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The boundary of a fibered face of the magic 3-manifold and the asymptotic behavior of minimal pseudo-Anosov dilatations

Eiko Kin and Mitsuhiko Takasawa

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Abstract. Let $\delta_{g,n}$ be the minimal dilatation of pseudo-Anosovs defined on an orientable surface of genus g with n punctures. It is proved by Tsai that for any fixed $g \geq 2$, there exists a constant c_g depending on g such that

$$\frac{1}{c_g} \cdot \frac{\log n}{n} < \log \delta_{g,n} < c_g \cdot \frac{\log n}{n} \quad \text{for any } n \geq 3.$$

This means that the logarithm of the minimal dilatation $\log \delta_{g,n}$ is on the order of $\log n / n$. We prove that if $2g+1$ is relatively prime to s or $s+1$ for each $0 \leq s \leq g$, then

$$\limsup_{n \to \infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2$$

holds. In particular, if $2g+1$ is prime, then the above inequality on $\delta_{g,n}$ holds. Our examples of pseudo-Anosovs ϕ's which provide the upper bound above have the following property: The mapping torus M_ϕ of ϕ is a single hyperbolic 3-manifold N called the magic manifold, or the fibration of M_ϕ comes from a fibration of N by Dehn filling cusps along the boundary slopes of a fiber.

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus g with n punctures and $\mathrm{Mod}(\Sigma)$ the mapping class group of Σ. By Thurston's classification theorem of surface automorphisms, elements of $\mathrm{Mod}(\Sigma)$ are either periodic, reducible, or pseudo-Anosov, see [20]. Pseudo-Anosov mapping classes have rich dynamical properties. The hyperbolization theorem by Thurston [21] relates the dynamics of pseudo-Anosovs and the geometry of hyperbolic fibered 3-manifolds. The theorem asserts that $\phi \in \mathrm{Mod}(\Sigma)$ is pseudo-Anosov if and only if the mapping torus M_ϕ of ϕ admits a complete hyperbolic metric of finite volume.

Each pseudo-Anosov element $\phi \in \mathrm{Mod}(\Sigma)$ has a representative $\Phi : \Sigma \to \Sigma$ called a pseudo-Anosov homeomorphism. Such a homeomorphism is equipped with a constant $\lambda = \lambda(\Phi) > 1$ called the *dilatation* of Φ. If we let $\mathrm{ent}(\Phi)$ be the *topological entropy* of Φ, then the equality $\mathrm{ent}(\Phi) = \log \lambda(\Phi)$ holds. Moreover

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$\mathrm{ent}(\Phi)$ attains the minimal entropy among all homeomorphisms which are isotopic to Φ, see [3, Exposé 10]. The *dilatation* $\lambda(\phi)$ of ϕ is defined to be $\lambda(\Phi)$. We call the quantities $\mathrm{ent}(\phi) = \log \lambda(\phi)$ and $\mathrm{Ent}(\phi) = |\chi(\Sigma)| \log \lambda(\phi)$ the *entropy* and *normalized entropy* of ϕ respectively, where $\chi(\Sigma)$ is the Euler characteristic of Σ.

If we fix Σ, the set of dilatations of pseudo-Anosovs defined on Σ is a closed discrete subset of $\mathbb{R}$, see [7] for example. In particular there exists a minimum. We denote by $\delta(\Sigma) > 1$, the minimal dilatation of pseudo-Anosov elements in $\mathrm{Mod}(\Sigma)$. The minimal dilatations are determined in only a few cases. (See for example [9] which is a survey on minimal pseudo-Anosov dilatations.)

Let us set $\delta_{g,n} = \delta(\Sigma_{g,n})$ and $\delta_g = \delta_{g,0}$. We write $A \asymp B$ if there exists a universal constant c such that $A/c < B < cA$. Penner proved in [17] that $\log \delta_g \asymp \frac{1}{g}$. This work by Penner was a starting point for the study of the asymptotic behavior of the minimal dilatations on surfaces varying topology. Later it was proved by Hironaka-Kin [6] that $\log \delta_{0,n} \asymp \frac{1}{n}$, and by Tsai [22] that $\log \delta_{1,n} \asymp \frac{1}{n}$. See also Valdivia [23]. The following theorem of Tsai is in contrast with the cases of genera 0 and 1.

THEOREM 1.1 ([22]). *For any fixed $g \geq 2$, there exists a constant c_g depending on g such that*

$$\frac{1}{c_g} \cdot \frac{\log n}{n} < \log \delta_{g,n} < c_g \cdot \frac{\log n}{n} \quad \text{for any } n \geq 3.$$

In particular for any fixed $g \geq 2$, we have

$$\log \delta_{g,n} \asymp \frac{\log n}{n}.$$

The following question is due to Tsai.

QUESTION 1.2. *What is the optimal constant c_g in Theorem 1.1?*

One can also ask the following.

QUESTION 1.3. *Given $g \geq 2$, does $\displaystyle\lim_{n\to\infty} \frac{n(\log \delta_{g,n})}{\log n}$ exist? What is its value?*

This is an analogous question, posed by McMullen, which is asking whether $\displaystyle\lim_{g\to\infty} g \log \delta_g$ exists or not, see [15]. Toward Questions 1.2 and 1.3, we prove the following.

THEOREM 1.4. *Given $g \geq 2$, there exists a sequence $\{n_i\}_{i=0}^{\infty}$ with $n_i \to \infty$ such that*

$$\limsup_{i\to\infty} \frac{n_i \log \delta_{g,n_i}}{\log n_i} \leq 2.$$

Theorem 1.4 improves the previous upper bound on $\log \delta_{g,n}$ by Tsai. In fact for any $g \geq 2$, Tsai's examples in [22] yield the upper bound $\displaystyle\limsup_{n\to\infty} \frac{n(\log \delta_{g,n})}{\log n} \leq$

$2(2g+1)$, which is proved by a similar computation in the proof of Theorem 1.4. As a corollary of Theorem 1.4, we have the following.

COROLLARY 1.5. *Given $g \geq 2$, the following set*

$$\{\tfrac{n}{\log n} \cdot \mathrm{ent}(\phi) \mid \phi \in \mathrm{Mod}(\Sigma_{g,n}) \textit{ is pseudo-Anosov}, \ n \geq 1\}$$

has an accumulation point 2.

To state other results which are related to Questions 1.2 and 1.3, we define a polynomial $B_{(g,p)}(t)$ for nonnegative integers g and p:

$$B_{(g,p)}(t) = t^{2p+1}(t^{2g+1} - 1) + 1 - 2t^{p+g+1} - t^{2g+1}.$$

We shall see that there exists a unique real root $r_{(g,p)}$ greater than 1 of $B_{(g,p)}(t)$, and these satisfy

$$\lim_{p \to \infty} \frac{p \log r_{(g,p)}}{\log p} = 1$$

(Lemma 4.1). The root $r_{(g,p)}$ gives the following upper bound.

THEOREM 1.6. *For $g \geq 2$ and $p \geq 0$, suppose that* $\gcd(2g+1, p+g+1) = 1$. *Then*

$$\delta_{g,2p+i} \leq r_{(g,p)} \quad \textit{for each } i \in \{1,2,3,4\}.$$

If g satisfies $(*)$ in the next Theorem 1.7, then one can take the sequence $\{n_i\}_{i=0}^{\infty}$ in Theorem 1.4 to be the sequence $\{n\}_{n=1}^{\infty}$ of natural numbers.

THEOREM 1.7. *Suppose that $g \geq 2$ satisfies*

$(*)$ $\gcd(2g+1, s) = 1$ *or* $\gcd(2g+1, s+1) = 1$ *for each* $0 \leq s \leq g$.

Then

$$\limsup_{n \to \infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2.$$

For example, $(*)$ holds for $g = 4$ since 9 is relatively prime to $1, 2, 4$ and 5; $(*)$ does not hold for $g = 7$ because $\gcd(15, 5) = 5$ and $\gcd(15, 6) = 3$. We point out that infinitely many g's satisfy $(*)$. In fact if $2g+1$ is prime, then $2g+1$ is relatively prime to s' for each $1 \leq s' \leq g+1$.

COROLLARY 1.8. *If $2g+1$ is prime for $g \geq 2$, then*

$$\limsup_{n \to \infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2.$$

REMARK 1.9. One can simplify $(*)$ in Theorem 1.7, since $2g+1$ is relative prime to $1, 2$ and g. In the case $g \geq 5$, $(*)$ is equivalent to

$(**)$ $\gcd(2g+1, s) = 1$ or $\gcd(2g+1, s+1) = 1$ or each $3 \leq s \leq g-2$.

Our results are proved by using the theory of fibered faces of hyperbolic and fibered 3-manifolds M, developed by Thurston [19], Fried [4], Matsumoto [14] and McMullen [15], see Section 2. We focus on a fibered face of a particular hyperbolic fibered 3-manifold, called the *magic manifold* N. This manifold is the exterior of the 3 chain link $\mathcal{C}_3$, see Figure 1. Our examples of pseudo-Anosovs ϕ's which provide the upper bounds in Theorems 1.4, 1.6 and 1.7 have the following property: The mapping torus M_ϕ of ϕ is homeomorphic to N, or the fibration of M_ϕ comes from a fibration of N by Dehn filling cusps along the boundary slopes of a fiber. An explicit construction of these examples is given by the first author, see [8, Example 4.8].

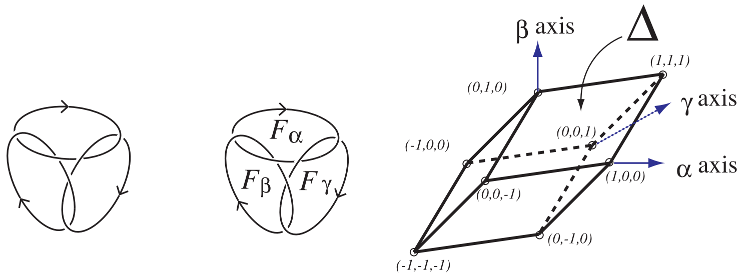

Figure 1. (left) 3 chain link $\mathcal{C}_3$. (center) F_α, F_β, F_γ. (right) Thurston norm ball U_N. (fibered face Δ is indicated.)

We turn to hyperbolic volumes of hyperbolic 3-manifolds. The set of volumes of hyperbolic 3-manifolds is a well-ordered closed subset in $\mathbb{R}$ of order type ω^ω, see [18]. In particular if we fix a surface Σ, then there exists a minimum among volumes of hyperbolic Σ-bundles over the circle. The proofs of Theorems 1.4, 1.7 immediately imply the following.

Proposition 1.10. *For each $g \geq 2$, there exists a sequence $\{n_i\}_{i=0}^{\infty}$ with $n_i \to \infty$ such that the minimal volume of Σ_{g,n_i}-bundles over the circle is less than or equal to* $\mathrm{vol}(N) \approx 5.3334$*, the volume of the magic manifold N. In particular, for any $g \geq 2$ satisfying $(*)$ and any $n \geq 3$, the minimal volume of $\Sigma_{g,n}$-bundles over the circle is less than or equal to* $\mathrm{vol}(N)$.

We close the introduction by asking

Question 1.11 **(cf. Theorems 1.4 and 1.7)**. *Does* $\limsup_{n\to\infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2$ *hold for all fixed $g \geq 2$?*

2. The Thurston norm and fibered 3-manifolds

Let M be an oriented hyperbolic 3-manifold with boundary ∂M (possibly $\partial M = \emptyset$). We recall the Thurston norm $\| \cdot \| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$. Let F be a finite union of oriented, connected surfaces. We define $\chi_-(F)$ to be

$$\chi_-(F) = \sum_{F_i \subset F} \max\{0, -\chi(F_i)\},$$

where F_i's are the connected components of F and $\chi(F_i)$ is the Euler characteristic of F_i. The Thurston norm $\| \cdot \|$ is defined for an integral class $a \in H_2(M, \partial M; \mathbb{Z})$ by

$$\|a\| = \min_F \{\chi_-(F) \mid a = [F]\},$$

where the minimum ranges over all oriented surfaces F embedded in M. A surface F which realizes this minimum is called a *minimal representative* of a, denoted by F_a. Then $\| \cdot \|$ defined on integral classes admits a unique continuous extension $\| \cdot \| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ which is linear on the ray through the origin. The unit ball U_M with respect to the Thurston norm is a compact, convex polyhedron. See [19] for more details.

Suppose that M is a surface bundle over the circle and let F be its fiber. The fibration determines a cohomology class $a^* \in H^1(M; \mathbb{Z})$, and hence a homology class $a \in H_2(M, \partial M; \mathbb{Z})$ by Poincaré duality. Thurston proved in [19] that there exists a top dimensional face Ω on ∂U_M such that $a = [F]$ is an integral class of $int(C_\Omega)$, where C_Ω is the cone over Ω with the origin and $int(C_\Omega)$ is its interior. Moreover the minimal representative F_a for any integral class a in $int(C_\Omega)$ becomes a fiber of the fibration associated to a. Such a face Ω is called a *fibered face*, and an integral class $a \in int(C_\Omega)$ is called a *fibered class*. This work of Thurston tells us that if M has second Betti number greater than 1, then M provides infinitely many pseudo-Anosov monodromies defined on surfaces with variable topology.

The set of integral and rational classes of $int(C_\Omega)$ are denoted by $int(C_\Omega(\mathbb{Z}))$ and $int(C_\Omega(\mathbb{Q}))$ respectively. When $a \in int(C_\Omega(\mathbb{Z}))$ is primitive, the associated fibration on M has a connected fiber represented by F_a. Let $\Phi_a : F_a \to F_a$ be the monodromy. Since M is hyperbolic, $\phi_a = [\Phi_a]$ is pseudo-Anosov. The *dilatation* $\lambda(a)$ and *entropy* $\mathrm{ent}(a) = \log \lambda(a)$ are defined as the dilatation and entropy of ϕ_a respectively. The entropy defined on primitive fibered classes can be extended to rational classes by homogeneity. It is shown by Fried in [4] that $\frac{1}{\mathrm{ent}} : int(C_\Omega(\mathbb{Q})) \to \mathbb{R}$ is concave, and in particular $\mathrm{ent} : int(C_\Omega(\mathbb{Q})) \to \mathbb{R}$ admits a unique continuous extension

$$\mathrm{ent} : int(C_\Omega) \to \mathbb{R}.$$

Moreover Fried proved that the restriction

$$\mathrm{ent}|_{int(\Omega)} (= \mathrm{Ent}|_{int(\Omega)}) : int(\Omega) \to \mathbb{R}$$

on the open face $int(\Omega)$ has the property that $\mathrm{ent}(a)$ goes to ∞ as $a \in int(\Omega)$ goes to a point on $\partial\Omega$. Thus we have a continuous function

$$\mathrm{Ent} = \| \cdot \| \, \mathrm{ent}(\cdot) : int(C_\Omega) \to \mathbb{R}$$

which is constant on each ray in $int(C_\Omega)$ through the origin.

These properties give us the following observation: Fix a hyperbolic fibered 3-manifold M with a fibered face Ω as above. For any compact set $\mathcal{D} \subset int(\Omega)$, there exists a constant $C = C_{\mathcal{D}} > 0$ satisfying the following. Let $a \in int(C_\Omega)$ be any integral class of $H_2(M, \partial M; \mathbb{Z})$. The normalized entropy $\mathrm{Ent}(a)(= \mathrm{Ent}(\phi_a))$ is bounded by C from above whenever $\overline{a} \in \mathcal{D}$, where $\overline{a}$ is the projection of a into $int(\Omega)$.

This observation enables us to investigate the following asymptotic behaviors of minimal dilatations.

(1) $\limsup_{n\to\infty} n \log \delta_{0,n} \leq 2\log(2+\sqrt{3})$, see [6, 11].
(2) $\limsup_{n\to\infty} n \log \delta_{1,n} \leq 2\log \lambda_0$, where $\lambda_0 \approx 2.2966$ is the largest real root of $t^4 - 2t^3 - 2t + 1$, see [10].
(3) $g \log \delta_g \leq \log(\frac{3+\sqrt{5}}{2})$, see [2, Appendix] and [5, 1, 12].

We note that for fixed $g \geq 2$, different methods for investigating the asymptotic behavior of $\delta_{g,n}$ varying n are necessary. Theorem 1.1 says that there exists no constant $C > 0$, independent of n so that $|\chi(\Sigma_{g,n})| \log \delta_{g,n} < C$. Thus if, for fixed $g \geq 2$, there exists a sequence of fibered classes $\{a_i\}$ with $a_i \in int(C_\Omega) \cap H_2(M, \partial M; \mathbb{Z})$ such that the fiber of the fibration associated to a_i is a surface of genus g having n_i boundary components with $n_i \to \infty$, then the accumulation points of the sequence of projective classes $\{\overline{a}_i\}$ must lie on the boundary of Ω. To prove Theorems 1.4, 1.6 and 1.7, we pay special attention to the magic manifold N. In Section 4.3, we choose such a sequence of fibered classes $\{a_i\}$ of N carefully. We analyze the asymptotic behavior of $\lambda(a_i)$'s by using a technique given in Section 3.

The Teichmüller polynomial, developed by McMullen[15] is a certain element Θ_Ω (associated to the fibered face Ω) in the group ring $\mathbb{Z}G$, where $G = H_1(M; \mathbb{Z})/\text{torsion}$, i.e, Θ_Ω is a finite sum

$$\Theta_\Omega = \sum_{g\in G} c_g g,$$

where c_g is an integer. For every fibered class $a \in int(C_\Omega)$, the *specialization* of Θ_Ω at the cohomology class $a^* \in H^1(M; \mathbb{Z})$ is defined by

$$\Theta_\Omega^{(a^*)}(t) = \sum_{g\in G} c_g t^{a^*(g)}$$

which is a polynomial with a variable t. It is a result in [15] that for all fibered class $a \in int(C_\Omega)$, the dilatation $\lambda(a)$ is equal to the largest real root of $\Theta_\Omega^{(a^*)}(t)$.

3. Roots of polynomials

This section concerns the asymptotic behavior of roots of families of polynomials. Let

$$g(t) = a_n t^{b_n} + a_{n-1} t^{b_{n-1}} + \cdots + a_1 t^{b_1} + a_0$$

be a polynomial with real coefficients $a_0, a_1, \cdots, a_n$ ($a_1, a_2, \cdots, a_n \neq 0$), where $g(t)$ is arranged in the order of descending powers of t. Let $\mathfrak{D}(g)$ be the number of variations in signs of the coefficients $a_n, a_{n-1}, \cdots, a_0$. For example if $g(t) = +t^4 + t^3 - 2t^2 + t - 1$, then $\mathfrak{D}(g) = 3$; if $h(t) = +t^4 + t^3 - 2t^2 + t + 1$, then $\mathfrak{D}(h) = 2$. Descartes's rule of signs (see [24]) says that the number of positive real roots of $g(t)$ (counted with multiplicities) is equal to either $\mathfrak{D}(g)$ or less than $\mathfrak{D}(g)$ by an even integer.

LEMMA 3.1. *Let $r \geq 0$, $s > 0$ and $u > 0$ be integers. Let*

$$\begin{aligned} P_m(t) &= t^{2m+r}(t^s - 1) + 1 - Q(t)t^m - t^u \\ &= t^{2m+r+s} - t^{2m+r} - Q(t)t^m - t^u + 1 \end{aligned}$$

be a polynomial for each $m \in \mathbb{N}$, where $Q(t)$ is a polynomial whose coefficients are positive integers. ($Q(t)$ could be a positive constant.)

(1) *Suppose that t^{2m+r+s} is the leading term of $P_m(t)$. Then $P_m(t)$ has a unique real root λ_m greater than 1.*

(2) *Given $0 < c_1 < 1$ and $c_2 > 1$, we have*

$$m^{\frac{c_1}{m}} < \lambda_m < m^{\frac{c_2}{m}} \quad \text{for } m \text{ large.}$$

In particular

$$\lim_{m\to\infty} \frac{m \log \lambda_m}{\log m} = 1.$$

(3) *For any real numbers $q \neq 0$ and v, we have*

$$\lim_{m\to\infty} \frac{(qm+v)\log \lambda_m}{\log(qm+v)} = q.$$

PROOF. *(1) Under the assumption on $P_m(t)$, we have $\mathfrak{D}(P_m) = 2$. By Descartes's rule of signs, the number of positive real roots of $P_m(t)$ is either 2 or 0. Since $P_m(0) = 1$ and $P_m(1) = -Q(1) < 0$, the number of positive real roots of $P_m(t)$ is exactly 2. Because $P_m(t)$ goes to ∞ as t does, $P_m(t)$ has a unique real root $\lambda_m > 1$.*

(2) We have

$$P_m(t)t^{-(2m+r)} = t^s - 1 + t^{-(2m+r)} - Q(t)t^{-(m+r)} - t^{-(2m+r-u)}.$$

We define $f_m(t)$ and $g_m(t)$ such that $P_m(t)t^{-(2m+r)} = f_m(t) + g_m(t)$ as follows.

$$\begin{aligned} f_m(t) &= t^s - 1 + t^{-(2m+r)}, \text{ and} \\ g_m(t) &= Q(t)t^{-(m+r)} + t^{-(2m+r-u)}. \end{aligned}$$

We let $t = m^{\frac{c}{m}}$ for $c > 0$. Then

$$\begin{aligned} f_m(m^{\frac{c}{m}}) &= (m^{\frac{c}{m}})^s - 1 + (m^{\frac{c}{m}})^{-(2m+r)} \\ &= \left((e^{\log m})^{\frac{c}{m}}\right)^s - 1 + m^{-c(2+\frac{r}{m})} \\ &= e^{\frac{sc\log m}{m}} - 1 + m^{-c(2+\frac{r}{m})}. \end{aligned}$$

By Maclaurin expansion of $e^{\frac{sc\log m}{m}}$*, we have*

$$e^{\frac{sc\log m}{m}} = 1 + \frac{sc\log m}{m} + R_2,$$

where

$$R_2 = \frac{e^w}{2}(\frac{sc\log m}{m})^2 \text{ for some } 0 < w < \frac{sc\log m}{m}.$$

Since $\frac{sc\log m}{m}$ *goes to* 0 *as* m *goes to* ∞*, we may assume that* $\frac{e^w}{2} < B$ *for some constant* $B > 0$*. Then*

$$\begin{aligned} f_m(m^{\frac{c}{m}}) &= \frac{sc\log m}{m} + R_2 + \frac{m^{1-c(2+\frac{r}{m})}}{m} \\ &< \frac{sc\log m}{m} + B(\frac{sc\log m}{m})^2 + \frac{m^{1-c(2+\frac{r}{m})}}{m} \\ &= \frac{sc\log m}{m} + Bs^2c^2(\frac{\log m}{m})^2 + \frac{m^{1-c(2+\frac{r}{m})}}{m} \\ &< \frac{sc\log m}{m} + Bs^2c^2(\frac{\log m}{m}) + \frac{m^{1-c(2+\frac{r}{m})}}{m} \\ &= \frac{(sc + Bs^2c^2)\log m + m^{1-c(2+\frac{r}{m})}}{m}. \end{aligned}$$

(The last inequality comes from $0 < \frac{\log m}{m} < 1$ *for* m *large.) Thus*

$$f_m(m^{\frac{c}{m}}) < \frac{(sc + Bs^2c^2)\log m + m^{1-c(2+\frac{r}{m})}}{m}. \tag{1}$$

The first equality $f_m(m^{\frac{c}{m}}) = \frac{sc\log m}{m} + R_2 + \frac{m^{1-c(2+\frac{r}{m})}}{m}$ *above together with* $R_2 > 0$ *and* $\frac{m^{1-c(2+\frac{r}{m})}}{m} > 0$ *tells us that*

$$f_m(m^{\frac{c}{m}}) > \frac{sc\log m}{m}. \tag{2}$$

Recall that all coefficients of $Q(t)$ *(appeared in* $P_m(t)$*) are positive integers. If we write* $Q(t) = \sum_{j=0}^{\ell} a_j t^j$*, where* $a_j \geq 0$*, then*

$$\begin{aligned} g_m(m^{\frac{c}{m}}) &= Q(m^{\frac{c}{m}})m^{-c(1+\frac{r}{m})} + m^{-c(2+\frac{r}{m}-\frac{u}{m})} \\ &= \Big(\sum_{j=0}^{\ell} a_j m^{-c(1+\frac{r}{m}-\frac{j}{m})}\Big) + m^{-c(2+\frac{r}{m}-\frac{u}{m})}. \end{aligned}$$

Thus we obtain

$$g_m(m^{\frac{c}{m}}) = \frac{\big(\sum_{j=0}^{\ell} a_j m^{1-c(1+\frac{r}{m}-\frac{j}{m})}\big) + m^{1-c(2+\frac{r}{m}-\frac{u}{m})}}{m}. \tag{3}$$

For the proof of the claim (1), it is enough to prove that for $0 < c_1 < 1$ *and* $c_2 > 1$*, we have* $f_m(m^{\frac{c_1}{m}}) < g_m(m^{\frac{c_1}{m}})$ *and* $f_m(m^{\frac{c_2}{m}}) > g_m(m^{\frac{c_2}{m}})$ *for* m *large.*

First, suppose that $0 < c < \frac{1}{2}$*. Let us consider how the following four terms grow.*

$$\log m,\ m^{1-c(2+\frac{r}{m})},\ m^{1-c(1+\frac{r}{m}-\frac{j}{m})} \text{ and } m^{1-c(2+\frac{r}{m}-\frac{u}{m})}. \tag{4}$$

The first two terms appear in (1), and the last two are coming from (3). All four terms go to ∞ *as* m *does, since the last three terms have the positive powers of* m*. Note that for any* $C > 0$*, we have* $\log m < m^C$ *for* m *large. Keeping this in mind, we observe that among the four terms in (4),* $m^{1-c(1+\frac{r}{m}-\frac{j}{m})}$ *is dominant. This is because*

$$1 - c(1 + \frac{r}{m} - \frac{j}{m}) > 1 - c(2 + \frac{r}{m} - \frac{u}{m}) \geq 1 - c(2 + \frac{r}{m})$$

for m *large. These imply that* $f_m(m^{\frac{c}{m}}) < g_m(m^{\frac{c}{m}})$ *holds for* m *large, since* $m^{1-c(1+\frac{r}{m}-\frac{j}{m})}$ *appears in the numerator of* $g_m(m^{\frac{c}{m}})$*, see (3).*

Next, we suppose that $\frac{1}{2} \leq c < 1$*. We can check that* $m^{1-c(1+\frac{r}{m}-\frac{j}{m})}$ *is still dominant among the four in (4). (The second and fourth terms are bounded as* m *goes to* ∞*.) Therefore we still have* $f_m(m^{\frac{c}{m}}) < g_m(m^{\frac{c}{m}})$ *for* m *large.*

Finally we suppose that $c > 1$*. Clearly, the last three terms in (4) go to* 0 *as* m *goes to* ∞*. Thus the numerator of* $g_m(m^{\frac{c}{m}})$*, see (3), goes to* 0 *as* m *tends to* ∞*. On the other hand,* $f_m(m^{\frac{c}{m}}) > \frac{sc\log m}{m}$ *holds (see (2)), and hence the numerator of*

$$\frac{sc\log m + mR_2 + m^{1-c(2+\frac{r}{m})}}{m} (= f_m(m^{\frac{c}{m}}))$$

goes to ∞ *as* m *does. Thus* $f_m(m^{\frac{c}{m}}) > g_m(m^{\frac{c}{m}})$ *for* m *large. This completes the proof of the first part of the claim (2).*

Taking logarithms on both sides of $m^{\frac{c_1}{m}} < \lambda_m < m^{\frac{c_2}{m}}$ *yields*

$$c_1 < \frac{m\log\lambda_m}{\log m} < c_2 \text{ for } m \text{ large.}$$

Since $0 < c_1 < 1$ *and* $c_2 > 1$ *are arbitrary, we have the desired limit. This completes the proof of the second half of the claim (2).*

(3) By the claim (2),

$$\frac{c_1\log m}{m} < \log\lambda_m < \frac{c_2\log m}{m} \text{ for } m \text{ large.}$$

Let us set $n = qm + v$*. We substitute* $m = \frac{n-v}{q}$ *above:*

$$\frac{c_1\log\left(\frac{n-v}{q}\right)}{\frac{n-v}{q}} < \log\lambda_m < \frac{c_2\log\left(\frac{n-v}{q}\right)}{\frac{n-v}{q}}.$$

Hence

$$\frac{qc_1\big(\log(n-v) - \log q\big)}{n-v} < \log\lambda_m < \frac{qc_2\big(\log(n-v) - \log q\big)}{n-v}.$$

We multiply all sides above by $\frac{n}{\log n} > 0$ *(for* n *large). Then*

$$\frac{qc_1 n\big(\log(n-v) - \log q\big)}{(n-v)\log n} < \frac{n(\log \lambda_m)}{\log n} < \frac{qc_2 n\big(\log(n-v) - \log q\big)}{(n-v)\log n}.$$

Note that $\frac{n\left(\log(n-v)-\log q\right)}{(n-v)\log n}$ *goes to* 1 *as* n *(and hence* m*) goes to* ∞*. Since* $0 < c_1 < 1$ *and* $c_2 > 1$ *are arbitrary, it follows that*

$$\lim_{m\to\infty} \frac{n(\log \lambda_m)}{\log n} = \lim_{m\to\infty} \frac{(qm+v)\log \lambda_m}{\log(qm+v)} = q.$$

□

4. The magic 3-manifold N

Monodromies of fibrations on N have been studied in [10, 11, 12]. In Sections 4.1 and 4.2, we recall some results which tell us that the topology of fibered classes a and the actual value of $\lambda(a)$. In Section 4.3, we find a family of fibered classes $a_{(g,p)}$ of N with two variables g and p, and we shall prove that it is a suitable family to prove theorems in Section 1 (cf. Remark 4.4).

Recall that $\Sigma_{g,n}$ is an orientable surface of genus g with n punctures. Abusing the notation, we sometimes denote by $\Sigma_{g,n}$, an orientable surface of genus g with n boundary components.

4.1. Fibered face Δ. Let K_α, K_β and K_γ be the components of the 3 chain link $\mathcal{C}_3$. They bound the oriented disks F_α, F_β and F_γ with 2 holes, see Figure 1. Let $\alpha = [F_\alpha]$, $\beta = [F_\beta]$, $\gamma = [F_\gamma] \in H_2(N, \partial N; \mathbb{Z})$. The set $\{\alpha, \beta, \gamma\}$ is a basis of $H_2(N, \partial N; \mathbb{Z})$. Figure 1 illustrates the Thurston norm ball U_N for N which is the parallelepiped with vertices $\pm\alpha$, $\pm\beta$, $\pm\gamma$, $\pm(\alpha+\beta+\gamma)$ ([19, Example 3 in Section 2]). Because of the symmetry of $\mathcal{C}_3$, every top dimensional face of U_N is a fibered face.

We denote a class $x\alpha + y\beta + z\gamma \in H_2(N, \partial N; \mathbb{R})$ by (x, y, z). We pick a fibered face Δ with vertices $\alpha = (1,0,0)$, $\alpha+\beta+\gamma = (1,1,1)$, $\beta = (0,1,0)$ and $-\gamma = (0,0,-1)$, see Figure 1. The open face $int(\Delta)$ is written by

$$int(\Delta) = \{(X,Y,Z) \mid X+Y-Z = 1,\ X > 0,\ Y > 0,\ X > Z,\ Y > Z\}.$$

A class $a = (x,y,z) \in H_2(N, \partial N; \mathbb{R})$ is an element of $int(C_\Delta)$ if and only if $x > 0$, $y > 0$, $x > z$ and $y > z$. In this case, we have $\|a\| = x + y - z$.

Let $a = (x,y,z)$ be a fibered class in $int(C_\Delta)$. The minimal representative of this class is denoted by F_a or $F_{(x,y,z)}$. We recall some formula which tells us that the number of the boundary components of F_a. We denote the tori $\partial\mathcal{N}(K_\alpha)$, $\partial\mathcal{N}(K_\beta)$, $\partial\mathcal{N}(K_\gamma)$ by T_α, T_β, T_γ respectively, where $\mathcal{N}(K)$ is a regular neighborhood of a knot K in S^3. Let us set $\partial_\alpha F_{(x,y,z)} = \partial F_{(x,y,z)} \cap T_\alpha$ which consists of the parallel simple closed curves on T_α. We define the subsets $\partial_\beta F_{(x,y,z)}$, $\partial_\gamma F_{(x,y,z)} \subset \partial F_{(x,y,z)}$ in the same manner. By [11, Lemma 3.1], the number of the boundary components

$$\sharp(\partial F_{(x,y,z)}) = \sharp(\partial_\alpha F_{(x,y,z)}) + \sharp(\partial_\beta F_{(x,y,z)}) + \sharp(\partial_\gamma F_{(x,y,z)})$$

is given by

$$\sharp(\partial F_{(x,y,z)}) = \gcd(x, y+z) + \gcd(y, z+x) + \gcd(z, x+y) \tag{5}$$

where $\sharp(\partial_\alpha F_{(x,y,z)}) = \gcd(x, y+z)$, $\sharp(\partial_\beta F_{(x,y,z)}) = \gcd(y, z+x)$, $\sharp(\partial_\gamma F_{(x,y,z)}) = \gcd(z, x+y)$ and $\gcd(0, w)$ is defined by $|w|$.

4.2. Dilatations and stable foliations of fibered classes a's. The Teichmüller polynomial associated to the fibered face Δ is computed in [11, Section 3.2], and it tells us that the dilatation $\lambda_{(x,y,z)}$ of a fibered class $(x,y,z) \in int(C_\Delta)$ is the largest real root of

$$f_{(x,y,z)}(t) = t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1,$$

see [11, Theorem 3.1]. (In fact, $\lambda_{(x,y,z)}$ is a unique real root greater than 1 of $f_{(x,y,z)}(t)$ by Descartes's rule of signs.)

Let $\Phi_{(x,y,z)} : F_{(x,y,z)} \to F_{(x,y,z)}$ be the monodromy of the fibration associated to a primitive class $(x,y,z) \in int(C_\Delta)$. Let $\mathcal{F}_{(x,y,z)}$ be the stable foliation of the pseudo-Anosov $\Phi_{(x,y,z)}$. The components of $\partial_\alpha F_{(x,y,z)}$ (resp. $\partial_\beta F_{(x,y,z)}$, $\partial_\gamma F_{(x,y,z)}$) are permuted cyclically by $\Phi_{(x,y,z)}$. In particular the number of prongs of $\mathcal{F}_{(x,y,z)}$ at a component of $\partial_\alpha F_{(x,y,z)}$ (resp. $\partial_\beta F_{(x,y,z)}$, $\partial_\gamma F_{(x,y,z)}$) is independent of the choice of the component. By [12, Proposition 3.3], the stable foliation $\mathcal{F}_{(x,y,z)}$ has the following properties.

- Each component of $\partial_\alpha F_{(x,y,z)}$ has $x/\gcd(x, y+z)$ prongs.
- Each component of $\partial_\beta F_{(x,y,z)}$ has $y/\gcd(y, x+z)$ prongs.
- Each component of $\partial_\gamma F_{(x,y,z)}$ has $(x+y-2z)/\gcd(z, x+y)$ prongs.
- $\mathcal{F}_{(x,y,z)}$ does not have singularities in the interior of $F_{(x,y,z)}$.

4.3. Proofs of theorems. Let $\mathfrak{a} = (1,1,0)$ and $\mathfrak{b} = (0,1,1)$. For $g \geq 0$ and $p \geq 0$, define a fibered class $a_{(g,p)}$ as follows.

$$a_{(g,p)} = (p+g+1)\mathfrak{a} + (p-g)\mathfrak{b} = (p+g+1, 2p+1, p-g) \in int(C_\Delta).$$

The class $a_{(g,p)}$ is primitive if and only if $2g+1$ and $p+g+1$ are relatively prime. One can check the identity

$$B_{(g,p)}(t) = f_{(p+g+1, 2p+1, p-g)}(t)$$

(see Section 1 for the definition of $B_{(g,p)}(t)$). We denote by $r_{(g,p)}$, the dilatation $\lambda(a_{(g,p)})$ of the fibered class $a_{g,p}$. (Thus the dilatation $r_{(g,p)} = \lambda(a_{(g,p)})$ of $a_{(g,p)}$ is a unique real root of $B_{(g,p)}(t)$ which is greater than 1, see Section 4.2.)

LEMMA 4.1. *We fix $g \geq 0$. Given $0 < c_1 < 1$ and $c_2 > 1$, we have*

$$p^{\frac{c_1}{p}} < r_{(g,p)} < p^{\frac{c_2}{p}} \quad \textit{for } p \textit{ large.}$$

In particular

$$\lim_{p\to\infty} \frac{p \log r_{(g,p)}}{\log p} = 1.$$

PROOF. *Apply Lemma 3.1 to the polynomial $B_{(g,p)}(t)$.* □

Lemma 4.2. *Suppose that $a_{(g,p)}$ is primitive. The minimal representative $F_{a_{(g,p)}}$ is a surface of genus g with $2p+4$ boundary components, and the stable foliation $\mathcal{F}_{a_{(g,p)}}$ has the following properties. If $p+g$ is odd (resp. even), then $\sharp(\partial_\alpha F_{a_{(g,p)}}) = 2$ (resp. 1) and $\sharp(\partial_\gamma F_{a_{(g,p)}}) = 1$ (resp. 2). A component of $\partial_\alpha F_{a_{(g,p)}}$ has $\frac{p+g+1}{2}$ prongs (resp. $(p+g+1)$ prongs), and a component of $\partial_\gamma F_{a_{(g,p)}}$ has $(p+3g+2)$ prongs (resp. $\frac{p+3g+2}{2}$ prongs).*

Proof. *By (5), we have that $\sharp(\partial_\beta F_{a_{(g,p)}}) = 2p+1$. We have*

$$\sharp(\partial_\alpha F_{a_{(g,p)}}) = \gcd(p+g+1, 3p-g+1) = \gcd(p+g+1, 2(2g+1)).$$

Since $a_{(g,p)}$ is primitive, $p+g+1$ and $2g+1$ must be relatively prime. Hence $\sharp(\partial_\alpha F_{a_{(g,p)}}) = 1$ (resp. 2) if $p+g$ is even (resp. odd). Let us compute $\sharp(\partial_\gamma F_{a_{(g,p)}})$. We have

$$\sharp(\partial_\gamma F_{a_{(g,p)}}) = \gcd(3p+g+2, p-g) = \gcd(2(2g+1), p-g).$$

Since $\gcd(2g+1, p-g) = \gcd(2g+1, p+g+1) = 1$, we have that $\sharp(\partial_\gamma F_{a_{(g,p)}}) = 2$ (resp. 1) if $p-g$ is even (resp. odd), equivalently $p+g$ is even (resp. odd). The genus of $F_{a_{(g,p)}}$ is computed from the identities $\|a_{(g,p)}\|(= |\chi(F_{a_{(g,p)}})|) = 2p+2g+2$ and $\sharp(\partial F_{a_{(g,p)}}) = 2p+4$.

The singularity data of $\mathcal{F}_{a_{(g,p)}}$ is obtained from the formula at the end of Section 4.2. □

By Lemma 4.2, it is straightforward to prove the following.

Lemma 4.3. *Suppose that $a_{(g,p)}$ is primitive. Then $(g,p) \notin \{(0,0),(0,1),(1,0)\}$ if and only if $\mathcal{F}_{a_{(g,p)}}$ does not have a 1 prong on each component of $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$. In particular if $g \geq 2$ and $p \geq 0$, then $\mathcal{F}_{a_{(g,p)}}$ does not have a 1 prong on each component of $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$.*

We are now ready to prove theorems in Section 1.

Proof of Theorem 1.4. There exists a sequence of primitive fibered classes $\{a_{(g,p_i)}\}_{i=0}^\infty$ with $p_i \to \infty$. (In fact, if we take $p_i = (g+1)+(2g+1)i$, then $2g+1$ and p_i+g+1 are relatively prime. Hence $a_{(g,p_i)}$ is primitive.) Then N is a $\Sigma_{g,2p_i+4}$-bundle over the circle whose monodromy of the fibration has the dilatation $r_{(g,p_i)}$. Therefore $\delta_{g,2p_i+4} \leq r_{(g,p_i)}$. If we set $n_i = 2p_i+4$, then

$$\frac{n_i \log \delta_{g,n_i}}{\log n_i} \leq \frac{n_i \log r_{(g,p_i)}}{\log n_i} = \frac{(2p_i+4) r_{(g,p_i)}}{\log(2p_i+4)}.$$

The right hand side goes to 2 as i goes to ∞, see Lemmas 3.1(3) and 4.1. This completes the proof. □

Proof of Theorem 1.6. The monodromy $\Phi_{a_{(g,p)}}$ of the fibration associated to the primitive fibered class $a_{(g,p)}$ is defined on the surface of genus g with $2p+4$ boundary components. It has the dilatation $r_{(g,p)}$, and hence $\delta_{g,2p+4} \leq r_{(g,p)}$.

Now let us prove $\delta_{g,2p+1} \leq r_{(g,p)}$. The fibration associated to $a_{(g,p)}$ extends naturally to a fibration on the manifold obtained from N by Dehn filling two cusps specified by the tori T_α and T_γ along the boundary slopes of the fiber. Then $\Phi_{a_{(g,p)}} : F_{a_{(g,p)}} \to F_{a_{(g,p)}}$ extends to the monodromy $\widehat{\Phi} : \widehat{F} \to \widehat{F}$ of the extended fibration, where the extended fiber $\widehat{F}$ is obtained from $F_{a_{(g,p)}}$ by filling each disk bounded by each component of $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$. Thus $\widehat{F}$ has the genus g with $2p+1$ boundary components, see Lemma 4.2. By Lemma 4.3, $\mathcal{F}_{a_{(g,p)}}$ does not have 1 prong at each component of $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$. Hence $\mathcal{F}_{a_{(g,p)}}$ extends canonically to the stable foliation $\widehat{\mathcal{F}}$ of $\widehat{\Phi}$. Therefore $\widehat{\phi} = [\widehat{\Phi}]$ is pseudo-Anosov with the same dilatation as $\Phi_{a_{(g,p)}}$. This implies that $\delta_{g,2p+1} \le r_{(g,p)}$.

The proofs of the rest of the bounds $\delta_{g,2p+2} \le r_{(g,p)}$ and $\delta_{g,2p+3} \le r_{(g,p)}$ are similar. In fact, the extended fiber of the fibration on the manifold obtained from N by Dehn filling a cusp specified by T_α or T_γ along the boundary slope of the fiber has the genus g with $2p+2$ or $2p+3$ boundary components, see Lemma 4.2. Lemma 4.3 ensures that the extended monodromy is pseudo-Anosov with the same dilatation as $\Phi_{a_{(g,p)}}$. □

Proof of Theorem 1.7. By Theorem 1.6 together with the assumption $(*)$ in Theorem 1.7, we have that for any $p \ge 0$ and for $j \in \{3,4\}$,

$$\delta_{g,2p+j} \le r_{(g,p)} \quad \text{or} \quad \delta_{g,2p+j} \le r_{(g,p+1)}.$$

Thus

$$\begin{aligned} \frac{(2p+j)\log \delta_{g,2p+j}}{\log(2p+j)} &\le \frac{(2p+j)\log r_{(g,p)}}{\log(2p+j)} \quad \text{or} \\ \frac{(2p+j)\log \delta_{g,2p+j}}{\log(2p+j)} &\le \frac{(2p+j)\log r_{(g,p+1)}}{\log(2p+j)}. \end{aligned}$$

By Lemma 3.1, it is easy to see that the both right hand sides in the above two inequalities go to 2 as p goes to ∞. Thus

$$\limsup_{p\to\infty} \frac{(2p+j)\log \delta_{g,2p+j}}{\log(2p+j)} \le 2.$$

Since this holds for $j \in \{3,4\}$, the proof is done. □

Proof of Proposition 1.10. We prove the claim in the second half. (The proof in the first half is similar.) If $g \ge 2$ satisfies $(*)$, then for any $p \ge 0$ there exist a $\Sigma_{g,2p+3}$-bundle and a $\Sigma_{g,2p+4}$-bundle over the circle obtained from N, see proof of Theorem 1.7. More precisely such a bundle is homeomorphic to N or it is obtained from N by Dehn filling cusps along the boundary slopes of the fiber. Thus Proposition 1.10 holds from the result which says that the hyperbolic volume decreases after Dehn filling, see [16, 18]. □

Remark 4.4. To address Question 1.3, we explored fibered classes of the magic manifold whose dilatations have a suitable asymptotic behavior. We found a family of primitive fibered classes $a_{(g,p)}$ by computer. By Lemma 4.2, most of the components of $\partial F_{a_{(g,p)}}$ lie on the torus T_β. The pseudo-Anosov stable foliation associated to $a_{(g,p)}$ has the property that each component of $\partial_\beta F_{a_{(g,p)}}$

has 1 prong. The striking property of $a_{(g,p)}$ is that the slope of the components of $\partial_\beta F_{a_{(g,p)}}$ is exactly equal to -1. Moreover, for any fixed g, the projective class $\overline{a}_{(g,p)}$ goes to a single point $(\frac{1}{2}, 1, \frac{1}{2}) \in \partial\Delta$ as p goes to ∞. It is proved by Martelli and Petronio[13] that the manifold $N(-1)$ obtained from N by Dehn filling a cusp along the boundary slope -1 is not hyperbolic. The property that each component of $\partial_\beta F_{a_{(g,p)}}$ has 1 prong can also be seen from the fact hat $N(-1)$ is a non hyperbolic manifold.

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Eiko Kin
Department of Mathematics, Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043, JAPAN
E-mail: kin@math.sci.osaka-u.ac.jp
URL: http://www.math.sci.osaka-u.ac.jp/∼kin/

Mitsuhiko Takasawa
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Ohokayama, Meguro, Tokyo 152-8552 Japan
E-mail: takasawa@is.titech.ac.jp
URL: http://www.is.titech.ac.jp/∼takasawa/