \(F_{(x,y)}\) for \(\phi_{(x,y)}\) is \(i([c_A], [\partial_{(b,A)} F_{(x,y)}])\)-pronged at \(\partial_{(b,A)} F_{(x,y)}\), and is \(i([c_i], [\partial_{(b,i)} F_{(x,y)}])\)-pronged at \(\partial_{(b,i)} F_{(x,y)}\), where \(i(\cdot, \cdot)\) means the geometric intersection number between homology classes of closed curves.

**Remark 6.2.** Every closed orbit of the suspension flow \(\phi_t^\iota\) on the mapping torus \(M_b\) travels around \(S^1\) direction at least once. This implies that \([c_A]\) has a non-zero first coordinate of the meridian and longitude basis for \(T_{(b,A)}\), i.e., we have \([c_A] = (k, \ell) \in \mathbb{Z}^2\) with \(k \neq 0\), since the meridian for \(T_{(b,A)}\) corresponds to the flow direction. Similarly, \([c_i]\) has a non-zero second coordinate of the meridian and longitude basis for \(T_{(b,i)}\), that is we have \([c_i] = (k', \ell') \in \mathbb{Z}^2\) with \(\ell' \neq 0\), since the longitude for \(T_{(b,i)}\) corresponds to the flow direction in this case.

Recall that given a braid \(b \in B_n\), we denote by \(S(b) \in SB_n\), the spherical \(n\)-braid with the same word as \(b\). For an \(i\)-increasing braid \(b\) of pseudo-Anosov type, consider the braid \((b\Delta^p)_1 = b[0,p,1]\) in Example 4.4(3). This is an \([i0,p,1]\)-increasing braid. Then we have its spherical braid \(S((b\Delta^p)_1)\). We now define other braids obtained from \((b\Delta^p)_1\). Let \((b\Delta^p)_1^\iota\) denote the braid obtained from \((b\Delta^p)_1\) by removing the strand of the index \(i[0,p,1]\). Let \(S((b\Delta^p)_1)\) and \(S((b\Delta^p)_1^\iota)\) be the spherical braids corresponding to \((b\Delta^p)_1\) and \((b\Delta^p)_1^\iota\) respectively. Then we have the following result.

**Lemma 6.3.** Suppose that \(b\) is an \(i\)-increasing braid of pseudo-Anosov type. For \(p\) large, the braid \((b\Delta^p)_1^\iota\) and the spherical braids \(S((b\Delta^p)_1), S((b\Delta^p)_1^\iota)\) are all pseudo-Anosov with the same dilatation as \((b\Delta^p)_1\).
Before proving Lemma 6.3, we recall a formula of the geometric intersection number $i([c], [c'])$ between two homology classes of simple closed curves $c$, $c'$ on a torus. Let $(p, q)$ and $(p', q')$ be primitive elements of $\mathbb{Z}^2$ which represent $[c]$ and $[c']$ respectively. Then

$$i([c], [c']) = |pq' - p'q|.$$ 

**Proof of Lemma 6.3.** The fibered class of $F$-surface of $(b\Delta^2p)_1$ is $(p + 1, 1) \in C(b,i)$. We have \([\partial(b,A)F_{(p+1,1)}] = (-1, p + 1)\) and \([\partial(b,i)F_{(p+1,1)}] = (-p + 1, 1)\), see (6.1). By Remark 6.2, one can write $[c_A] = (k, \ell)$ with $k \neq 0$ and $[c_i] = (k', \ell')$ with $\ell' \neq 0$. Then $i([c_A], [\partial(b,A)F_{(p+1,1)}]) = k(p + 1) + \ell$ and $i([c_i], [\partial(b,i)F_{(p+1,1)}]) = |k' + \ell'(p + 1)|$. Since $k \neq 0$ and $\ell' \neq 0$, these intersection numbers are increasing with respective to $p$ and they are clearly greater than 1 when $p$ is large. Then Lemma 6.1 says that when $p$ is large, the stable foliation $F_{(p+1,1)}$ for the monodromy $\phi_{(p+1,1)}$ is not 1-pronged at each component of $\partial(b,A)F_{(p+1,1)} \cup \partial(b,i)F_{(p+1,1)}$. By the discussion in Section 2.4, we are done. \(\square\)

7. Properties of $F$-surfaces and $E$-surfaces

The aim of this section is to study properties of $E_\gamma$, $F$-surfaces and to present the technique used in the last section.

**Lemma 7.1.** For an $i$-increasing braid $b \in B_n$ with $u(b,i) = u$, we set $\beta = b\Delta^2 \in B_n$. Then there is an $n$-increasing braid $\gamma \in B_{n+u}$ such that

$$(\text{br}(\beta), \text{cl}(\beta(i)), A_\beta) \sim (\text{br}(\gamma), A_\gamma, \text{cl}(\gamma(u))).$$

In particular $M_b \simeq M_\gamma \simeq M_{\beta}$ and $E_{(\beta,\gamma)} = F_\gamma$, $F_\beta = E_{(\gamma,n)}$ up to isotopy in $M_\beta$. Moreover if $b$ is pseudo-Anosov, then $\gamma$ is also pseudo-Anosov.

A similar claim holds for $i$-decreasing braids.

**Proof.** By Lemma 4.6 we may assume that $b \in B_n$ is an $n$-increasing braid of a standard form $b = (w_1\sigma_{n-1}^2) \cdots (w_u\sigma_{n-1}^2)$ containing $u$ subwords $\sigma_{n-1}^2$. Using the identity $\Delta^2 = \Delta_{n-1}^2 \sigma_{n-1} \cdots \sigma_2 \sigma_{n-1} \cdots \sigma_{n-1} \in B_n$, we have (Figure 15(1))

$$\text{br}(\beta) = \text{br}(b\Delta^2) = \text{br}(w_1\sigma_{n-1}^2 \cdots w_u\sigma_{n-1}^2 \Delta_{n-1}^2 \sigma_{n-1} \cdots \sigma_2 \sigma_{n-1} \cdots \sigma_{n-1}).$$

We first deform $\text{br}(\beta)$ into a link as in Figure 15(3). The same figure(1)(2)(3) tells us the process to get the desired link in (3). Then we perform the local moves in the shaded regions containing $u$ subwords $\sigma_{n-1}^2$ in $b$ so that the link in question is a union of the closure of some $n$-increasing braid $\gamma \in B_{n+u}$ and its braided axis, namely a braided link, see Figure 15(3)(4)(5). As a result,

$$(\text{br}(\beta), \text{cl}(\beta(u)), A_\beta) \sim (\text{br}(\gamma), A_\gamma, \text{cl}(\gamma(u))).$$

This expression says that $M_\beta \simeq M_{\gamma}$ and the $E_\gamma$, $F$-surfaces for $\beta$ are equal to the $F_\gamma$, $E$-surfaces for $\gamma$. Since $M_b \simeq M_\beta$ we are done. \(\square\)
Figure 15. Demonstration of Lemma 7.1 when $b$ is $n$-increasing with $u(b,n) = 2$. (1) $\text{br}(\beta) = w_1 \sigma_{n-1}^2 w_2 \sigma_{n-1}^2 \Delta^2$. (5)(6) $\text{br}(\gamma)$ of $\gamma = \kappa_0 \kappa_1 \kappa_2 \kappa_3 \Delta_{n-1}^2$.

Here we introduce a simple representative of $\gamma \in B_{n+u}$ in Lemma 7.1. By the deformation as in (5)(6) of Figure 15, we can take the following representative of $\gamma$.

$$\gamma = \kappa_0 \kappa_1 \cdots \kappa_{u+1} \Delta_{n-1}^2,$$

where

$$\begin{align*}
\kappa_0 &= \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 \sigma_2 \cdots \sigma_{n+u-1}, \\
\kappa_j &= w_j \sigma_{n-1} \sigma_{n} \cdots \sigma_{n+u-j-1} \sigma_{n+u-j-2} \cdots \sigma_{n-1}^{-1} \quad \text{if } 1 \leq j \leq u-1, \\
\kappa_u &= w_u \sigma_{n-1}, \\
\kappa_{u+1} &= \sigma_{n-1}^{-1} \quad \text{if } u = 1, \\
\kappa_{u+1} &= \sigma_{n+u-1}^{-1} \sigma_{n+u-2}^{-1} \cdots \sigma_{n}^{-1} \quad \text{if } u \geq 2.
\end{align*}$$
For example if \((n, u) = (3, 2)\), then

\begin{equation}
\gamma = \kappa_0 \kappa_1 \kappa_2 \kappa_3 \Delta_2^2 = \sigma_2 \sigma_2 \sigma_3 \sigma_4 w_1 \sigma_2 \sigma_3 \sigma_2^{-1} w_2 \sigma_2 \sigma_4^{-1} \sigma_3^{-1} \sigma_1^2.
\end{equation}

If \((n, u) = (3, 3)\), then \(\gamma = \kappa_0 \kappa_1 \kappa_2 \kappa_3 \kappa_4 \Delta_2^2\), that is

\begin{equation}
\gamma = \sigma_2 \sigma_2 \sigma_3 \sigma_5 w_1 \sigma_2 \sigma_3 \sigma_3^{-1} \sigma_2^{-1} w_2 \sigma_2 \sigma_3 \sigma_2^{-1} w_3 \sigma_2 \sigma_5^{-1} \sigma_4^{-1} \sigma_3^{-1} \sigma_2^2.
\end{equation}

Lemma 7.1 is used in the following situation. Suppose that \(\alpha \in B_{n+u}\) is a \(j\)-increasing braid and our task is to prove that \(\alpha\) is pseudo-Anosov and its \(E\)-surface \(E_{(\alpha, j)}\) is a fiber of a fibration on \(M_{\alpha} \to S^1\). (The conditions are needed to apply Theorem 5.2(1) for \(\alpha\).) To do this, we need to find an \(i\)-increasing and pseudo-Anosov braid \(b \in B_n\) with \(u = u(b, i)\) and need to check the resulting \(n\)-increasing braid \(\gamma \in B_{n+u}\) in Lemma 7.1 satisfies the property

\[(br(\gamma), A_\gamma, cl(\gamma(n))) \sim (br(\alpha), A_\alpha, cl(\alpha(j))],\]

i.e. \(\gamma\) is conjugate to \(\alpha\) preserving the corresponding strand. If this equivalence holds, then by Lemma 7.1 together with the above equivalence \(\sim\), our task is done. As a result \(\{\alpha_p\}\) has a small normalized entropy by Theorem 5.2(1).

8. Application

In the last section we prove Theorems C, D and E. We first recall a study of pseudo-Anosov 3-braids [14, 24]. Let \(w\) be a word in \(\sigma_1^{-1}\) and \(\sigma_2\). If both \(\sigma_1^{-1}\) and \(\sigma_2\) occur at least once in \(w\), then we say that \(w\) is a pA word. It is known that the 3-braid represented by a pA word is pseudo-Anosov. Conversely a 3-braid \(b\) is pseudo-Anosov, then there is a pA word \(w\) such that the braid represented by \(w\) is conjugate to \(b\) up to a power of the full twist.

The stable foliation \(\mathcal{F}_b\) is 1-pronged at each boundary component of \(F_b\) for each pseudo-Anosov 3-braid \(b\). Figure 17(3) exhibits a train track automaton. A train track map for the 3-braid represented by a pA word \(w\) is obtained from the closed loop corresponding to \(w\) in the automaton. For more details, see Ham-Song [13].

8.1. Palindromic/Skew-palindromic braids. We define an anti-homomorphism

\[rev: B_n \to B_n, \quad \sigma_{i_1}^{\mu_1} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_k}^{\mu_k} \mapsto \sigma_{i_k}^{\mu_k} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_1}^{\mu_1}, \quad \mu_j = \pm 1.\]

A braid \(b \in B_n\) is palindromic if \(rev(b) = b\). Clearly \(b \cdot rev(b)\) is palindromic for any \(b \in B_n\). Let us consider another anti-homomorphism

\[skew: B_n \to B_n, \quad \sigma_{i_1}^{\mu_1} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_k}^{\mu_k} \mapsto \sigma_{n-i_k}^{\mu_k} \sigma_{n-i_2}^{\mu_2} \cdots \sigma_{n-i_1}^{\mu_1}, \quad \mu_j = \pm 1.\]

A braid \(b \in B_n\) is skew-palindromic if \(skew(b) = b\). Clearly \(b \cdot skew(b)\) is skew-palindromic for any \(b \in B_n\).

We now prove Theorems C and D which indicate the asymptotic behaviors of minimal entropies among these subsets are quite distinct.

Proof of Theorem C. For the surjective homomorphism \(\pi: B_n \to S_n\) we write \(\pi_j = \pi(\sigma_j)\). Suppose that an \(n\)-braid \(b = \sigma_{i_1}^{\mu_1} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_k}^{\mu_k}\) is palindromic. Since \(rev(b) = b\) we have

\(\pi_{rev(b)} = \pi_{i_k} \cdots \pi_{i_2} \pi_{i_1} = \pi_{i_1} \pi_{i_2} \cdots \pi_{i_k} (= \pi_b).\)
Multiply the both side by $i_1 i_2 \cdots i_k$ from the left:

$$(i_1 i_2 \cdots i_k)(i_k \cdots i_2 i_1) = (i_2 i_3 \cdots i_k) = \sigma_b^2.$$

Since $\pi_j^2 = id$ the left-hand side equals $id$. Hence $id = \pi_b^2$ which means that the square $b^2$ is pure. A theorem by Song [28] states that for a pseudo-Anosov pure element $b' \in B_n$, its dilatation has a uniform lower bound $2 + \sqrt{5} \leq \lambda(b')$. In particular if $b' = b^2$, then $2 + \sqrt{5} \leq \lambda(b^2) = (\lambda(b))^2$. This completes the proof.

**Proof of Theorem D.** We separate the proof into two cases, depending on the parity of the braid degree. We first prove $\log \delta(PA_{2n}) = 1/n$. Let us take $\xi = \sigma_1 \sigma_2^2 \sigma_3 \sigma_4 \in B_5$ (Figure 16). The braid $\xi$ is 3-increasing with $u(\xi, 3) = 2$. We consider the disk twist about $D_{(\xi, 3)}$. We obtain the braid $\xi_p$ which is $(3 + 2p)$-increasing for each $p \geq 1$. Observe that $\xi_p^\bullet$ is a skew-palindromic braid with even degree for each $p \geq 1$:

$$\xi_p^\bullet = (\sigma_1 \cdots \sigma_{1+2p})(\sigma_3 \cdots \sigma_{3+2p}) \in B_{4+2p}.$$
(For the definition of $\xi_p^*$, see Section 5.) By the lower bound of dilatations by Penner, it is enough to prove that the sequence $\{\xi_p^*\}$ has a small normalized entropy. We prove this in the following two steps. In Step 1 we prove that $\{\xi_p\}$ has a small normalized entropy. In Step 2 we prove that the stable foliation $F_{\xi_p}$ is not 1-pronged at $\partial(\xi_p,3+2p)F_{\xi_p}$ for $p \geq 1$. This tells us that $\xi_p^*$ is pseudo-Anosov with the same dilatation as $\xi_p$. By Step 1 it follows that $\{\xi_p^*\}$ has a small normalized entropy.

**Step 1.** The sequence $\{\xi_p\}$ has a small normalized entropy.

By Theorem 5.2(1) it suffices to prove that $\xi$ is pseudo-Anosov and $[E_{(\xi,3)}]$ is a fibered class. Consider a pseudo-Anosov braid $b = \sigma_1^{-1}\sigma_2^2\sigma_1^{-1}\sigma_2^2 \in B_3$. It is 3-increasing with $u(b,3) = 2$. For $\beta = b\Delta^2$ we have $M_\beta \simeq M_\gamma$. By Lemma 7.1 $(\text{br}(\beta), \text{cl}(\beta(3)), A_\beta) \sim (\text{br}(\gamma), A_\gamma, \text{cl}(\gamma(3)))$, where $\gamma \in B_5$ is the braid in (7.1) substituting $\sigma_1^{-1}$ for $w_1$ and $\sigma_1^{-1}$ for $w_2$. It is not hard to check that $\gamma$ is conjugate to $\xi$ in $B_5$ and their permutations have a common fixed point $3$. Hence

\[(8.1) \quad (\text{br}(\beta), \text{cl}(\beta(3)), A_\beta) \sim (\text{br}(\xi), A_\xi, \text{cl}(\xi(3))).\]

In particular $E_{(\xi,3)} = F_\beta$ which means that $E_{(\xi,3)}$ is a fiber of a fibration on the hyperbolic mapping torus $M_\beta \simeq M_\xi$ over $S^1$. Thus $\xi$ is pseudo-Anosov.

**Step 2.** $F_{\xi_p}$ is $(p+1)$-pronged at $\partial(\xi_p,3+2p)F_{\xi_p}$ for $p \geq 1$.

We read the singularity data of $F_{\xi_p}$ from the monodromy $\phi_\beta : F_\beta \to F_\beta$ of the fibration on $M_\beta \to S^1$. First consider the suspension flow $\phi_\beta$ on the mapping torus $M_\beta$. Since $F_\beta$ is 1-pronged at each component of $F_\beta$, we have simple closed curves $c_A \subset \mathcal{T}(b,A)$ and $c_3 \subset \mathcal{T}(b,3)$ such that $[c_A] = (1,0), [c_3] = (2,1) \in \mathbb{Z}^2$ (Figure 17(1)(2)).

Next we turn to $\beta = b\Delta^2 \in B_3$ and the suspension flow $\phi_\beta$ on $M_\beta \simeq M_\delta$. We have simple closed curves $c_{(\beta,A)} \subset \mathcal{T}(\beta,A)$ and $c_{(\beta,3)} \subset \mathcal{T}(\beta,3)$. Since $\beta$ is the product of $b$ and $\Delta^2$, we get $[c_{(\beta,3)}] = (1,0) + (0,1) = (1,1)$. The first term $(1,0)$ comes from $[c_A]$ and the second one $(0,1)$ comes from $\Delta^2$. Similarly we have $[c_{(\beta,3)}] = (2,1) + (1,0) = (3,1)$. By (8.1) we have $F_\beta = E_{(\xi,3)}$ and $E_{(\beta,3)} = F_\xi$. We also have $\mathcal{T}(\beta,A) = \mathcal{T}(\xi,3)$ and $\mathcal{T}(\beta,3) = \mathcal{T}(\xi,A)$. Since

$$p[F_\beta] + [E_{(\beta,3)}] = [F_\xi] + p[E_{(\xi,3)}] = [F_\xi + pE_{(\xi,3)}] = (1,p) \in C_{(\xi,3)},$$

the stable foliation $F_{(1,p)}$ associated with an integral class $(1,p) \in C_{(\xi,3)}$ is the stable foliation associated with $(p,1) \in C_{(\beta,3)}$. By (6.1) for $(x,y) = (p,1)$

$$[\partial_{(\beta,A)}(F_\xi + pE_{(\xi,3)})] = (-1,p), \quad [\partial_{(\beta,3)}(F_\xi + pE_{(\xi,3)})] = (-p,1) \in \mathbb{Z}^2.$$  

From $i([c_{(\beta,3)}], [\partial_{(\beta,3)}(F_\xi + pE_{(\xi,3)})]) = p+1$ and $i([c_{(\beta,3)}], [\partial_{(\beta,3)}(F_\xi + pE_{(\xi,3)})]) = p+3$ together with Lemma 6.1, one sees that $F_{(1,p)}$ associated with $(1,p) \in C_{(\xi,3)}$ is $(p+1)$-pronged at $\partial_{(\beta,3)}F_{(1,p)}(= \partial_{(\xi,3)}F_{(1,p)})$, and $(p+3)$-pronged at $\partial_{(\beta,3)}F_{(1,p)}(= \partial_{(\xi,3)}F_{(1,p)})$.

Since $g_p : M_\xi \to M_\xi$ sends $F_{(1,p)}$ to $F_{\xi_p}$ the stable foliation $F_{(1,p)}$ associated with $(1,p) \in C_{(\xi,3)}$ is identified with $F_{\xi_p}$ via $g_p$. The boundary components $\partial_{(\xi,3)}F_{(1,p)}$ and $\partial_{(\beta,3)}F_{(1,p)}$ have $\partial_{(\beta,A)}(F_\xi + pE_{(\xi,3)})$.

\[\text{There is a solution for the conjugacy problem on } B_n \text{ [6]. The software Braiding [12] can be used to determine whether two braids are conjugate.}\]
and \( \partial(\xi,3) F_{(1,p)} \) correspond to \( \partial(\xi_p,4) F_{\xi_p} \) and \( \partial(\xi_p,3+2p) F_{\xi_p} \) respectively via \( g_p \). Thus \( F_{\xi_p} \) is \((p+1)\)-pronged at \( \partial(\xi_p,3+2p) F_{\xi_p} \). This completes the proof of Step 2.

Next we prove \( \log(\delta(PA_{2n+1})) > 1/n \) following the above arguments in Steps 1, 2. Take an initial braid

\[
\eta = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \in B_8.
\]

It is 4-increasing with \( u(\eta, 4) = 2 \). Consider \( \eta_p \in B_{8+2p} \) obtained from \( \eta \) by the disk twist. Then \( \eta_p^\bullet \) is a skew-palindromic braid with odd degree for each \( p \geq 1 \):

\[
\eta_p^\bullet = (\sigma_1 \sigma_2 \cdots \sigma_4 \sigma_3 2 \sigma_1 \sigma_2 \cdots \sigma_4 \sigma_3 \sigma_4 \sigma_5 \sigma_1 \sigma_2 \cdots \sigma_4 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7) \in B_{7+2p}.
\]

For our purpose it suffices to prove that \( \{\eta_p^\bullet\} \) has a small normalized entropy. Following Step 1 we first prove that \( \eta \) is pseudo-Anosov and \( [E_{(\eta,4)}] \) is a fibered class. Consider a pseudo-Anosov braid \( b = \sigma^{-1} \sigma_3^2 \Delta^2 \in B_3 \) which is 3-increasing with \( u(b, 3) = 5 \). For \( \beta = b \Delta^2 \) Lemma 7.1 tells us that \( (\text{br}(\beta), \text{cl}(\beta(3)), A_\beta) \sim (\text{br}(\beta), A_\gamma, \text{cl}(\gamma(3))) \), where \( \gamma = \kappa_0 \kappa_1 \cdots \kappa_q \Delta_3^2 \in B_8 \). One sees that \( \gamma \) is conjugate to \( \eta \) in \( B_8 \). Since the permutation \( \pi_\eta \) has a unique fixed point it follows that \( (\text{br}(\beta), \text{cl}(\beta(3)), A_\beta) \sim (\text{br}(\eta), A_\eta, \text{cl}(\eta(4))) \). This expression says that \( E_{(\eta,4)} = F_{\beta} \) is a fiber of a fibration on the hyperbolic \( M_8 \simeq M_\eta \) over \( S^1 \). Hence \( \eta \) is pseudo-Anosov. We conclude that \( \{\eta_p\} \) has a small normalized entropy.

Following Step 2 one sees that \( F_{\eta_p} \) is \((p+2)\)-pronged at \( \partial(\eta_p,4+2p) F_{\eta_p} \) for \( p \geq 1 \). Thus \( \eta_p^\bullet \) is pseudo-Anosov with the same dilatation as \( \eta_p \). This completes the proof. \( \square \)

### 8.2. Spin mapping class groups.

In this section we prove Theorem E. We first recall a connection between \( \mathcal{H}(\Sigma_g) \) and \( \text{Mod}(\Sigma_{0,2g+2}) \). Let \( t_j \in \text{Mod}(\Sigma_g) \) for \( 1 \leq j \leq 2g + 1 \) be the right-handed Dehn twist about the simple closed curve \( C_j \) as in Figure 18. Birman-Hilden [3] proved that \( \mathcal{H}(\Sigma_g) \) is generated by \( t_1, t_2, \ldots, t_{2g+1} \). In fact they prove that

\[
Q : \mathcal{H}(\Sigma_g) \rightarrow \text{Mod}(\Sigma_{0,2g+2}) \quad t_j \mapsto t_j
\]

sending \( t_j \) to the right-handed half twist \( t_j \) (see Section 2.3) is well-defined and it is a surjective homomorphism whose kernel is generated by the involution \( \iota = [I] \) as in Figure 5. Using the relation between \( \text{Mod}(\Sigma_{0,2g+2}) \) and \( SB_{2g+2} \) we have

\[
\mathcal{H}(\Sigma_g)/\langle \iota \rangle \simeq \text{Mod}(\Sigma_{0,2g+2}) \simeq SB_{2g+2}/\langle \Delta^2 \rangle.
\]
Suppose that $t$. Note that $braid$ relations one verifies that $\text{Mod}$. Thus Lemma 8.1 tells us that $t$. 

**Proof.** It is well-known that $\phi \in \mathcal{H}(\Sigma_g)$ is pseudo-Anosov if and only if $Q(\phi)$ is pseudo-Anosov and in this case $\lambda(\phi) = \lambda(Q(\phi))$ holds. The following lemma is useful to find elements of the odd/even spin mapping class groups.

**Lemma 8.1** (Theorem 6.1 in [18] for (1), Theorem 3.1 in [17] for (2)). Suppose that $g \geq 3$.

1. $t_j \in \text{Mod}_g[q_1]$ for $4 \leq j \leq 2g + 1$.
2. $t_j t_j^{-1} \in \text{Mod}_g[q_0]$ for $1 \leq j \leq 2g + 1$.

By the above result of Birman-Hilden, all mapping classes in Lemma 8.1 are elements of $\mathcal{H}(\Sigma_g)$. Using the braid relations: $t_j t_j = t_j t_j$, if $|i - j| \geq 2$ and $t_j t_j = t_j t_j$ for $1 \leq j \leq 2g$, we have

$$t_j t_j t_j^{-1} = t_j^{-1} t_j t_j^{-1} = t_j^{-1} (t_j t_j^{-1}) t_j^{-1}.$$ 

Thus Lemma 8.1 tells us that $t_j t_j t_j^{-1} \in \text{Mod}_g[q_1]$ for $4 \leq j \leq 2g$ and $t_j t_j t_j^{-1} \in \text{Mod}_g[q_0]$ for $1 \leq j \leq 2g$.

The following spin mapping classes are used in the proof of Theorem E.

**Lemma 8.2.** Let $p \geq 1$ be an integer.

1. $t_2 t_3 t_5 \cdots t_{5+2p}^2 \in \text{Mod}_g[q_1]$ for any $g \geq p + 2$.
2. $(t_3 t_5 t_7 \cdots t_{5+2p})^2 t_3 t_5 t_7 + 2 \in \text{Mod}_g[q_0]$ for any $g \geq p + 2$.

**Proof.** We prove the lemma by the induction on $p$. We first prove (1). When $p = 1$

$$t_2 t_3 t_5 t_6 t_7^2 t_7 = t_2 \cdot t_3 \cdot t_4 t_5 t_6^2 \cdot t_4^2 \cdot t_6 t_7 t_6^2 \cdot t_6 t_5^2 \cdot t_5^2 \cdot t_5^2$$

which is an element of $\text{Mod}_g[q_1]$ for $g \geq 3$ by Lemma 8.1(1).

Assume that $t_2 t_3 t_5 t_7 \cdots t_{5+2p} \in \text{Mod}_g[q_1]$ for $g \geq p + 1 + 2$. By the braid relations one verifies that

$$t_2 t_3 t_5 t_7 \cdots t_{5+2(p-1)} t_{5+2(p-1)} t_{5+2p} = t_2 t_3 t_5 t_7 \cdots t_{5+2(p-1)} t_{5+2(p-1)} t_{5+2(p-1)} t_{5+2p} t_{5+2p}.$$

Thus the assumption together with Lemma 8.1(1) implies that $t_2 t_3 t_5 t_7 \cdots t_{5+2p} \in \text{Mod}_g[q_1]$ for $g \geq p + 2$. 

![Figure 19](image-url)
Let us turn to (2). When $p = 1$

$$(t_2t_3t_4t_5t_6t_7)^2t_7^2 = t_2t_3t_2^{-1} \cdot t_4^2 \cdot t_4t_3t_4^{-1} \cdot t_4^2 \cdot t_6t_7t_6^{-1} \cdot t_6^2 \cdot t_7^2 \cdot t_7^2$$

which is an element of $\text{Mod}_g[q_0]$ for $g \geq 3$.

Assume that $(t_2t_3 \cdots t_{5+2(p-1)})^2t_5^2 \in \text{Mod}_g[q_0]$ for any $g \geq p - 1 + 2$. By the braid relations again, we have

$$(t_2t_3 \cdots t_{5+2(p-1)})^2t_5^2 = (t_2t_3 \cdots t_{5+2(p-1)})^2t_5^2 \cdot t_4^2 \cdot t_4t_3t_4^{-1} \cdot t_4^2 \cdot t_4+2pt_5^2 \cdot t_5+2p-1t_4^2 \cdot t_5^2 \cdot t_5^2.$$  

By the assumption together with Lemma 8.1(2) we have $(t_2t_3 \cdots t_{5+2p})^2t_5^2 \in \text{Mod}_g[q_0]$ for $g \geq p + 2$. This completes the proof.

The shift map $sh : B_n \rightarrow B_{n+1}$ is an injective homomorphism sending $\sigma_j$ to $\sigma_{j+1}$ for $1 \leq j \leq n - 1$. Suppose that $b \in B_n$ is pseudo-Anosov. Then $S(sh(b)) \in SB_{n+1}$ is pseudo-Anosov with the same dilatation as $b$ since $\Gamma(S(sh(b)))$ is conjugate to $f_b = c(\Gamma(b))$ in $\text{Mod}(\Sigma_0,n+1)$. (See Section 2.3 for definitions $\Gamma, \Gamma.$) We finally prove Theorem E.

Proof of Theorem E(1). Consider $o = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_2^2\sigma_4\sigma_5\sigma_3\sigma_5 \in B_6$ which is $4$-increasing with $u(o, 4) = 2$ (Figure 19). The braid $o_p$ is obtained from $o$ by disk twist for each $p \geq 1$. Then

$$o_p^* = \sigma_1\sigma_2\sigma_3\sigma_4\cdots\sigma_{4+2p}^2\sigma_{4+2p} \in B_{5+2p},$$

$$S(sh(o_p^*)) = \sigma_2\sigma_3\sigma_4\cdots\sigma_{5+2p}^2\sigma_{5+2p} \in SB_{6+2p}.$$  

By Lemma 8.2(1) $(t_2t_3(t_4t_5 \cdots t_{5+2p}))^2t_5+2p \in \text{Mod}_{p+2}[q_1]$ for $p \geq 1$, and it is pseudo-Anosov if $S(sh(o_p^*))$ is pseudo-Anosov. In this case they have the same dilatation. Thus by the relation between $o_p^*$ and $S(sh(o_p^*))$ it is enough to prove that $\{o_p^*\}$ has a small normalized entropy. We first claim that $\{o_p\}$ has a small normalized entropy.

By Theorem 5.2(1) it suffices to prove that $o$ is a pseudo-Anosov and $[E_{\langle o, 4 \rangle}]$ is a fibered class. Consider a 3-braid $b = \sigma_1^2\sigma_2^2 \cdot \sigma_2^3$ which is 3-increasing with $u(b, 3) = 3$. Let $\beta$ denote $\beta B^2$. By Lemma 7.1 $(\text{br}(\beta), \text{cl}(\beta(3)), A_3) \sim (\text{br}(\gamma), A_3, \text{cl}(\gamma(3)))$, where $\gamma \in B_6$ is the braid in (7.2) substituting $\sigma_1^2, \emptyset, \emptyset$ for $w_1, w_2, w_3$ respectively. In this case $\gamma$ is conjugate to $o$ in $B_6$. Since the permutation $\pi_o$ has a unique fixed point 4, it follows that $(\text{br}(\beta), \text{cl}(\beta(3)), A_3) \sim (\text{br}(\alpha), A_3, \text{cl}(\alpha(4)))$. This tells us that $M_\beta \simeq M_\alpha$ and $[E_{\langle o, 4 \rangle}] = [F_3]$ is a fibered class. On the other hand $\beta$ is conjugate to $\sigma_1^3\sigma_2^{-2}\Delta^4$ in $B_3$ which means that $\beta$ is pseudo-Anosov. Thus $M_\beta \simeq M_\alpha$ is hyperbolic and $o$ is pseudo-Anosov.

Next we prove that $o_p^*$ is pseudo-Anosov with the same dilatation as $o_p$ for $p \geq 1$. By the same argument as in the proof of Theorem D one sees that $F_{o_p}$ is $(p + 2)$-pronged at $\partial_{(o_p, 4+2p)}F_{o_p}$. Thus $o_p^*$ has the desired property for $p \geq 1$. We finish the proof of (1).

We turn to (2). Let us consider $v = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5^2\sigma_1\sigma_2\sigma_3^3 \in B_6$ which is 3-increasing with $u(v, 3) = 2$. Let $v_p \in B_{6+2p}$ be the braid obtained from $v$ by the disk twist. Then $v_p$ is $(3 + 2p)$-increasing and

$$v_p^* = (\sigma_1\sigma_2\cdots\sigma_{4+2p})^2\sigma_{4+2p} \in B_{5+2p},$$

$$S(sh(v_p^*)) = (\sigma_2\sigma_3\cdots\sigma_{5+2p})^2\sigma_{5+2p} \in SB_{6+2p}.$$
By Lemma 8.2(2) it is enough to prove that \( \{v_p^*\} \) has a small normalized entropy. To do this we first prove that \( \{v_p\} \) has a small normalized entropy. Consider a pseudo-Anosov 3-braid

\[
b = \sigma_1^2 \sigma_2^{-2} \Delta^4 = \sigma_1^3 \sigma_2^2 \sigma_1 \Delta^2 = \sigma_1^3 \sigma_2^2 \cdot \sigma_1 \sigma_2^2 \]

which is 3-increasing with \( u(b, 3) = 3 \). Lemma 7.1 tells us that for \( \beta = b \Delta^2 \) we have \( (br(\beta), cl(\beta(3))) \sim (br(\gamma), A_\gamma, cl(\gamma(3))) \), where \( \gamma \in B_6 \) is the braid in (7.2) substituting \( \sigma_1^3 \) for \( w_1 \), \( \sigma_2^2 \) for \( w_2 \) and \( \sigma_1 \) for \( w_3 \). One sees that \( \gamma \) is conjugate to \( v \) in \( B_6 \). Thus \( (br(\beta), cl(\beta(3)), A_\beta) \sim (br(v), A_v, cl(v(3))) \). This implies that \( [E_{\{v, 3\}}] = [F_\beta] \) is a fibered class of the hyperbolic \( M_\beta \simeq M_v \), and hence \( v \) is pseudo-Anosov. By Theorem 5.2(1), \( \{v_p\} \) has a small normalized entropy.

One sees that \( F_{v_p} \) is \((p+3)\)-pronged at \( \partial_{(v_p, 3+2p)} F_{v_p} \). Thus \( v_p^* \) is pseudo-Anosov with the same dilatation as \( v_p \) for \( p \geq 1 \). This completes the proof. \( \square \)

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