THE ASYMPTOTIC BEHAVIOR OF THE MINIMAL PSEUDO-ANOSOV DILATATIONS IN THE HYPERELLIPTIC HANDLEBODY GROUPS

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Dedicated to Professors Taizo Kanenobu, Yasutaka Nakanishi and Makoto Sakuma for their sixtieth birthdays

Abstract. We consider the hyperelliptic handlebody group on a closed surface of genus g. This is the subgroup of the mapping class group on a closed surface of genus g consisting of isotopy classes of homeomorphisms on the surface that commute with some fixed hyperelliptic involution and that extend to homeomorphisms on the handlebody. We prove that the logarithm of the minimal dilatation (i.e, the minimal entropy) of all pseudo-Anosov elements in the hyperelliptic handlebody group of genus g is comparable to $1/g$. This means that the asymptotic behavior of the minimal pseudo-Anosov dilatation of the subgroup of genus g in question is the same as that of the ambient mapping class group of genus g. We also determine finite presentations of the hyperelliptic handlebody groups.

1. Introduction

Let Σ_g be a closed, orientable surface of genus g, and let $\mathrm{Mod}(\Sigma_g)$ be the mapping class group on Σ_g. The *hyperelliptic mapping class group* $\mathcal{H}(\Sigma_g)$ is the subgroup of $\mathrm{Mod}(\Sigma_g)$ consisting of isotopy classes of orientation preserving homeomorphisms on Σ_g that commute with some fixed hyperelliptic involution $\mathcal{S} : \Sigma_g \to \Sigma_g$. If $g \geq 3$, then $\mathcal{H}(\Sigma_g)$ is of infinite index in $\mathrm{Mod}(\Sigma_g)$, and it is a particular subgroup in some sense. Despite such a property, $\mathcal{H}(\Sigma_g)$ plays a significant role to study the mapping class group $\mathrm{Mod}(\Sigma_g)$. Especially, elements of $\mathcal{H}(\Sigma_g)$ have a handy description via the spherical braid group SB_{2g+2} with $2g+2$ strings, which is proved by Birman-Hilden:

$$\mathcal{H}(\Sigma_g)/\langle \iota \rangle \simeq SB_{2g+2}/\langle \Delta^2 \rangle,$$

where $\iota = [\mathcal{S}] \in \mathcal{H}(\Sigma_g)$ is the mapping class of $\mathcal{S}$, and $\Delta \in SB_{2g+2}$ is a half twist braid. Here $\langle \iota \rangle$ and $\langle \Delta^2 \rangle$ are the subgroups generated by ι and Δ^2 respectively. There exists a natural surjective homomorphism from SB_{2g+2} to the mapping class group $\mathrm{Mod}(\Sigma_{0,2g+2})$ on a sphere with $2g+2$ punctures:

$$\Gamma : SB_{2g+2} \to \mathrm{Mod}(\Sigma_{0,2g+2})$$

with the kernel generated by Δ^2.

Let G be a subgroup of $\mathrm{Mod}(\Sigma_g)$. Whenever $G \cap \mathcal{H}(\Sigma_g)$ contains a non-trivial element, it is worthwhile to consider the subgroup $G \cap \mathcal{H}(\Sigma_g)$ of $\mathrm{Mod}(\Sigma_g)$. The group $G \cap \mathcal{H}(\Sigma_g)$ would be an intriguing one in its own right. Also we may have a chance to find new examples or phenomena on G by using a handy braid description of $G \cap \mathcal{H}(\Sigma_g)$. In the case G is the Torelli group $\mathcal{I}(\Sigma_g)$

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consisting of elements of $\mathrm{Mod}(\Sigma_g)$ which act trivially on $H_1(\Sigma_g;\mathbb{Z})$, the hyperelliptic Torelli group $\mathcal{I}(\Sigma_g)\cap\mathcal{H}(\Sigma_g)$ is studied by Brendle-Margalit, see [6] and references therein. In this paper, we consider the *handlebody group* $\mathrm{Mod}(\mathbb{H}_g)$ as G. This is the subgroup of $\mathrm{Mod}(\Sigma_g)$ consisting of isotopy classes of orientation preserving homeomorphisms on Σ_g that extend to homeomorphisms on the handlebody $\mathbb{H}_g$ of genus g. The main subgroup of $\mathrm{Mod}(\Sigma_g)$ in this paper is the *hyperelliptic handlebody group*

$$\mathcal{H}(\mathbb{H}_g)=\mathrm{Mod}(\mathbb{H}_g)\cap\mathcal{H}(\Sigma_g).$$

We prove a version of Birman-Hilden's theorem about $\mathcal{H}(\mathbb{H}_g)$, and identify the subgroup of SB_{2g+2} corresponding to $\mathcal{H}(\mathbb{H}_g)$. More precisely, we prove in Theorem 2.11 that

$$\mathcal{H}(\mathbb{H}_g)/\langle\iota\rangle\simeq SW_{2g+2}/\langle\Delta^2\rangle,$$

where SW_{2g+2} is so called the *wicket group.* (See Section 2.5.1.) Hilden introduced a subgroup SH_{2g+2} of $\mathrm{Mod}(\Sigma_{0,2g+2})$ in [15], which is now called the *(spherical) Hilden group.* The group SH_{2g+2} is isomorphic to the image $\Gamma(SW_{2g+2})$ under $\Gamma: SB_{2g+2}\to\mathrm{Mod}(\Sigma_{0,2g+2})$ (Theorem 2.6). As an application of the above relation between $\mathcal{H}(\mathbb{H}_g)$ and SW_{2g+2}, we determine a finite presentation of $\mathcal{H}(\mathbb{H}_g)$ in Appendix A, see Theorem A.8

We are interested in the asymptotic behavior of the minimal dilatations of all pseudo-Anosov elements in $\mathcal{H}(\mathbb{H}_g)$ varying g. To state our results, we need some setup. Let Σ be an orientable, connected surface possibly with punctures. A homeomorphism $\Phi:\Sigma\to\Sigma$ is *pseudo-Anosov* if there exist a pair of transverse measured foliations $(\mathcal{F}^u,\mu^u)$ and $(\mathcal{F}^s,\mu^s)$ and a constant $\lambda=\lambda(\Phi)>1$ such that

$$\Phi(\mathcal{F}^u,\mu^u)=(\mathcal{F}^u,\lambda\mu^u)\quad\text{and}\quad\Phi(\mathcal{F}^s,\mu^s)=(\mathcal{F}^s,\lambda^{-1}\mu^s).$$

Then $\mathcal{F}^u$ and $\mathcal{F}^s$ are called the *unstable* and *stable foliations*, and λ is called the *dilatation* or *stretch factor* of Φ. The topological entropy $\mathrm{ent}(\Phi)$ is precisely equal to $\log\lambda(\Phi)$. A significant property of pseudo-Anosov homeomorphisms is that $\mathrm{ent}(\Phi)$ attains the minimal entropy among all homeomorphisms on Σ which are isotopic to Φ, see [11, Exposé 10]. An element ϕ of the mapping class group $\mathrm{Mod}(\Sigma)$ of Σ is called *pseudo-Anosov* if ϕ contains a pseudo-Anosov homeomorphism $\Phi:\Sigma\to\Sigma$ as a representative. In this case, we let $\lambda(\phi)=\lambda(\Phi)$ and $\mathrm{ent}(\phi)=\mathrm{ent}(\Phi)$, and we call them the *dilatation* and *entropy* of ϕ respectively. We call

$$\mathrm{Ent}(\phi)=|\chi(\Sigma)|\mathrm{ent}(\phi)$$

the *normalized entropy* of ϕ, where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Let $f:\Sigma\to\Sigma$ be a representative of a given mapping class $\phi\in\mathrm{Mod}(\Sigma)$. The mapping torus $\mathbb{T}_\phi=\mathbb{T}_{[f]}$ is defined by

$$\mathbb{T}_\phi=\Sigma\times\mathbb{R}/\sim,$$

where $\sim$ identifies $(x,t+1)$ with $(f(x),t)$ for $x\in\Sigma$ and $t\in\mathbb{R}$. We call ϕ the *monodromy* of $\mathbb{T}_\phi$. We sometimes call the representative $f\in\phi$ the monodromy of $\mathbb{T}_\phi$. The *suspension flow* f^t on $\mathbb{T}_\phi$ is a flow induced by the vector field $\frac{\partial}{\partial t}$. The hyperbolization theorem by Thurston [33] states that when a 3-manifold M is a surface bundle over the circle, that is $M\simeq\mathbb{T}_\phi$ for some mapping class ϕ, M admits a hyperbolic structure if and only if ϕ is pseudo-Anosov.

We fix a surface Σ, and consider the set of dilatations of all pseudo-Anosov elements on Σ:

$$\mathrm{dil}(\Sigma)=\{\lambda(\phi)\mid\phi\in\mathrm{Mod}(\Sigma)\text{ is pseudo-Anosov}\}.$$

This is a closed, discrete subset of $\mathbb{R}$, see [20] for example. In particular, given a subgroup G of $\mathrm{Mod}(\Sigma)$ which contains pseudo-Anosov elements, there exists a minimum $\delta(G)>1$ among dilatations of all pseudo-Anosov elements in G. Clearly we have $\delta(G)\geq\delta(\mathrm{Mod}(\Sigma))$. Let $\Sigma_{g,n}$ be a closed, orientable surface of genus g removed n punctures. We denote by δ_g and $\delta_{g,n}$, the minimal dilatations $\delta(\mathrm{Mod}(\Sigma_g))$ and $\delta(\mathrm{Mod}(\Sigma_{g,n}))$ respectively.

By pioneering work of Penner [27], the asymptotic equality

$$\log \delta_g \asymp \frac{1}{g}$$

holds. Here $A \asymp B$ means that there exists a universal constant $c > 0$ so that $\frac{A}{c} < B < cA$. In this case, we say that A is comparable to B. Penner proves this claim by using his lower bound $\log \delta_{g,n} \geq \frac{\log 2}{12g+4n-12}$ ([27]). After work of Penner, one can ask the following.

Question 1.1. *Which sequence of subgroups $G_{(g)}$'s of $\mathrm{Mod}(\Sigma_g)$ satisfies $\log \delta(G_{(g)}) \asymp \frac{1}{g}$?*

Hironaka also studied Question 1.1 in [16]. To prove $\log \delta(G_{(g)}) \asymp \frac{1}{g}$, thanks to the Penner's lower bound $\log \delta_g \geq \frac{\log 2}{12g-12}$, it suffices to construct a sequence of pseudo-Anosov elements $\phi_{(g)} \in G_{(g)}$ for $g \geq 2$ whose normalized entropies $\mathrm{Ent}(\phi_{(g)}) = (2g-2)\mathrm{ent}(\phi_{(g)})$ are uniformly bounded from above.

It is a result by Farb-Leininger-Margalit that the dilatation of any pseudo-Anosov element in the Torelli group $\mathcal{I}(\Sigma_g)$ has a uniform lower bound ([9, Theorem 1.1]). See also Agol-Leininger-Margalit [1]. On the other hand, the two subgroups $\mathcal{H}(\Sigma_g)$ and $\mathrm{Mod}(\mathbb{H}_g)$ are examples of answers to Question 1.1. In fact, Hironaka-Kin prove in [18, Theorem 1.1],

$$g \log \delta(\mathcal{H}(\Sigma_g)) < \log(2+\sqrt{3}) \approx 1.3169 \text{ for } g \geq 2.$$

Hironaka proves in [16, Section 3.1],

$$\limsup_{g\to\infty} g \log \delta(\mathrm{Mod}(\mathbb{H}_g)) \leq \log(33+8\sqrt{17}) \approx 4.1894. \tag{1.1}$$

The main result of this paper is to prove that $\log \delta(\mathcal{H}(\mathbb{H}_g))$ is still comparable to $1/g$.

Theorem 1.2. *We have $\log \delta(\mathcal{H}(\mathbb{H}_g)) \asymp \frac{1}{g}$ and $\log \delta(SH_{2n}) \asymp \frac{1}{n}$.*

Proposition 1.3. *There exists a sequence of pseudo-Anosov braids $w_{2n} \in SW_{2n}$ $(n \geq 3)$ such that*

$$\lim_{n\to\infty} n \log(\lambda(w_{2n})) = 2 \log \kappa,$$

where $\kappa = \frac{1+\sqrt{5}}{2} + \frac{\sqrt{2+2\sqrt{5}}}{2} \approx 2.89005$ equals the largest root of

$$t^4 - 2t^3 - 2t^2 - 2t + 1 = (t^2 - (1+\sqrt{5})t + 1)(t^2 - (1-\sqrt{5})t + 1).$$

The braids w_{2n}'s are written by the standard generators of the spherical braid groups concretely (Section 3). Theorem 1.2 follows from Proposition 1.3 as we explain now. We say that a braid $b \in SB_{2g+2}$ is pseudo-Anosov if $\Gamma(b) \in \mathrm{Mod}(\Sigma_{0,2g+2})$ is a pseudo-Anosov mapping class. In this case, the dilatation $\lambda(b)$ is defined by the dilatation $\lambda(\Gamma(b))$ of the pseudo-Anosov element $\Gamma(b)$. On the other hand, there exists a surjective homomorphism $Q : \mathcal{H}(\mathbb{H}_g) \to SH_{2g+2}$ with the kernel $\langle \iota \rangle$ (Theorem 2.11). If $\phi \in \mathcal{H}(\mathbb{H}_g)$ is pseudo-Anosov, then $Q(\phi) \in SH_{2g+2}$ is also pseudo-Anosov. If $\Phi : \Sigma_{0,2g+2} \to \Sigma_{0,2g+2}$ is a pseudo-Anosov homeomorphism which represents $Q(\phi)$, then one can take a pseudo-Anosov homeomoprhism $\tilde{\Phi} : \Sigma_g \to \Sigma_g$ which is a lift of Φ such that $\phi = [\tilde{\Phi}]$. Two pseudo-Anosov homeomorphisms Φ and $\tilde{\Phi}$ have the same dilatation, since their local dynamics are the same. Hence we have $\lambda(\phi) = \lambda(Q(\phi))$. In particular we have $\delta(\mathcal{H}(\mathbb{H}_g)) = \delta(SH_{2g+2})$ for $g \geq 2$ (Lemma 2.12). Proposition 1.3 says that there exists a sequence of pseudo-Anosov elements $\Gamma(w_{2n}) \in SH_{2n}$ whose normalized entropies $\mathrm{Ent}(\Gamma(w_{2n}))$ are uniformly bounded from above. Thus the same thing occurs in $\mathcal{H}(\mathbb{H}_g)$. See Section 2.6.

By Proposition 1.3 together with $\delta(\mathcal{H}(\mathbb{H}_g)) = \delta(SH_{2g+2})$, the following holds.

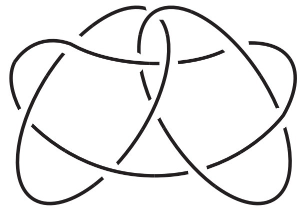

FIGURE 1. Link L_0 which gets the name $L10n95$ in the Thistlethwaite link table, see the Knot Atlas [22].

Theorem 1.4. *We have* $\limsup_{g\to\infty} g \log \delta(\mathcal{H}(\mathbb{H}_g)) \leq 2\log\kappa \approx 2.12255$.

Since $\mathcal{H}(\mathbb{H}_g)$ is the subgroup of $\mathrm{Mod}(\mathbb{H}_g)$, we have $\delta(\mathrm{Mod}(\mathbb{H}_g)) \leq \delta(\mathcal{H}(\mathbb{H}_g))$. Comparing Theorem 1.4 with (1.1), we find that Theorem 1.4 improves the previous upper bound of $\delta(\mathrm{Mod}(\mathbb{H}_g))$ by Hironaka. In this sense, the sequence of pseudo-Anosov elements of $\mathcal{H}(\mathbb{H}_g)$ used for the proof of Theorem 1.4 is a new example for $\mathrm{Mod}(\mathbb{H}_g)$ whose normalized entropies are uniformly bounded from above.

Let us mention a property of the sequence of pseudo-Anosov braids w_{2n}'s in Proposition 1.3 and give an outline of its proof. (See Section 3 for more details.) Let L_0 be a link with 3 components as in Figure 1. The mapping torus of $\Gamma(w_6) \in \mathrm{Mod}(\varSigma_{0,6})$ is homeomorphic to $S^3 \setminus L_0$, that is the complement of L_0 in a 3-sphere S^3. Thus once we prove that w_6 is a pseudo-Anosov braid, it follows that $S^3 \setminus L_0$ is a hyperbolic fibered 3-manifold. The sequence $w_8, w_{10}, w_{12}, \cdots$ has a property such that if $k = 4n+8$, then the mapping torus of $\Gamma(w_k)$ is homeomorphic to $S^3 \setminus L_0$, and if $k = 4n+6$, then the fibration of the mapping torus of $\Gamma(w_k)$ comes from a fibration of $S^3 \setminus L_0$ by Dehn filling cusps along the boundary slopes of a fiber (which depends on k). A technique about *disk twists* (see Section 2.7) provides a method of constructing sequences of mapping classes on punctured spheres whose mapping tori are homeomorphic to each other. We use this technique for the construction of the sequence $w_8, w_{12}, \cdots, w_{4n+8}, \cdots$ from the mapping torus of $\Gamma(w_6)$. We conclude that the braids $w_8, w_{12}, \cdots, w_{4n+8}, \cdots$ are pseudo-Anosov from the fact that $S^3 \setminus L_0$ is hyperbolic. We point out that our method by using disk twists quite suit to construct elements in the Hilden groups whose mapping tori are homeomorphic to each other. Now let $\Phi = \Phi_6 : \varSigma_{0,6} \to \varSigma_{0,6}$ be the pseudo-Anosov homeomorphism which represents $\Gamma(w_6)$, and let τ_6 and $\mathfrak{p}_6 : \tau_6 \to \tau_6$ be the invariant train track and the train track representative for $\Gamma(w_6)$ respectively. We find that $\lambda(w_6)$ is equal to the constant κ in Proposition 1.3. An analysis by using the suspension flow Φ^t on $S^3 \setminus L_0$ and the train track representative $\mathfrak{p}_6 : \tau_6 \to \tau_6$ tells us the dynamics of the pseudo-Anosov homeomorphism which represents $\Gamma(w_{4n+8})$ for each $n \geq 0$. In particular one can construct the train track representative $\mathfrak{p}_{4n+8} : \tau_{4n+8} \to \tau_{4n+8}$ for $\Gamma(w_{4n+8})$ concretely. From the 'shape' of the invariant train track τ_{4n+8}, we see that w_{4n+6} is a pseudo-Anosov braid with the same dilatation as w_{4n+8}. By a study of a particular fibered face for the exterior of the link L_0, we see that the normalized entropy of $\Gamma(w_{4n+8})$ converges to the one of $\Gamma(w_6)$, which implies that Proposition 1.3 holds.

From view point of fibered faces of fibered 3-manifolds, the sequence of mapping classes $\Gamma(w_{4n+8})$'s are obtained from a certain deformation of the monodromy $\Gamma(w_6)$ on the $\varSigma_{0,6}$-fiber of the fibration on $S^3 \setminus L_0$. See also Hironaka [17] and Valdivia [34] for other constructions in which fibered faces on hyperbolic 3-manifolds are used crucially.

By using Penner's lower bound $\log \delta_{0,n} \geq \frac{\log 2}{4n-12}$, Hironaka-Kin prove that $\log \delta_{0,n} \asymp \frac{1}{n}$ ([18]). In fact, it is shown in [18] that the subgroup $\Gamma(SB_{(n-1)})$ of $\mathrm{Mod}(\Sigma_{0,n})$ which consists of all mapping classes on an $(n-1)$-punctured disk D_{n-1} satisfies $\log \delta(\Gamma(SB_{(n-1)})) \asymp \frac{1}{n}$. (See Section 2.3 for the definition of $SB_{(n-1)}$.) By Theorem 1.2, we have another example, namely the Hilden group SH_{2n}, with the same property, that is the asymptotic behavior of the minimal dilatation of SH_{2n} is the same as that of the ambient group $\mathrm{Mod}(\Sigma_{0,2n})$. On the other hand, it is proved by Song that the dilatation of any pseudo-Anosov element of the pure braid groups has a uniform lower bound ([28]). We ask the following.

Question 1.5. *Which sequence of subgroups $G_{(n)}$'s of $\mathrm{Mod}(\Sigma_{0,n})$ satisfies $\log \delta(G_{(n)}) \asymp \frac{1}{n}$?*

The organization of this paper is as follows. In Section 2, we review basic facts on the Thurston norm and fibered faces on hyperbolic fibered 3-manifolds. We recall the connection between the spherical braid groups and the mapping class groups on punctured spheres. Then we recall the definitions of the Hilden groups and the wicket groups, and we describe a connection between them. We also introduce the hyperelliptic handlebody groups and give a relation between the hyperelliptic handlebody groups and the wicket groups. Lastly, we introduce the disk twists. In Section 3, we prove Proposition 1.3. In Apppendix A, we prove some claims given in Sections 2.5 and 2.6, and we determine a finite presentation of $\mathcal{H}(\mathbb{H}_g)$.

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2. Preliminaries

2.1. Mapping class groups.

Let Σ be a compact, connected, orientable surface removed the set of finitely many points P in its interior. The mapping class group $\mathrm{Mod}(\Sigma)$ is the group of isotopy classes of homeomorphisms on Σ which fix both P and the boundary $\partial\Sigma$ as *sets*. We apply elements of $\mathrm{Mod}(\Sigma)$ from right to left.

2.2. Thurston norm, fibered faces and entropy functions.

Let M be an oriented hyperbolic 3-manifold possibly with boundary. We recall some properties of the Thurston norm $\|\cdot\| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$. For more details, see [31] by Thurston. See also [7, Sections 5.2, 5.3] by Calegari.

Let F be a finite union of oriented, connected surfaces. We define $\chi_-(F)$ to be

$$\chi_-(F) = \sum_{F_i \subset F} \max\{0, -\chi(F_i)\},$$

where F_i's are the connected components of F. The Thurston norm $\|\cdot\|$ is defined for an integral class $a \in H_2(M, \partial M; \mathbb{Z})$ by

$$\|a\| = \min_F \{\chi_-(F) \mid a = [F]\},$$

where the minimum ranges over all oriented surfaces F embedded in M. A surface F which realizes the minimum is called a *minimal representative* or *norm-minimizing* of a. Then $\|\cdot\|$ defined on all integral classes admits a unique continuous extension $\|\cdot\| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ which is linear on rays through the origin. A significant property of $\|\cdot\|$ is that the unit ball U_M with respect to $\|\cdot\|$ is a finite-sided polyhedron.

We take a top dimensional face Ω on the boundary ∂U_M. Let C_Ω be the cone over Ω with the origin, and let $int(C_\Omega)$ be its open cone, that is the interior of C_Ω. When M is a hyperbolic fibered 3-manifold, the Thurston norm provides deep information about fibrations on M.

Theorem 2.1 (Thurston [31])**.** *Suppose that M fibers over the circle S^1 with fiber F. Then there exists a top dimensional face Ω on ∂U_M so that $[F] \in int(C_\Omega)$. Moreover given any integral class $a \in int(C_\Omega)$, its minimal representative F_a becomes a fiber of a fibration on M.*

Such a face Ω and such an open cone $int(C_\Omega)$ are called a *fibered face* and *fibered cone* respectively, and an integral class $a \in int(C_\Omega)$ is called a *fibered class.*

Now we take any primitive fibered class $a \in int(C_\Omega)$. The minimal representative F_a is a connected fiber of the fibration associated to a. If we let $\Phi_a : F_a \to F_a$ be the monodromy of this fibration, then the mapping class $\phi_a = [\Phi_a]$ is necessarily pseudo-Anosov, since M is hyperbolic. One can define the dilatation $\lambda(a)$ and entropy $\mathrm{ent}(a)$ to be the dilatation and entropy of pseudo-Anosov ϕ_a. The entropy function defined on primitive fibered classes a's can be extended to the entropy function on rational classes by homogeneity. An important property of such entropies, studied by Fried, Matsumoto and McMullen is that the function $a \mapsto \mathrm{ent}(a)$ defined for rational classes $a \in int(C_\Omega)$ extends to a real analytic convex function on the fibered cone $int(C_\Omega)$, see [24] for example. Moreover the *normalized entropy function*

$$\mathrm{Ent} = \| \cdot \| \mathrm{ent} : int(C_\Omega) \to \mathbb{R}$$

is constant on each ray in $int(C_\Omega)$ through the origin.

Since M fibers over S^1 with fiber F, M is homeomorphic to a mapping torus $\mathbb{T}_{[\Phi]}$, where $\Phi : F \to F$ is the monodromy of the fibration associated to $[F] \in int(C_\Omega)$. We may assume that $\Phi : F \to F$ is a pseudo-Anosov homeomorphism with the stable and unstable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. A surface F' is called a *cross-section* to the suspension flow Φ^t on M if F' is transverse to Φ^t and F' intersects every flow line.

Let J_1 and J_2 be embedded arcs in M which are transverse to Φ^t. We say that J_1 *is connected to* J_2 if there exists a positive continuous function $\mathfrak{g} : J_1 \to \mathbb{R}$ which satisfies the following. For any $x \in J_1$, we have $\Phi^{\mathfrak{g}(x)} \in J_2$ and $\Phi^t(x) \notin J_2$ for $0 < t < \mathfrak{g}(x)$. Moreover the map : $J_1 \to J_2$ given by $x \mapsto \Phi^{\mathfrak{g}(x)}(x)$ is a homeomorphism. In this case, we let

$$[J_1, J_2] = \{\Phi^t(x) \mid x \in J_1, 0 \leq t \leq \mathfrak{g}(x)\},$$

and we call $[J_1, J_2]$ a *flowband.* We use flowbands in the proof of Proposition 1.3.

Theorem 2.2 (Fried [13] for (1)(2), Thurston [31] for (3))**.** *Let $\Phi : F \to F$, $M \simeq \mathbb{T}_{[\Phi]}$ and Ω be as above. Let $\widehat{\mathcal{F}^s}$ and $\widehat{\mathcal{F}^u}$ denote the suspensions of $\mathcal{F}^s$ and $\mathcal{F}^u$ by Φ in $M \simeq \mathbb{T}_{[\Phi]}$. For any minimal representative F_a of any fibered class $a \in int(C_\Omega)$, we can modify F_a by an isotopy which satisfies the following.*

1. *F_a is transverse to Φ^t, and the first return map : $F_a \to F_a$ is precisely the pseudo-Anosov monodromy $\Phi_a : F_a \to F_a$ of the fibration on M associated to a. Moreover F_a is unique up to isotopy along flow lines.*
2. *The stable and unstable foliations of the pseudo-Anosov homeomorphism Φ_a are given by $\widehat{\mathcal{F}^s} \cap F_a$ and $\widehat{\mathcal{F}^u} \cap F_a$ respectively.*
3. *If $a' \in H_2(M, \partial M; \mathbb{R})$ is represented by some cross-section to Φ^t, then $a' \in int(C_\Omega)$.*

2.3. **Spherical braid groups.** Let SB_m be the spherical braid group with m strings. We depict braids vertically in this paper. We define the product of braids as follows. Given $b, b' \in SB_m$, we stuck b on b', and concatenate the bottom ith endpoint of b with the top ith endpoint of b' for each $1 \leq i \leq m$. Then we get m strings, and the product $bb' \in SB_m$ is the resulting braid (after rescaling such m strings), see Figure 2. We often label the numbers $1, \cdots, m$ (from left to right) at the bottom of a given braid. Let σ_i denote a braid of SB_m obtained by crossing the ith string under the $(i+1)$st string, see Figure 3(1). (Here the ith string means the string labeled i at the

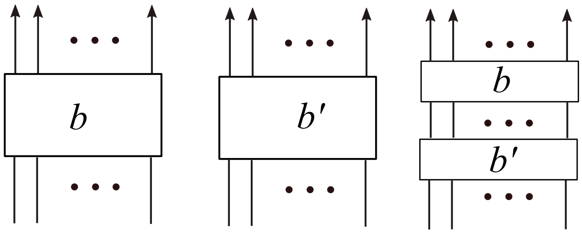


Figure 2. Braids b, b' and bb'.

bottom.) It is well-known that SB_m is a group generated by $\sigma_1, \sigma_2, \cdots, \sigma_{m-1}$, and its relations are given by

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$,
(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, \cdots, m-2$,
(3) $\sigma_1\sigma_2 \cdot \cdots \cdot \sigma_{m-2}\sigma_{m-1}^2\sigma_{m-2} \cdots \sigma_2\sigma_1 = 1$.

We recall a connection between SB_m and $\mathrm{Mod}(\Sigma_{0,m})$. Let $c_1, \cdots, c_m$ be the punctures of $\Sigma_{0,m}$. Let h_i be the left-handed half twist about the arc between the ith and $(i+1)$st punctures c_i and c_{i+1}, see Figure 3(2). We define a homomorphism

$$\Gamma : SB_m \to \mathrm{Mod}(\Sigma_{0,m})$$

which sends σ_i to h_i for $i \in \{1, \cdots, m-1\}$. Since $\mathrm{Mod}(\Sigma_{0,m})$ is generated by $h_1, \cdots, h_{m-1}$, Γ is surjective. If we let

$$\Delta = \Delta_m = (\sigma_1\sigma_2 \cdots \sigma_{m-1})(\sigma_1\sigma_2 \cdots \sigma_{m-2}) \cdots (\sigma_1\sigma_2)\sigma_1$$

which is a half twist braid, then the kernel of Γ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is generated by a full twist braid Δ^2. Thus

$$SB_m/\langle\Delta^2\rangle \simeq \mathrm{Mod}(\Sigma_{0,m}).$$

Given a braid $b \in SB_m$, the mapping torus $\mathbb{T}_{\Gamma(b)}$ of $\Gamma(b)$ is denoted by $\mathbb{T}_b$ for simplicity.

Remark 2.3. *Each m-braid as in Figure 3(1) with the orientation from the bottom of strings to the top induces the motion of m points on the sphere. This gives rise to the above homomorphism Γ, which maps σ_i to h_i. In this paper, we denote the braid in Figure 3(1) by σ_i.*

We say that a braid $b \in SB_m$ is *pseudo-Anosov* if $\Gamma(b)$ is a pseudo-Anosov mapping class. In this case, we define the dilatation $\lambda(b)$ of b to be the dilatation $\lambda(\Gamma(b))$. Also, we let $\Phi_b : \Sigma_{0,m} \to \Sigma_{0,m}$ be the pseudo-Anosov homeomorphism which represents $\Gamma(b)$, and let $\mathcal{F}_b$ be the unstable foliation for Φ_b.

Let $SB_{(m-1)}$ be the subgroup of SB_m which is generated by $\sigma_1, \cdots, \sigma_{m-2}$. (Hence a braid $b \in SB_{(m-1)}$ is represented by a word without $\sigma_{m-1}^{\pm 1}$.) As we will see in Section 2.4, $SB_{(m-1)}$ is closely related to the $(m-1)$-braid group B_{m-1}.

2.4. **Braid groups.** We recall a connection between the two groups $(m-1)$-braid group B_{m-1} on a disk and the mapping class group $\mathrm{Mod}(D_{m-1})$, where D_{m-1} is a disk with $m-1$ punctures $c_1, \cdots, c_{m-1}$. By abusing notations, we denote by σ_i, the braid of B_{m-1} obtained by crossing the

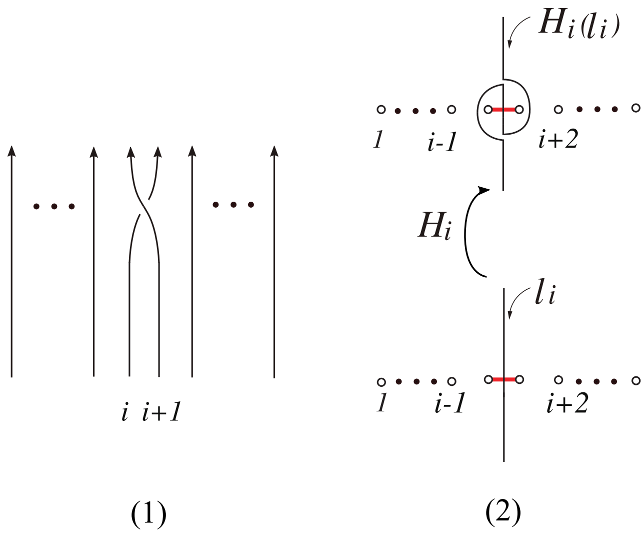


FIGURE 3. (1) σ_i. (2) Action of a representative $H_i \in h_i$ on ℓ_i, where ℓ_i is a vertical arc which passes through the horizontal arc between the punctures c_i and c_{i+1}, see Remark 2.4.

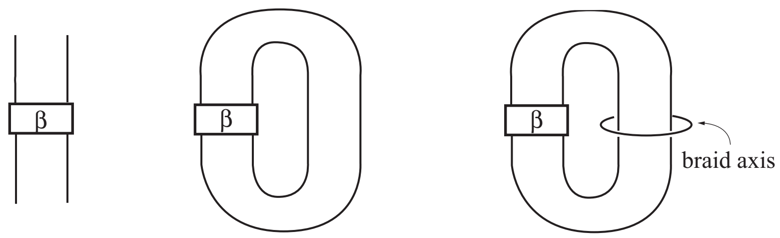


FIGURE 4. Braid β, closure $\mathrm{cl}(\beta)$, and braided link $\mathrm{br}(\beta)$ from left to right.

ith string under the $(i+1)$st string. The braid group B_{m-1} with $m-1$ strings is the group generated by $\sigma_1, \cdots, \sigma_{m-2}$ having the following relations.

(1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$,

(2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, \cdots, m-3$.

Abusing notations again, we denote by h_i, the left-handed half twist about the arc between the ith and $(i+1)$st punctures of D_{m-1}. We also use Γ for the surjective homomorphism

$$\Gamma : B_{m-1} \to \mathrm{Mod}(D_{m-1})$$

which sends σ_i to h_i for $i \in \{1, \cdots, m-2\}$. In this case, the kernel of Γ is an infinite cyclic group generated by the full twist braid $\Delta^2 = \Delta^2_{m-1}$.

We have a homomorphism

$$\begin{array}{rcl} c : \mathrm{Mod}(D_{m-1}) & \to & \mathrm{Mod}(\Sigma_{0,m}) \\ h_i & \mapsto & h_i \end{array}$$

which is induced by the map that sends the boundary of the disk to the mth puncture of $\Sigma_{0,m}$. Observe that

$$c(\Gamma(B_{m-1})) = c(\mathrm{Mod}(D_{m-1})) = \Gamma(SB_{(m-1)}).$$

Given $\beta \in B_{m-1}$, we denote the mapping torus $\mathbb{T}_{c(\Gamma(\beta))}$ of $c(\Gamma(\beta))$ by $\mathbb{T}_\beta$ for simplicity. Let $\mathrm{cl}(\beta)$ be the closure of β (or the closed braid of β). We have $\mathbb{T}_\beta \simeq S^3 \setminus \mathrm{br}(\beta)$, (that is $\mathbb{T}_\beta$ is homeomorphic to $S^3 \setminus \mathrm{br}(\beta)$) where $\mathrm{br}(\beta)$ is the *braided link* of β which is a union of $\mathrm{cl}(\beta)$ and its *braid axis*, see Figure 4.

Remark 2.4. *Recall that we apply elements of the mapping class groups from right to left. This convention together with the homomorphism* Γ *from* B_{m-1} *to* $\mathrm{Mod}(D_{m-1})$ *gives rise to an orientation of strings of* β *from the bottom to the top, which is compatible with the direction of the suspension flow on* $\mathbb{T}_\beta = S^3 \setminus \mathrm{br}(\beta)$.

We say that $\beta \in B_{m-1}$ is *pseudo-Anosov* if $c(\Gamma(\beta))$ is pseudo-Anosov. In this case, we define the dilatation $\lambda(\beta)$ of β to be the dilatation $\lambda(c(\Gamma(\beta)))$.

By definition, an m-braid $b \in SB_{(m-1)}$ is represented by a word without $\sigma_{m-1}^{\pm 1}$. Removing the last string of b, we get an $(m-1)$-braid on a sphere. If we regard such a braid as the one on a disk, we have an $(m-1)$-braid $\underline{b}$ with the same word as b. By definition of $\underline{b}$, we have

$$c(\Gamma(\underline{b})) = \Gamma(b).$$

Since $\mathbb{T}_{\underline{b}} = \mathbb{T}_{c(\Gamma(\underline{b}))} = \mathbb{T}_{\Gamma(b)} = \mathbb{T}_b$, we have $\mathbb{T}_{\underline{b}} = \mathbb{T}_b$. We get the following lemma immediately.

Lemma 2.5. *A braid* $b \in SB_{(m-1)}$ *is pseudo-Anosov if and only if* $\underline{b} \in B_{m-1}$ *is pseudo-Anosov. In this case, the equality* $\lambda(b) = \lambda(\underline{b})$ *holds, and* $\mathbb{T}_b (= \mathbb{T}_{\underline{b}})$ *is a hyperbolic fibered* 3*-manifold.*

2.5. **Hilden groups and wicket groups.**

2.5.1. *Relations between Hilden groups and wicket groups.* First of all, we define a subgroup of $\mathrm{Mod}(\Sigma_{0,2n})$ which was introduced by Hilden [15]. Let $A_1, \cdots, A_n$ be n disjoint trivial arcs properly embedded in a unit ball D^3 as in Figure 5(1). More precisely, each A_i is unknotted and the union $\mathbf{A} = \mathbf{A}_n = A_1 \cup \cdots \cup A_n$ is unlinked. Such A_i's are called *wickets*. Let $\mathrm{Homeo}_+(D^3, \mathbf{A})$ be the set of orientation preserving homeomorphisms on D^3 preserving $\mathbf{A}$ setwise. For each $\Psi \in \mathrm{Homeo}_+(D^3, \mathbf{A})$, we have the restriction

$$\Psi|_{\partial D^3} : (\partial D^3, \partial \mathbf{A}) \to (\partial D^3, \partial \mathbf{A})$$

which is an orientation preserving homeomorphism on a 2-sphere $S^2 = \partial D^3$ preserving $2n$ points of $\partial \mathbf{A}$ setwise. Its isotopy class $[\Psi|_{\partial D^3}]$ gives rise to an element of $\mathrm{Mod}(\Sigma_{0,2n})$. We define a homomorphism

$$\mathrm{Mod}(D^3, \mathbf{A}) \to \mathrm{Mod}(\Sigma_{0,2n})$$

which sends a mapping class $[\Psi]$ of $\Psi \in \mathrm{Homeo}_+(D^3, \mathbf{A})$ to the mapping class $[\Psi|_{\partial D^3}]$. This homomorphism is injective, see for example [5, p.484] or [15, p.157]. We prove this claim in Appendix A for the convenience of readers, see Proposition A.4.

The group $\mathrm{Mod}(D^3, \mathbf{A})$ or its homomorphic image into $\mathrm{Mod}(\Sigma_{0,2n})$ is called the *(spherical) Hilden group* SH_{2n}. Let us describe SH_{2n} by using certain subgroup of the spherical braid group SB_{2n} of $2n$ strings. Given a braid $b \in SB_{2n}$, we stuck b on $\mathbf{A} = A_1 \cup \cdots \cup A_n$, and concatenate the bottom endpoints of b with the endpoints of $\mathbf{A}$, see Figure 5(2). Then we obtain n disjoint smooth arcs ${}^b\mathbf{A}$ properly embedded in D^3. We may suppose that the arcs ${}^b\mathbf{A}$ have the same endpoints as $\mathbf{A}$. The *(spherical) wicket group* SW_{2n} is the subgroup of SB_{2n} generated by braids b's such that ${}^b\mathbf{A}$ is isotopic to $\mathbf{A}$ relative to $\partial \mathbf{A}$. For example, the following braids are elements of SW_{2n}.

$$\begin{aligned}
r_i &= \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}^{-1}\sigma_{2i}^{-1} \quad (i \in \{1, \cdots, n-1\}),\\
s_i &= \sigma_{2i}^{-1}\sigma_{2i+1}^{-1}\sigma_{2i-1}^{-1}\sigma_{2i}^{-1} \quad (i \in \{1, \cdots, n-1\}),\\
t_j &= \sigma_{2j-1}^{-1} \quad (j \in \{1, \cdots, n\}),
\end{aligned}$$

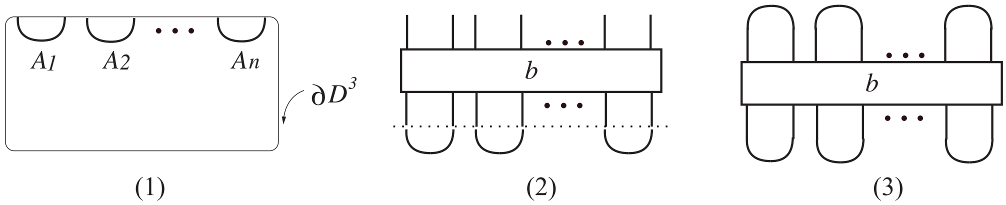


FIGURE 5. (1) $\mathbf{A} = A_1 \cup \cdots \cup A_n$. (2) ${}^b\mathbf{A}$. (3) $\mathrm{pl}(b)$.

see Figure 6(1)(2)(3). Now we recall the homomorphism $\Gamma : SB_{2n} \to \mathrm{Mod}(\Sigma_{0,2n})$. We claim that $\Gamma(r_i)$, $\Gamma(s_i)$ and $\Gamma(t_j)$ are elements of SH_{2n}. Indeed, $\Gamma(r_i)$ (resp. $\Gamma(s_i)$) interchanges the ith and $(i+1)$st wickets A_i, A_{i+1} by passing A_i through (resp. around) A_{i+1}. $\Gamma(t_j)$ rotates the jth wicket A_j 180 degrees around its vertical axis of the symmetry, see Figure 6(4)(5)(6).

Theorem 2.6. *The Hilden group SH_{2n} is the image of the homomorphism $\Gamma|_{SW_{2n}} : SW_{2n} \to \mathrm{Mod}(\Sigma_{0,2n})$ whose kernel is equal to $\langle \Delta^2 \rangle$. In particular,*

$$SW_{2n}/\langle \Delta^2 \rangle \simeq SH_{2n}.$$

We shall prove Theorem 2.6 in Appendix A by using a finite generating set of SW_{2n} (resp. SH_{2n}) given by Brendle-Hatcher [5] (resp. Hilden [15]). We note that the definition of the spherical wicket groups in [5] is different from the one in this paper. We shall claim in Appendix A that these two definitions give rise to the same group, see Proposition A.1.

The wicket groups are closely related to the *loop braid groups* which arise naturally in the different fields of mathematics. For more details of loop braid groups, see Damiani [8].

For a finite presentation of the *Hilden group* on a *plane*, see Tawn [30].

2.5.2. *Plat closures of braids.* In this section, we prove that SW_{2n} is of infinite index in SB_{2n} for $n \geq 2$. (We do not use this claim in the rest of the paper.) To do this, we turn to the plat closures of braids which were introduced by Birman. Given $b \in SB_{2n}$, the *plat closure* of b, denoted by $\mathrm{pl}(b)$, is a link in S^3 obtained from b putting trivial n arcs on n pairs of consecutive, bottom (resp. top) $2n$ endpoints of b, see Figure 5(3). Observe that given two braids $w, w' \in SW_{2n}$, the plat closures $\mathrm{pl}(b)$ and $\mathrm{pl}(wbw')$ represent the same link. Moreover the plat closure of any element $w \in SW_{2n}$, $\mathrm{pl}(w)$, is a disjoint union of n unknots. Every link in S^3 can be represented by the plat closure of some braid with even strings [2, Theorem 5.1]. Birman characterizes two braids with the same strings whose plat closures yield the same link [2, Theorem 5.3]. Fore more discussion on plat closures of braids, see [2, Chapter 5].

Lemma 2.7. *SW_{2n} is of infinite index in SB_{2n} for $n \geq 2$.*

Proof. We take a braid $b = \sigma_2\sigma_2 \notin SW_{2n}$. Given $w, w' \in SW_{2n}$, we have $\mathrm{pl}(b^k) = \mathrm{pl}(b^k w') = \mathrm{pl}(wb^k)$ for each integer k, and the link $\mathrm{pl}(b^k)$ contains the $(2, 2k)$ torus link (as components) which is not a disjoint union of unknots for each $k \neq 0$. In particular both $b^k w', wb^k \notin SW_{2n}$. This implies that SW_{2n} is of infinite index in SB_{2n} for $n \geq 2$. □

By Lemma 2.7, the Hilden group SH_{2n} is of infinite index in $\mathrm{Mod}(\Sigma_{0,2n})$ for $n \geq 2$, since $SW_{2n}/\langle \Delta^2 \rangle \simeq SH_{2n}$ and $SB_{2n}/\langle \Delta^2 \rangle \simeq \mathrm{Mod}(\Sigma_{0,2n})$.

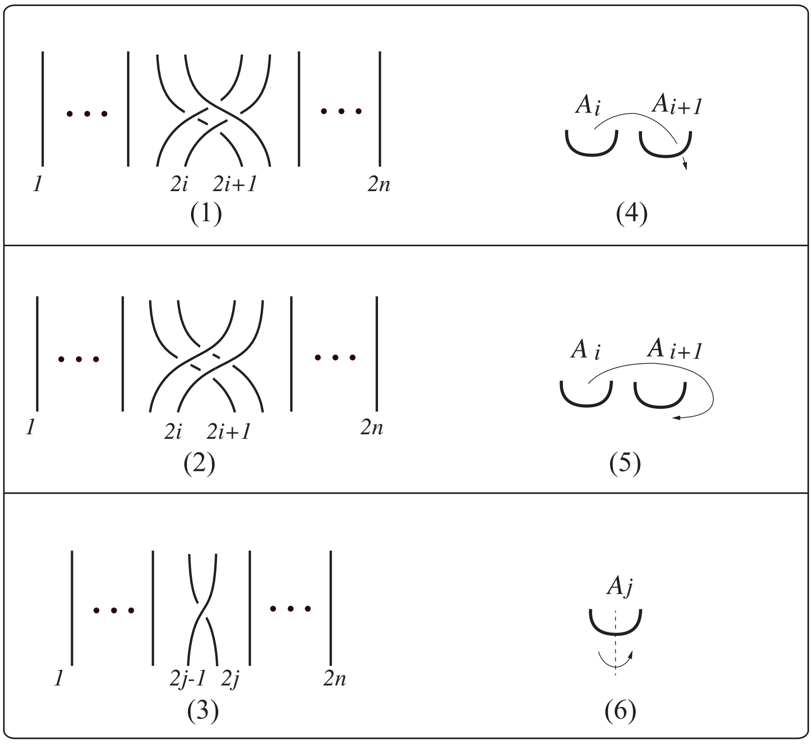


FIGURE 6. (1) r_i, (2) s_i and (3) $t_j \in SW_{2n}$. (4) $\Gamma(r_i)$, (5) $\Gamma(s_i)$ and (6) $\Gamma(t_j) \in SH_{2n}$. See also Remark 2.3. (cf. [5, Figure 2].)

2.6. **Hyperelliptic handlebody groups.** Let $\mathbb{H}_g$ be a handlebody of genus g, i.e, $\mathbb{H}_g$ is an oriented 3-manifold obtained from a 3-ball attaching g copies of a 1-handle. We take an involution $\mathcal{S} : \mathbb{H}_g \to \mathbb{H}_g$ whose quotient space $\mathbb{H}_g/\mathcal{S}$ is a 3-ball D^3 with a union of wickets $\mathbf{A} = A_1 \cup \cdots \cup A_{g+1}$ as the image of the fixed point sets of $\mathcal{S}$ under the quotient, see Figure 7. We call $\mathcal{S}$ the *hyperelliptic involution* on $\mathbb{H}_g$. The restriction $\mathcal{S}|_{\partial\mathbb{H}_g} : \partial\mathbb{H}_g \to \partial\mathbb{H}_g$ defines an involution on $\partial\mathbb{H}_g \simeq \Sigma_g$. For simplicity, we denote such an involution $\mathcal{S}|_{\partial\mathbb{H}_g}$ by the same notation $\mathcal{S}$, and also call it the hyperelliptic involution on $\partial\mathbb{H}_g$. The quotient space $\partial\mathbb{H}_g/\mathcal{S}$ is a 2-sphere with $2g+2$ marked points that are the image of the fixed points set of $\mathcal{S} : \partial\mathbb{H}_g \to \partial\mathbb{H}_g$ under the quotient.

Let $\mathcal{H}(\Sigma_g)$ be the subgroup of $\mathrm{Mod}(\Sigma_g)$ consisting of isotopy classes of orientation preserving homeomorphisms on Σ_g that commute with $\mathcal{S} : \partial\mathbb{H}_g \to \partial\mathbb{H}_g$. Such a group $\mathcal{H}(\Sigma_g)$ is called the *hyperelliptic mapping class group* or *symmetric mapping class group*. Note that $\mathrm{Mod}(\Sigma_2) = \mathcal{H}(\Sigma_2)$. If $g \geq 3$, then $\mathcal{H}(\Sigma_g)$ is of infinite index in $\mathrm{Mod}(\Sigma_g)$. By the fundamental result by Birman-Hilden [3], one has a handy description of $\mathcal{H}(\Sigma_g)$ via braids, as we explain now. Note that any homeomorphism on $\partial\mathbb{H}_g$ that commute with $\mathcal{S}$ fixes the fixed points set of $\mathcal{S} : \partial\mathbb{H}_g \to \partial\mathbb{H}_g$ as a set. Hence via the quotient of $\partial\mathbb{H}_g$ by $\mathcal{S}$, such a homeomoprhism on $\partial\mathbb{H}_g$ descends to a homeomorphism on a sphere $\partial\mathbb{H}_g/\mathcal{S}$ which preserves the $2g+2$ marked points of $\partial\mathbb{H}_g/\mathcal{S}$. Thus we have a map

$$q : \mathcal{H}(\Sigma_g) \to \mathrm{Mod}(\Sigma_{0,2g+2})$$

by using a representative of each mapping class of $\mathcal{H}(\Sigma_g)$ which commutes with $\mathcal{S}$. Let $\iota \in \mathcal{H}(\Sigma_g)$ denote a mapping class of $\mathcal{S} : \partial\mathbb{H}_g \to \partial\mathbb{H}_g$ which is of order 2.

Theorem 2.8 (Birman-Hilden)**.** *For $g \geq 2$, the map $q : \mathcal{H}(\Sigma_g) \to \mathrm{Mod}(\Sigma_{0,2g+2})$ is well-defined, and it is a surjective homomorphism with the kernel $\langle \iota \rangle$. In particular,*

$$\mathcal{H}(\Sigma_g)/\langle \iota \rangle \simeq \mathrm{Mod}(\Sigma_{0,2g+2}) \simeq SB_{2g+2}/\langle \Delta^2 \rangle.$$

Thurston's classification theorem of surface homeomorphisms states that every mapping class $\phi \in \mathrm{Mod}(\Sigma)$ is one of the three types: periodic, reducible, pseudo-Anosov ([32]). The following well-known lemma says that q preserves these types.

Lemma 2.9. *If $\phi \in \mathcal{H}(\Sigma_g)$ is pseudo-Anosov (resp. periodic, reducible), then so is $q(\phi) \in \mathrm{Mod}(\Sigma_{0,2g+2})$, i.e, $q(\phi)$ is pseudo-Anosov (resp. periodic, reducible). When $\phi \in \mathcal{H}(\Sigma_g)$ is pseudo-Anosov, the equality $\lambda(\phi) = \lambda(q(\phi))$ holds.*

Proof. It is not hard to see that if ϕ is periodic (resp. reducible), then $q(\phi)$ is periodic (resp. reducible). Suppose that $\phi \in \mathcal{H}(\Sigma_g)$ is pseudo-Anosov. Then we see that $q(\phi)$ is pseudo-Anosov. If not, then it is periodic or reducible. Assume that $q(\phi)$ is periodic. (The proof in the reducible case is similar.) We take a periodic homeomorphism $f : \Sigma_{0,2g+2} \to \Sigma_{0,2g+2}$ which represents $q(\phi)$. Consider a lift $\tilde{f} : \Sigma_g \to \Sigma_g$ of f. Then $\tilde{f}$ is a periodic homeomorphism which represents ϕ. Thus $\phi = [\tilde{f}]$ is a periodic mapping class, which contradicts the assumption that ϕ is pseudo-Anosov.

We consider a pseudo-Anosov homeomorphism $\Phi : \Sigma_{0,2g+2} \to \Sigma_{0,2g+2}$ which represents the pseudo-Anosov mapping class $q(\phi)$. Take a lift $\tilde{\Phi}$ of Φ which represents $\phi \in \mathcal{H}(\Sigma_g)$. Then $\tilde{\Phi}$ is a pseudo-Anosov homeomorphism whose stable/unstable foliations are lifts of the stable/unstable foliations of Φ. In particular, we have $\lambda(\tilde{\Phi}) = \lambda(\Phi)$, since Φ and $\tilde{\Phi}$ have the same dynamics locally. □

By Theorem 2.8 and Lemma 2.9, we have the following.

Corollary 2.10. *We have $\delta(\mathcal{H}(\Sigma_g)) = \delta_{0,2g+2}$ for $g \geq 2$.*

Let $\mathrm{Mod}(\mathbb{H}_g)$ be the group of isotopy classes of orientation preserving homeomorphisms on $\mathbb{H}_g$. We call $\mathrm{Mod}(\mathbb{H}_g)$ the *handlebody group*. We denote by $\mathrm{SHomeo}_+(\mathbb{H}_g)$, the group of orientation preserving homeomorphisms on $\mathbb{H}_g$ which commute with $\mathcal{S} : \mathbb{H}_g \to \mathbb{H}_g$. Let $\mathcal{H}(\mathbb{H}_g)$ be the subgroup of $\mathrm{Mod}(\mathbb{H}_g)$ consisting of isotopy classes of elements in $\mathrm{SHomeo}_+(\mathbb{H}_g)$. We call $\mathcal{H}(\mathbb{H}_g)$ the *hyperelliptic handlebody group.* Abusing the notation, we also denote by $\iota \in \mathcal{H}(\mathbb{H}_g)$, the mapping class of $\mathcal{S} : \mathbb{H}_g \to \mathbb{H}_g$. One can define a homomorphism

$$\mathrm{Mod}(\mathbb{H}_g) \to \mathrm{Mod}(\Sigma_g)$$

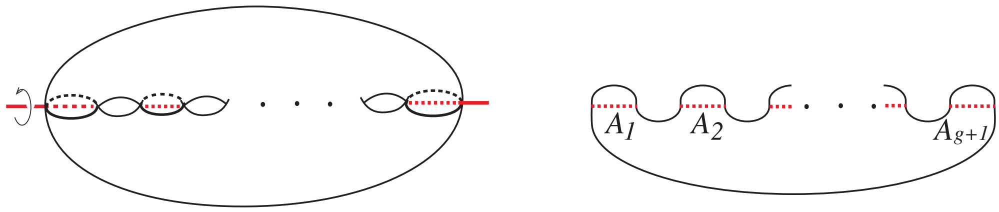
which sends a mapping class $[\Psi]$ of an orientation preserving homeomorphism $\Psi : \mathbb{H}_g \to \mathbb{H}_g$ to the mapping class $[\Psi|_{\partial \mathbb{H}_g}]$ of $\Psi|_{\partial \mathbb{H}_g} : \partial \mathbb{H}_g \to \partial \mathbb{H}_g$. This homomorphism is injective ([12, Theorem 3.7]), and not surjective ([29, Section 3.12]). We also call the homomorphic image of $\mathrm{Mod}(\mathbb{H}_g)$ in $\mathrm{Mod}(\Sigma_g)$ the handlebody group, and also call the homomorphic image of $\mathcal{H}(\mathbb{H}_g)$ in $\mathrm{Mod}(\Sigma_g)$, the hyperelliptic handlebody group. As subgroups of $\mathrm{Mod}(\Sigma_g)$, we have

$$\mathcal{H}(\mathbb{H}_g) = \mathrm{Mod}(\mathbb{H}_g) \cap \mathcal{H}(\Sigma_g).$$

We have $\mathrm{Mod}(\mathbb{H}_2) = \mathcal{H}(\mathbb{H}_2)$ since $\mathrm{Mod}(\Sigma_2) = \mathcal{H}(\Sigma_2)$ holds. If $g \geq 2$, then $\mathrm{Mod}(\mathbb{H}_g)$ is of infinite index in $\mathrm{Mod}(\Sigma_g)$; If $g \geq 3$, then $\mathcal{H}(\mathbb{H}_g)$ is of infinite index in $\mathrm{Mod}(\mathbb{H}_g)$, see Remark A.7 in Appendix A.

In the end of this section, we give a description of $\mathcal{H}(\mathbb{H}_g)$ via SW_{2g+2}. Any element of $\mathrm{SHomeo}_+(\mathbb{H}_g)$ fixes the fixed points set of $\mathcal{S} : \mathbb{H}_g \to \mathbb{H}_g$ as a set, and hence such an element descends to a homeomorphism on $\mathbb{H}_g/\mathcal{S} \simeq D^3$ which preserves $\mathbf{A}$ as a set. Thus a map

$$Q : \mathcal{H}(\mathbb{H}_g) \to SH_{2g+2}$$



(1) (2)

FIGURE 7. (1) Hyperelliptic involution $\mathcal{S} : \mathbb{H}_g \to \mathbb{H}_g$ (on a handlebody $\mathbb{H}_g$) which is a rotation by 180 degrees about the indicated axis. The fix points set of $\mathcal{S}$ is illustrated by dotted segments. The restriction $\mathcal{S} = \mathcal{S}|_{\partial \mathbb{H}_g} : \partial \mathbb{H}_g \to \partial \mathbb{H}_g$ defines an involution on $\partial \mathbb{H}_g \simeq \Sigma_g$. (2) $\mathbb{H}_g / \mathcal{S} \simeq D^3$ with $\mathbf{A} = A_1 \cup A_2 \cup \cdots \cup A_{g+1}$.

is obtained. When we think $\mathcal{H}(\mathbb{H}_g)$ as the subgroup of $\mathrm{Mod}(\Sigma_g)$ (resp. SH_{2g+2} as the subgroup of $\mathrm{Mod}(\Sigma_{0,2g+2})$), we have the restriction of the homomorphism q in Theorem 2.8

$$Q = q|_{\mathcal{H}(\mathbb{H}_g)} : \mathcal{H}(\mathbb{H}_g) \to SH_{2g+2}. \tag{2.1}$$

The following theorem, which is a version of Birman-Hilden's theorem 2.8, is useful.

Theorem 2.11. *For $g \geq 2$, the map $Q : \mathcal{H}(\mathbb{H}_g) \to SH_{2g+2}$ is well-defined, and it is a surjective homomorphism with the kernel $\langle \iota \rangle$. In particular,*

$$\mathcal{H}(\mathbb{H}_g)/\langle \iota \rangle \simeq SH_{2g+2}.$$

For a proof of Theorem 2.11, see Appendix A. By Theorems 2.6 and 2.11, we have

$$\mathcal{H}(\mathbb{H}_g)/\langle \iota \rangle \simeq SW_{2g+2}/\langle \Delta^2 \rangle \simeq SH_{2g+2}.$$

Lemma 2.9 and Theorem 2.11 imply the following.

Lemma 2.12. *We have $\delta(\mathcal{H}(\mathbb{H}_g)) = \delta(SH_{2g+2})$ for $g \geq 2$.*

2.7. **Disk twists.** We will discuss a method of constructing links in S^3 whose complements are the same. Let L be a link in S^3. We denote a tubular neighborhood of L by $\mathcal{N}(L)$, and the exterior of L, that is $S^3 \setminus \mathrm{int}(\mathcal{N}(L))$ by $\mathcal{E}(L)$. Suppose that L contains an unknot $K \subset L$. Then $\mathcal{E}(K)$ (resp. $\partial \mathcal{E}(K)$) is homeomorphic to a solid torus (resp. torus). We denote the link $L \setminus K$ by L_K. We take a disk D bounded by the longitude of $\mathcal{N}(K)$. By using D, we define two homeomorphisms

$$T = T_D : \mathcal{E}(K) \to \mathcal{E}(K)$$

called the *(left-handed) disk twist about* D and

$$H = H_D : \mathcal{E}(L)(= \mathcal{E}(K \cup L_K)) \to \mathcal{E}(K \cup T_D(L_K))$$

as follows. We cut $\mathcal{E}(K)$ along D. We have resulting two sides obtained from D. Then we reglue the two sides by rotating either of the sides 360 degrees so that the mapping class of the restriction $T|_{\partial \mathcal{E}(K)} : \partial \mathcal{E}(K) \to \partial \mathcal{E}(K)$ defines the left-handed Dehn twist about ∂D, see Figure 8(1). Such an operation defines the former homeomorphism $T_D : \mathcal{E}(K) \to \mathcal{E}(K)$. If m segments of L_K pass through D, then $T(L_K)$ is obtained from L_K by adding a full twist braid Δ_m^2 near D. In the case $m = 2$, see Figure 8(2). Notice that $T_D : \mathcal{E}(K) \to \mathcal{E}(K)$ determines the latter homeomorphism

$$H = H_D : \mathcal{E}(L)(= \mathcal{E}(K \cup L_K)) \to \mathcal{E}(K \cup T(L_K)).$$

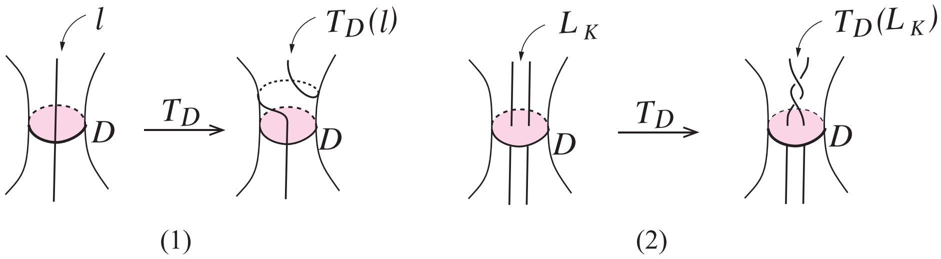


FIGURE 8. (1) Action of T_D on ℓ, where ℓ is an arc on $\partial\mathcal{E}(K)$ which passes through ∂D. (2) Local picture of L_K and its image $T_D(L_K)$.

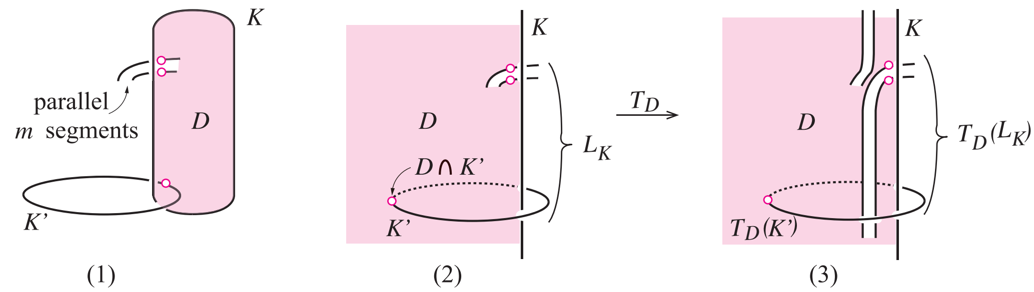


FIGURE 9. Small circles in (1), (2) (resp. (3)) indicate the intersection between L_K and D (resp. between $T_D(L_K)$ and D).

For any integer $\ell \neq 0$, we have a homeomorphism of the ℓth power $T^\ell = T_D^\ell : \mathcal{E}(K) \to \mathcal{E}(K)$ so that $T^\ell|_{\partial\mathcal{E}(K)} : \partial\mathcal{E}(K) \to \partial\mathcal{E}(K)$ is the ℓth power of the left-handed Dehn twist about ∂D. Observe that $T^\ell = T_D^\ell$ converts $L = K \cup L_K$ into a link $K \cup T^\ell(L_K)$ in S^3 such that $S^3 \setminus L$ is homeomorphic to $S^3 \setminus (K \cup T^\ell(L_K))$. We denote by H_D^ℓ, a homeomorphism: $\mathcal{E}(L)(= \mathcal{E}(K \cup L_K)) \to \mathcal{E}(K \cup T^\ell(L_K))$.

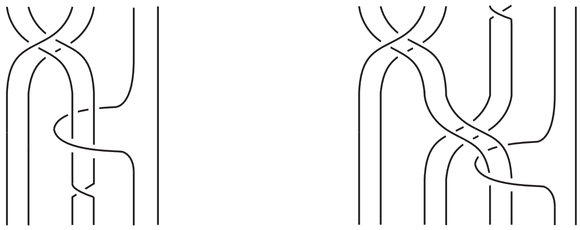
The following remark is used in the proof of Proposition 1.3.

Remark 2.13. *Let L be a link in S^3. Suppose that L contains two unknotted components K and K' such that $K \cup K'$ is the Hopf link. Let D be a disk bounded by the longitude of $\mathcal{N}(K)$. We assume that parallel $m \geq 1$ segments of $L_K \setminus K'$ pass through D, see Figure 9(1) in the case $m = 2$. ($L_K \setminus K'$ may intersect with the disk bounded by the longitude of $\mathcal{N}(K')$.) Pushing D along the meridian of $\mathcal{N}(K)$, one can put the resulting disk D as in Figure 9(2). The small circles in Figure 9(2) indicate the intersection between L_K and D. Now we consider the disk twist T about D, that is, we cut $\mathcal{E}(K)$ along D and we reglue the two sides obtained from D by rotating one of the sides by 360 degrees. In this case, one can choose the intersection point $D \cap K'$ as an origin of the rotation of D. As a result, we get a local diagram of the link $T(L_K)$ shown in Figure 9(3) so that $T = T_D$ fixes K'. (See K' and $T_D(K')$ in Figure 9(2)(3).)*

3. PROOF OF PROPOSITION 1.3

We introduce a sequence of braids $w_{2n} \in SW_{2n}$. Let

$$w_6 = \sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2\sigma_4\sigma_3^2\sigma_4\sigma_3 = \sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2\sigma_4\sigma_3\sigma_4\sigma_3\sigma_4 \in SB_{(5)},$$



(1) (2)

FIGURE 10. (1) $w_6 \in SB_{(5)} \cap SW_6$. (2) $w_8 \in SB_{(7)} \cap SW_8$.

see Figure 10(1). (For the definition of the subgroup $SB_{(m-1)}$ of SB_m, see Section 2.3.) Since ${}^{w_6}\mathbf{A}$ is isotopic to $\mathbf{A}$ relative to $\partial\mathbf{A}$, we have $w_6 \in SW_6$. In order to define a sequence of braids $w_8, w_{10}, \cdots$, we introduce $x_{4n+8}, y_{4n+8} \in SB_{(4n+7)}$ for each $n \geq 0$ as follows.

$$\begin{aligned} x_{4n+8} &= \sigma_5\sigma_2^{-1}\sigma_1^{-1}(\sigma_3\sigma_4\cdots\sigma_{4n+5})(\sigma_2\sigma_3\cdots\sigma_{4n+4})\sigma_{4n+6}(\sigma_{4n+5})^2\sigma_{4n+6}, \\ y_{4n+8} &= (\sigma_1\sigma_2\cdots\sigma_{4n+5})^4\sigma_{4n+6}\sigma_{4n+5}\sigma_{4n+4}(\sigma_{4n+3})^2\sigma_{4n+4}\sigma_{4n+5}\sigma_{4n+6}, \end{aligned}$$

see Figure 11(1)(2). It is straightforward to check that ${}^{b}\mathbf{A}$ is isotopic to $\mathbf{A}$ relative to $\partial\mathbf{A}$ when $b = x_{4n+8}$, y_{4n+8}. Thus x_{4n+8}, $y_{4n+8} \in SW_{4n+8}$. We let

$$w_{4n+8} = x_{4n+8}(y_{4n+8})^n \in SB_{(4n+7)} \cap SW_{4n+8},$$

where $(y_8)^0 = 1 \in SB_{(7)}$. For example, in the case of $n = 0$,

$$w_8 = x_8(y_8)^0 = \sigma_5\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_4\sigma_5\sigma_2\sigma_3\sigma_4\sigma_6\sigma_5^2\sigma_6,$$

see Figure 10(2). Notice that the last two strings ($(4n+7)$th and $(4n+8)$th strings) of both x_{4n+8} and y_{4n+8} define the identity $1 \in SB_2$, see Figure 11(1)(2). Thus we obtain the $(4n+6)$-spherical braid by removing the last two strings from w_{4n+8}. In the case $n \geq 1$, we denote by w_{4n+6}, the resulting $(4n+6)$-braid. Said differently if we let x_{4n+6} (resp. y_{4n+6}) be the $(4n+6)$-spherical braid obtained from x_{4n+8} (resp. y_{4n+8}) by removing the last two strings from x_{4n+8} (resp. y_{4n+8}), then w_{4n+6} is given by

$$w_{4n+6} = x_{4n+6}(y_{4n+6})^n,$$

see Figure 11(3)(4). Clearly $w_{4n+6} \in SW_{4n+6}$, since $w_{4n+8} \in SW_{4n+8}$.

Remark 3.1. *The braid $w_6 \in SW_6$ is not the same as the braid which is obtained from w_8 as above. The latter braid is not used in the rest of the paper.*

In the proof of the next lemma, we use some basic facts on *train tracks* of pseudo-Anosov homeomorphisms. See [4, 26] for more details. For a quick review about train tracks, see [21, Section 2.1] which contains terms and basic facts needed in this paper.

Lemma 3.2. *The braid $\underline{w_6} \in B_5$ is pseudo-Anosov, and $\lambda(\underline{w_6})$ equals κ, where κ is the constant given in Proposition 1.3.*

Proof. We choose a train track $\tau \subset D_5$ with non-loop edges $p_1, \cdots, p_6$ as in Figure 12(1), where $c_1, \cdots, c_5$ are punctures of D_5. Each component of $D_5 \setminus \tau$ is either a 1-gon with one puncture, a 3-gon (without punctures), or a 1-gon containing the boundary of the disk (see the illustration of

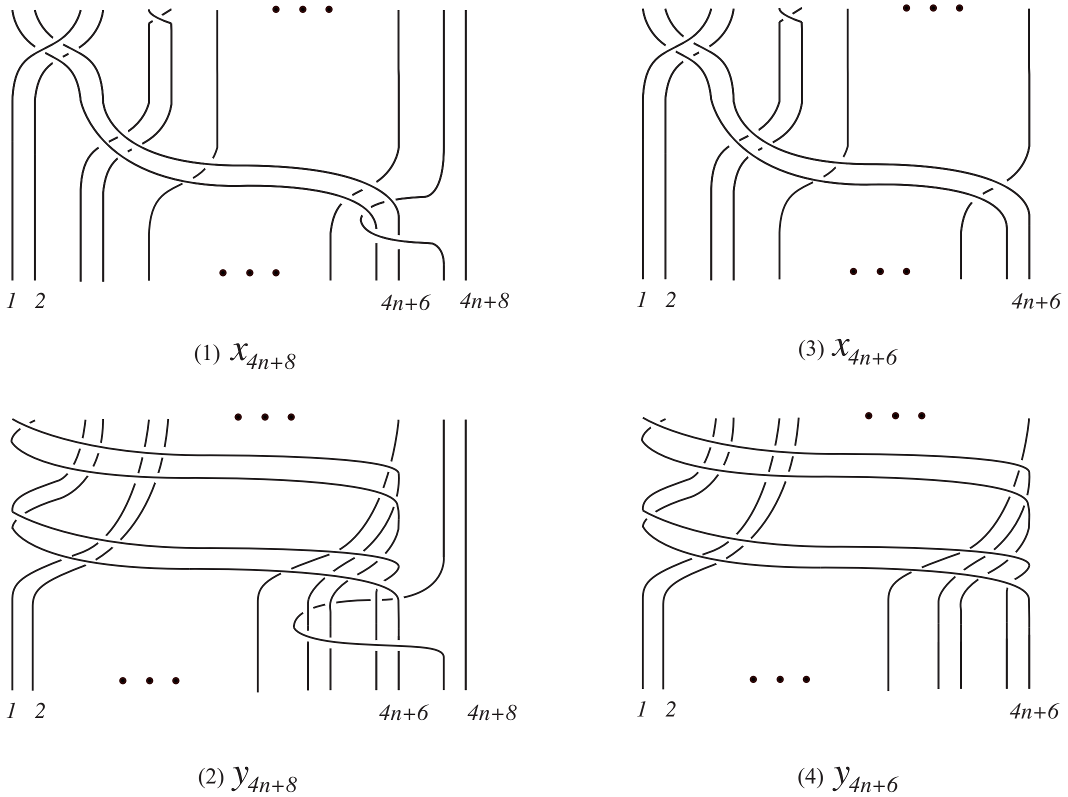


FIGURE 11. (1) x_{4n+8} and (2) $y_{4n+8} \in SB_{(4n+7)} \cap SW_{4n+8}$. (3) x_{4n+6} and (4) $y_{4n+6} \in SW_{4n+6}$. (In (1)–(4), dots indicate parallel strings.)

$\tau \subset D_5 \times \{0\}$ on the bottom of Figure 12(3)). We consider the braid $\underline{w_6}$ with base points $c_1, \cdots, c_5$. We push τ on D_5 along the suspension flow on $S^3 \setminus \mathrm{br}(\underline{w_6})$, then we get the train track τ' on $D_5 \times \{1\}$ illustrated in Figure 12(2). This implies that there exists a representative $f \in \Gamma(\underline{w_6})$ such that $\tau' = f(\tau)$. Here, the edge (p_i) of τ' in Figure 12(2) denotes the image of p_i under f.

We see that $f(\tau)$ is carried by τ, and hence τ is an invariant train track[1] for $\Gamma(\underline{w_6})$. Let $N(\tau)$ be a fibered (tie) neighborhood of τ whose fibers (ties) are segment given by a retraction $\mathfrak{R} : \mathcal{N}(\tau) \to \tau$. Then we get a train track representative $\mathfrak{p} = \mathfrak{R} \circ f|_\tau : \tau \to \tau$ for $\Gamma(\underline{w_6})$. It turns out that $p_1, \cdots, p_6$ are [2]real edges for $\mathfrak{p}$, and other loop edges of τ are periodic under $\mathfrak{p}$, and hence they are infinitesimal edges. The incident matrix $M_{\mathfrak{p}}$ with respect to real edges is given by

$$M_{\mathfrak{p}} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

[1] One can use the software Trains [14] to find invariant train tracks for pseudo-Anosov braids.

[2] We recall the terminology in Bestvina-Handel [4]. An edge e of τ for a train track representative $\mathfrak{p} : \tau \to \tau$ is called *infinitesimal* if e is eventually periodic under $\mathfrak{p}$. Otherwise e is called *real*.

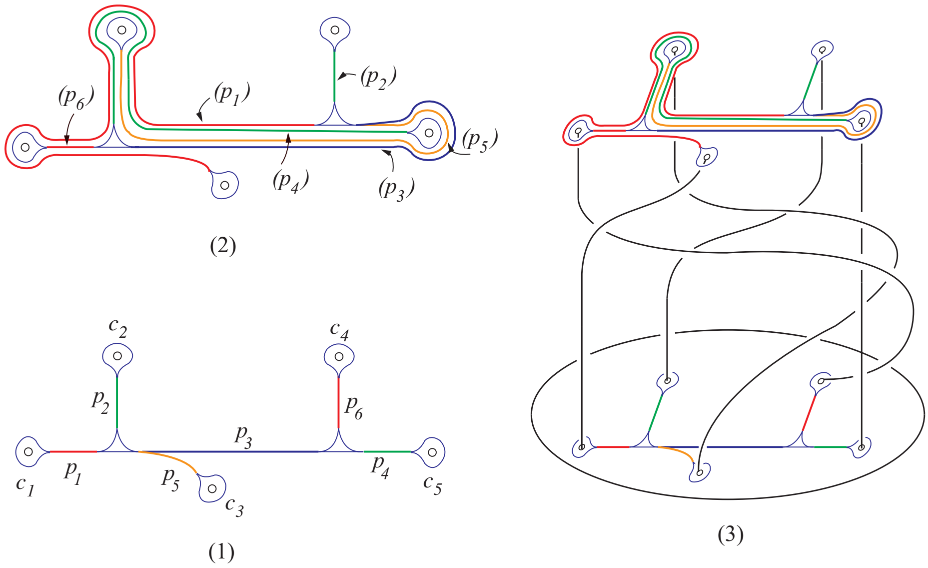


FIGURE 12. (1) $\tau \subset D_5$. (2) $\tau' \subset D_5$. (3) We get $\tau' \subset D_5 \times \{1\}$ by pushing $\tau \subset D_5 \times \{0\}$ along the suspension flow on $S^3 \setminus \mathrm{br}(\underline{w_6})$.

For example, we get ${}^t\begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}$ for the 3rd column of $M_{\mathfrak{p}}$, since $f(p_3)$ passes through p_3 once and p_4 twice in either direction. (See the edge path (p_3) in Figure 12(2).) Since the 5th power $M_{\mathfrak{p}}^5$ is positive, $M_{\mathfrak{p}}$ is Perron-Frobenius and we conclude that $\underline{w_6}$ is pseudo-Anosov. The characteristic polynomial of $M_{\mathfrak{p}}$ equals

$$(t-1)^2(t^4 - 2t^3 - 2t^2 - 2t + 1),$$

and the largest root κ of the second factor gives us $\lambda(\underline{w_6})$. $\square$

The type of singularities of the (un)stable foliation for the pseudo-Anosov homeomorphism $\Phi = \Phi_{w_6} : \Sigma_{0,6} \to \Sigma_{0,6}$ can be read from the topological types of components of $\Sigma_{0,6} \setminus \tau$. See [4, Section 3.4] which describes a construction of invariant measured foliations. Notice that two component of $\Sigma_{0,6} \setminus \tau$ are non punctured 3-gons. The other components are once punctured 1-gons. Thus exactly two points in the interior of $\Sigma_{0,6}$ have 3 prongs and each puncture of $\Sigma_{0,6}$ has a 1 prong.

Observe that $\mathrm{br}(\underline{w_6})$ is the link with 3 components. The following lemma says that complements of both links $\mathrm{br}(\underline{w_6})$ and L_0 (Figure 1) are the same.

Lemma 3.3. *$\mathbb{T}_{\underline{w_6}}$ is homeomorphic to $S^3 \setminus L_0$. In particular $S^3 \setminus L_0$ is a hyperbolic fibered 3-manifold.*

Proof. We use another diagram of L_0 illustrated in Figure 13(1). The link L_0 contains two unknots K and K^0 so that $K \cup K^0$ is the Hopf link. We take the disk D bounded by the longitude of $\mathcal{N}(K)$. We may assume that $L_0 \setminus K (= (L_0)_K)$ intersect with D at the three points indicated by small circles in the same figure. We apply the argument (in the case $m = 2$) of Remark 2.13, and consider the disk twist about D, see Figure 13(2). It turns out that $K \cup T_D(L_0 \setminus K)$ is of the form $\mathrm{br}(\underline{w_6})$, see Figure 13(3). Thus $S^3 \setminus L_0$ is homeomorphic to $S^3 \setminus \mathrm{br}(\underline{w_6}) (= S^3 \setminus (K \cup T_D(L_0 \setminus K)))$.

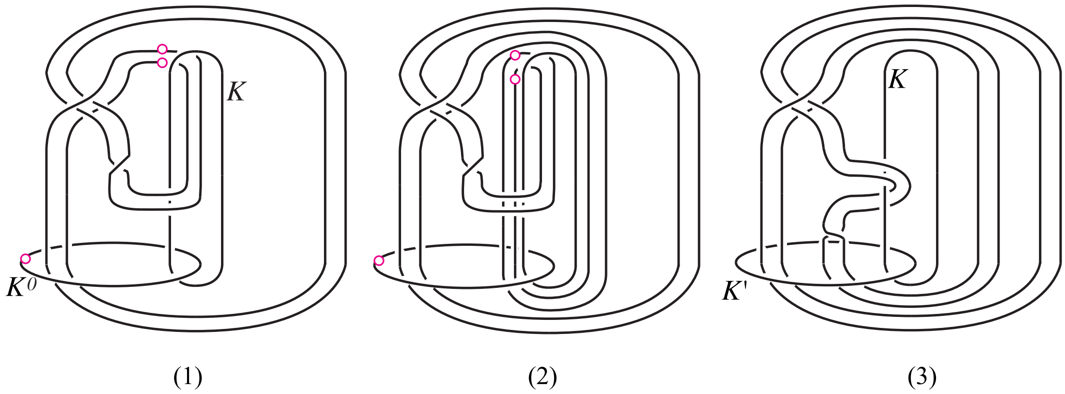


FIGURE 13. (1) A diagram of L_0. ($K \cup K^0$ is the Hopf link.) (2) $K \cup T_D(L_0 \setminus K)$. (3) $\mathrm{br}(\underline{w_6})$.

Since $\mathbb{T}_{\underline{w_6}}$ is homeomorphic to $S^3 \setminus \mathrm{br}(\underline{w_6})$ and $\underline{w_6}$ is pseudo-Anosov by Lemma 3.2, we complete the proof. □

Let $F_{g,n}$ be a compact, connected, orientable surface of genus g with n boundary components. Let M_0 be the exterior of the link L_0. By Lemma 3.3, we let $a_6 \in H_2(M_0, \partial M_0; \mathbb{Z})$ be the homology class of the $F_{0,6}$-fiber of the fibration on M_0 whose monodromy is described by $w_6 \in SB_6$.

Lemma 3.4. *$\mathbb{T}_{\underline{w_{4n+8}}}$ is homeomorphic to $\mathbb{T}_{\underline{w_6}}$ for each $n \geq 0$. In particular $\underline{w_{4n+8}} \in B_{4n+7}$ is pseudo-Anosov for each $n \geq 0$.*

Proof. We prove that $S^3 \setminus \mathrm{br}(\underline{w_{4n+8}})$ is homeomorphic to $S^3 \setminus \mathrm{br}(\underline{w_6})$. To do this, we use Remark 2.13 twice. The braided link $\mathrm{br}(\underline{w_6})$ contains two unknotted components, the braid axis K' and the closure of the last string of $\underline{w_6}$, say K so that $K \cup K'$ is the Hopf link. Let D' be the disk bounded by the longitude of $\mathcal{N}(K')$. Consider the nth power of the disk twist $T^n_{D'}$ for $n \geq 0$. Following Remark 2.13, we take the point $D' \cap K$ as an origin of the rotation of D' for the disk twists $T^n_{D'}$. Then we have the diagram of $K' \cup T^n_{D'}(\mathrm{br}(\underline{w_6}) \setminus K') = K' \cup T^n_{D'}(\mathrm{cl}(\underline{w_6}))$ shown in Figure 14(2). Note that $T^n_{D'}(\mathrm{cl}(\underline{w_6}))$ is isotopic to $\mathrm{cl}(\underline{w_6}\Delta^{2n})$, that is the closure of $\underline{w_6}\Delta^{2n} = \underline{w_6}(\sigma_1\sigma_2\sigma_3\sigma_4)^{5n}$. Thus

$$K' \cup T^n_{D'}(\mathrm{br}(\underline{w_6}) \setminus K') = \mathrm{br}(\underline{w_6}\Delta^{2n}),$$

and $H^n_{D'}$ is a homeomorphism from $\mathcal{E}(\mathrm{br}(\underline{w_6}))$ to $\mathcal{E}(K' \cup T^n_{D'}(\mathrm{br}(\underline{w_6}) \setminus K')) \simeq \mathcal{E}(\mathrm{br}(\underline{w_6}\Delta^{2n}))$. (See Section 2.7 for $H^n_{D'}$.) The closure of the last string of $\underline{w_6}\Delta^{2n} \in B_5$ is an unknot, say K'' which bounds a disk D''. (In the case $n = 0$, we have $K = K''$, $D = D''$ and $H^n_{D'}$ equals the identity map.) We apply the argument of Remark 2.13, and consider the disk twist $T_{D''}$ taking the point of $D'' \cap K'$ as an origin of the rotation of the disk D'' for $T_{D''}$. It turns out that

$$K'' \cup T_{D''}(\mathrm{br}(\underline{w_6}\Delta^{2n}) \setminus K'') = \mathrm{br}(\underline{x_{4n+8}}\ \underline{(y_{4n+8})^n})(= \mathrm{br}(\underline{w_{4n+8}})).$$

To see the equality, we first note that $\mathrm{br}(\underline{w_6}\Delta^{2n}) \setminus (K' \cup K'')$ intersects with D'' at $2 + 4n$ points, see Figure 14(2). We arrange, by an isotopy, $2 + 4n$ intersection points in a line which is parallel to K''. Then view the image $T_{D''}(\mathrm{br}(\underline{w_6}\Delta^{2n}) \setminus K'')$ following the local move under the disk twist $T_{D''}$, see Figure 9(2)(3). (Here m in Figure 9(1) is equal to $2 + 4n$.) Figure 15 explains this procedure in the case $n = 2$.

The above equality implies that $H_{D''}$ is a homeomorphism from $\mathcal{E}(\mathrm{br}(\underline{w_6}\Delta^{2n}))$ to $\mathcal{E}(K'' \cup T_{D''}(\mathrm{br}(\underline{w_6}\Delta^{2n}) \setminus K'')) \simeq \mathcal{E}(\mathrm{br}(\underline{w_{4n+8}}))$. The composition of the maps $H_{D''} \circ H^n_{D'}$ sends $\mathcal{E}(\mathrm{br}(\underline{w_6}))$

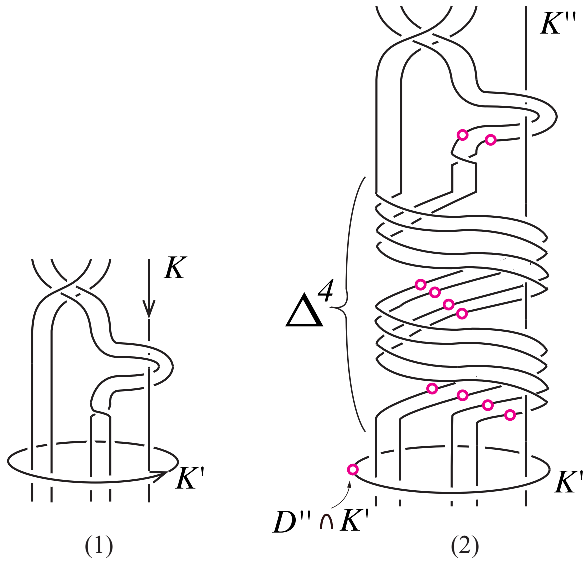


FIGURE 14. Identifying the top and bottom of strings, we get (1) $\mathrm{br}(\underline{w_6})$ and (2) $\mathrm{br}(\underline{w_6}\Delta^{2n})$, where $n = 2$ in this figure. Figure (1) indicates orientations of K and K'. Figure (2) explains $\mathrm{br}(\underline{w_6}\Delta^{2n}) \setminus K''$ intersects with D'' at $3 + 4n$ points (indicated by small circles). Hence $\mathrm{br}(\underline{w_6}\Delta^{2n}) \setminus (K' \cup K'')$ intersects with D'' at $2 + 4n$ points.

to $\mathcal{E}(\mathrm{br}(\underline{w_{4n+8}}))$. Thus $S^3 \setminus \mathrm{br}(\underline{w_6})$ is homeomorphic to $S^3 \setminus \mathrm{br}(\underline{w_{4n+8}})$. Since $\mathbb{T}_{\underline{w_6}} = S^3 \setminus \mathrm{br}(\underline{w_6})$ is hyperbolic, we conclude that by Lemma 2.5, the braid $\underline{w_{4n+8}}$ is pseudo-Anosov. □

By Lemma 3.4, we let $a_{4n+8} \in H_2(M_0, \partial M_0; \mathbb{Z})$ be the homology class of the $F_{0,4n+8}$-fiber of the fibration on M_0 whose monodromy is described by w_{4n+8}. To study properties of such a fibered class, we take a 3-punctured disk $F \simeq \Sigma_{0,4}$ embedded in $S^3 \setminus \mathrm{br}(\underline{w_6})$ so that F is bounded by the unknotted component $K \subset \mathrm{br}(\underline{w_6})$, i.e, F is an interior of the disk D removed the set of 3 points $(\mathrm{br}(\underline{w_6}) \setminus K) \cap D$. To choose an orientation of F, we consider an orientation of the last string of $\underline{w_6}$ from the *top* to the *bottom*. This determines an orientation of K (see Figure 14(1)), and we have an orientation of F induced by K. Let $\bar{F}$ be the oriented disk with 3 holes (i.e, sphere with 4 boundary components) embedded in $\mathcal{E}(\mathrm{br}(\underline{w_6}))$, which is obtained from D removed the interiors of the 3 disks whose centers are the above 3 points. The fibered class a_{4n+8} can be expressed by using a_6 and $[\bar{F}] \in H_2(M_0, \partial M_0; \mathbb{Z})$ as follows.

Lemma 3.5. *We have $a_{4n+8} = (n+1)a_6 + [\bar{F}] \in H_2(M_0, \partial M_0; \mathbb{Z})$ for each $n \geq 0$. In particular, the ray of a_{4n+8} through the origin goes to the ray of a_6 as n goes to ∞.*

Proof. Recall that F_{a_6} is a minimal representative of a_6. In other words, F_{a_6} is a $F_{0,6}$-fiber of the fibration on M_0 associated to a_6. We consider the *oriented sum* $nF_{a_6} + \bar{F}$ which is an oriented surface embedded in M_0. This surface is obtained by the cut and paste construction of parallel n copies of F_{a_6} and a copy of $\bar{F}$. (For the construction of the oriented sum, see [31, p104] or [7, Section 5.1.1].) We take a surface F'' embedded in M_0, which is a disk with $(3+4n)$ holes as follows. Consider the disk D'' bounded by the longitude of $\mathcal{N}(K'')$. Then remove the interiors of small $(3+4n)$ disks from D'' whose centers are the $(3+4n)$ intersection points $(\mathrm{br}(\underline{w_6}\Delta^{2n}) \setminus K'') \cap D''$,

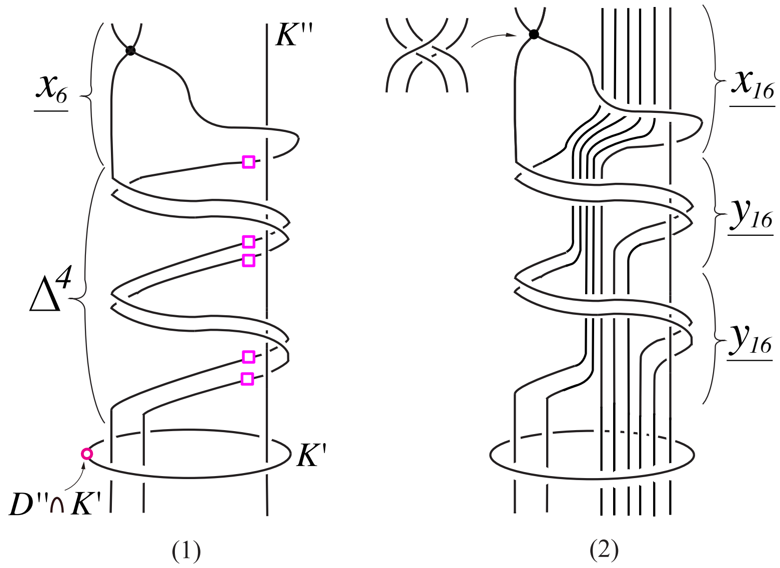


FIGURE 15. Illustrations of two braided links (1) $\mathrm{br}(\underline{w_6}\Delta^4)$ and (2) $\mathrm{br}(\underline{x_{16}}\ \underline{(y_{16})}^2)(= \mathrm{br}(\underline{w_{16}}))$. To get actual two braided links, we duplicate each of the braid strings except the last one. Each small rectangle $\square$ in (1) represents some two small circles given in Figure 14(2). The 'virtual' crossings $\bullet$ in both figures (1)(2) mean $\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2$.

see Figure 14(2). We denote by F'', the resulting disk with $(3+4n)$ holes. We see that the homeomorphism $H^n_{D'} : \mathcal{E}(\mathrm{br}(\underline{w_6})) \to \mathcal{E}(\mathrm{br}(\underline{w_6}\Delta^{2n}))$ in the proof of Lemma 3.4 sends $nF_{a_6} + \bar{F}$ to F''. Hence

$$[F''] = [nF_{a_6} + \bar{F}] = na_6 + [\bar{F}] \in H_2(M_0, \partial M_0; \mathbb{Z}).$$

Obviously $[H^n_{D'}(F_{a_6})] = [F_{a_6}](= a_6)$. We now consider the oriented sum $H^n_{D'}(F_{a_6}) + F''$. Then $H_{D''} : \mathcal{E}(\mathrm{br}(\underline{w_6}\Delta^{2n})) \to \mathcal{E}(\mathrm{br}(\underline{w_{4n+8}}))$ in the proof of Lemma 3.4 sends $H^n_{D'}(F_{a_6}) + F''$ to the $F_{0,4n+8}$-fiber of the fibration on M_0 associated to a_{4n+8}. Putting all things together, we have

$$a_{4n+8} = [H^n_{D'}(F_{a_6}) + F''] = [H^n_{D'}(F_{a_6})] + [F''] = a_6 + na_6 + [\bar{F}] = (n+1)a_6 + [\bar{F}].$$

Thus

$$\lim_{n\to\infty} \tfrac{a_{4n+8}}{n+1} = \lim_{n\to\infty} (a_6 + \tfrac{[\bar{F}]}{n+1}) = a_6.$$

This completes the proof. $\square$

Since the normalized entropy function Ent is constant on each ray through the origin in the fibered cone, Lemmas 3.2, 3.3, 3.4 and 3.5 tell us that

(3.1) $$\lim_{n\to\infty} |\chi(F_{0,4n+8})| \log(\lambda(w_{4n+8})) = |\chi(F_{0,6})| \log(\lambda(w_6)) = 4\log\kappa.$$

Since $|\chi(F_{0,4n+8})| = 4n+6$ goes to ∞ as n does, and the right-hand side is constant, we conclude that

(3.2) $$\lim_{n\to\infty} \log\lambda(w_{4n+8}) = 0.$$

Let Ω be a fibered face of M_0 such that $a_6 \in int(C_\Omega)$. Lemma 3.5 implies that $a_{4n+8} \in int(C_\Omega)$ for n large. We shall prove in Lemma 3.6 that the fibered class a_{4n+8} lies in $int(C_\Omega)$ for *each* $n \geq 0$. Recall that K and K' are the unknotted components of $\mathrm{br}(\underline{w_6})$. We choose an orientation of K' as in Figure 14(1). For an embedded surface $\hat{S}$ in $M_0 = \mathcal{E}(\mathrm{br}(\underline{w_6}))$, we denote by $\partial_K(\hat{S})$ and $\partial_{K'}(\hat{S})$, the components of the boundary $\partial\hat{S}$ of $\hat{S}$ which lie on $\partial\mathcal{N}(K)$ and $\partial\mathcal{N}(K')$ respectively. Let $\hat{\Phi} : F_{0,6} \to F_{0,6}$ be a pseudo-Anosov homeomorphism whose mapping class $[\hat{\Phi}]$ is described by w_6. (Thus $\mathbb{T}_{[\hat{\Phi}]}$ is homeomorpphic to $\mathcal{E}(\mathrm{br}(\underline{w_6})) \simeq M_0$.)

Lemma 3.6. *We have $a_{4n+8} \in int(C_\Omega)$ for each $n \geq 0$.*

Proof. The minimal representative F_{a_6} is transverse to the suspension flow $\hat{\Phi}^t$ obviously, but $\bar{F}$ is not, since both $\partial_K\bar{F}$ and $\partial_{K'}\bar{F}$ are parallel to flow lines of $\hat{\Phi}^t$. (See also Figure 16(3) and its caption.) We prove that the oriented sum $(n+1)F_{a_6} + \bar{F}$ for $n \geq 0$ is transverse to $\hat{\Phi}^t$ (up to isotopy) and it intersects every flow line. This means that $(n+1)F_{a_6} + \bar{F}$ is a cross-section to $\hat{\Phi}^t$ for $\mathcal{E}(\mathrm{br}(\underline{w_6}))$. By Theorem 2.2(3), we have $a_{4n+8} = [(n+1)F_{a_6} + \bar{F}] \in int(C_\Omega)$.

By the proof of Lemma 3.5, $(n+1)F_{a_6} + \bar{F}$ becomes a fiber of the fibration on M_0 associated to a_{4n+8}. Hence we may assume that

$$F_{a_{4n+8}} = (n+1)F_{a_6} + \bar{F}. \tag{3.3}$$

We have the meridian and longitude basis $\{m_K, \ell_K\}$ for $\partial\mathcal{N}(K)$ and $\{m_{K'}, \ell_{K'}\}$ for $\partial\mathcal{N}(K')$. It follows that

$$\begin{aligned} [\partial_K F_{a_{4n+8}}] &= (n+1)m_K + \ell_K \neq \pm\ell_K \in H_1(\partial\mathcal{N}(K)) \text{ and} \\ [\partial_{K'} F_{a_{4n+8}}] &= (n+1)\ell_{K'} + m_{K'} \neq \pm m_{K'} \in H_1(\partial\mathcal{N}(K')). \end{aligned}$$

This implies that both $\partial_K F_{a_{4n+8}}$ and $\partial_{K'} F_{a_{4n+8}}$ are transverse to every flow line of $\hat{\Phi}^t$, since $[\partial_K\bar{F}] = \ell_K$ and $[\partial'_K\bar{F}] = m_{K'}$. By (3.3), $F_{a_{4n+8}}$ is an oriented sum obtained from the $(n+1)$ copies of F_{a_6} and the surface $\bar{F}$. Hence the shape of the embedded surface $F_{a_{4n+8}}$ in M_0 is of a 'spiral staircase' which turns round $(n+1)$ times along ℓ_K. Therefore $F_{a_{4n+8}}$ is transverse to $\hat{\Phi}^t$. Moreover $F_{a_{4n+8}}$ intersects every flow line of $\hat{\Phi}^t$ (by construction of $(n+1)F_{a_6} + \bar{F}(= F_{a_{4n+8}})$), since so does F_{a_6}. This completes the proof. □

Lemma 3.7. *If $n \geq 1$, then w_{4n+6} is pseudo-Anosov and the equality $\lambda(w_{4n+6}) = \lambda(w_{4n+8})$ holds. In particular $\mathbb{T}_{w_{4n+6}}$ is a hyperbolic fibered 3-manifold obtained from $\mathbb{T}_{w_{4n+8}}$ by Dehn fillings about the two cusps along the boundary slopes of the fiber associated to $a_{w_{4n+8}}$.*

We work on the cusped 3-manifold $\mathbb{T}_{w_6} = \mathbb{T}_{\underline{w_6}} \simeq S^3 \setminus L_0$ instead of M_0 with boundary. To prove Lemma 3.7, we shall construct an invariant train track for $\Gamma(w_{4n+8})$ concretely, and study types of singularities of the unstable foliation $\mathcal{F}_{w_{4n+8}}$ of the pseudo-Anosov homeomorphism $\Phi_{w_{4n+8}} : \Sigma_{0,4n+8} \to \Sigma_{0,4n+8}$ which represents $\Gamma(w_{4n+8})$. The same idea in [21, Section 3] for the construction of train tracks can be used. We repeat a similar argument modifying some claims of [21] in a suitable way for the present paper. Hereafter we use basic properties on branched surfaces. See [25] for more details on the theory of branched surfaces.

By using the pseudo-Anosov homeomorphism $\Phi = \Phi_{w_6} : \Sigma_{0,6} \to \Sigma_{0,6}$, we build the mapping torus

$$\mathbb{T}_{w_6} = \Sigma_{0,6} \times \mathbb{R}/_{(x,t+1)=(\Phi(x),t)}$$

for $x \in \Sigma_{0,6}$ and $t \in \mathbb{R}$. Given a subset $U \subset \Sigma_{0,6}$, we define $U^t \subset \mathbb{T}_{w_6}$ to be the image $U \times \{t\}$ under the projection $p : \Sigma \times \mathbb{R} \to \mathbb{T}_{w_6}$. We have an orientation preserving homeomorphism

$$h : S^3 \setminus \mathrm{br}(\underline{w_6}) \to \mathbb{T}_{w_6}.$$

Recall that F is an oriented 4-punctured sphere in $S^3 \setminus \mathrm{br}(\underline{w_6})$. Choose an orientation of $\Sigma_{0,6}^v = \Sigma_{0,6} \times \{v\}$ for each $v \in \mathbb{R}$ so that its normal direction coincides with the flow Φ^t direction. We shall capture the image $h(F)$ in $\mathbb{T}_{w_6}$. To do this, let s be a segment between the punctures c_5 and c_6. Since Φ fixes c_5 and c_6 pointwise (see the last two strings of w_6 in Figure 10(1)), $s \cup \Phi(s)$ bounds a 2-gon. Viewing the image $\Phi(s)$, we see that the 2-gon contains the punctures c_1 and c_2, see Figure 16. Such a 2-gon removed c_1 and c_2 is denoted by $S \subset \Sigma_{0,6}$. The segment $s^0 = s \times \{0\}$ is connected to $s^1 = s \times \{1\}$, and these segments make a flowband $J = [s^0, s^1]$ which is illustrated in Figure 16(3). Since $s^1 = (\Phi(s))^0$ in $\mathbb{T}_{w_6}$, the union

$$S^0 \cup J (= (S \times \{0\}) \cup J) \subset \mathbb{T}_{w_6}$$

defines a 4-punctured sphere, see Figure 16(3). The set of punctures of F maps to the set of punctures of $S^0 \cup J$ under h. This tells us that $h(F) = S^0 \cup J$ up to isotopy. For simplicity, $S^0 \cup J$ in $\mathbb{T}_{w_6}$ is denoted by F.

We choose $0 < \epsilon < 2\epsilon < 1$. We push $F(= S^0 \cup J) \subset \mathbb{T}_{w_6}$ along the flow lines for ϵ times so that the resulting 4-punctured sphere, denoted by F^ϵ, satisfies

$$F^\epsilon \cap \Sigma_{0,6}^\epsilon = S^\epsilon (= S \times \{\epsilon\}),$$

see Figure 18(2) for S^ϵ. By using F^ϵ and $\Sigma_{0,6}^{2\epsilon} = \Sigma_{0,6} \times \{2\epsilon\}$ which corresponds to a fiber F_{a_6} of the fibration on M_0, we set

$$\widehat{\mathcal{B}} = F^\epsilon \cup \Sigma_{0,6}^{2\epsilon}.$$

We get the branched surface $\mathcal{B}$ from $\widehat{\mathcal{B}}$ (which agrees with the orientations of F^ϵ and $\Sigma_{0,6}^{2\epsilon}$) after we modify the flowband

$$[s^\epsilon, (\Phi(s))^\epsilon] = [s^\epsilon, s^1] \cup [(\Phi(s))^0, (\Phi(s))^\epsilon]$$

of F^ϵ near the segment $s^{2\epsilon} = F^\epsilon \cap \Sigma_{0,6}^{2\epsilon} \in [s^\epsilon, s^1]$. (cf. For the illustration of this modification, see [21, Figure 14].) By Lemmas 3.4 and 3.5, there exists a $\Sigma_{0,4n+8}$-fiber of the fibration on $\mathbb{T}_{w_6}$ with the monodromy $\Gamma(w_{4n+8})$. We denote such a fiber by $\Sigma_{w_{4n+8}}$. By (3.3), we have

$$\Sigma_{w_{4n+8}} = F + (n+1)\Sigma_{0,6}. \tag{3.4}$$

By the construction of $\mathcal{B}$, we see that $\Sigma_{w_{4n+8}}$ is carried by $\mathcal{B}$.

Let $\widehat{\mathcal{F}} \subset \mathbb{T}_{w_6}$ be the suspension of the unstable foliation $\mathcal{F}$ for Φ. We may assume that the train track $\tau \subset D_5$ in the proof of Lemma 3.2 lies on $\Sigma_{0,6}$. Then τ is an invariant train track for $\Gamma(w_6) = [\Phi]$. Theorem 2.2(1)(2) and Lemma 3.6 imply the following.

Lemma 3.8. *The pseudo-Anosov homeomorphism $\Phi_{w_{4n+8}} : \Sigma_{w_{4n+8}} \to \Sigma_{w_{4n+8}}$ is precisely the first return map: $\Sigma_{w_{4n+8}} \to \Sigma_{w_{4n+8}}$ of Φ^t. Moreover $\mathcal{F}_{w_{4n+8}} = \widehat{\mathcal{F}} \cap \Sigma_{w_{4n+8}}$.*

We turn to the construction of the branched surface $\mathcal{B}_\Omega$ which carries $\widehat{\mathcal{F}}$. First of all, we note that τ is obtained from $\Phi(\tau)$ by *folding* edges (or *zipping* edges), see Figure 17. We choose a family of train tracks $\{\tau_t\}_{0 \le t \le 1}$ on $\Sigma_{0,6}$ as follows.

(1) $\tau_0 = \Phi(\tau)$.
(2) τ_t at $t = \epsilon$ is a train track illustrated in Figure 17(middle in the left column).
(3) $\tau_t = \tau$ for $2\epsilon \le t \le 1$.
(4) If $0 \le s < t \le 2\epsilon$, then $\tau_t = \tau_s$ or τ_t is obtained from τ_s by folding edges between a cusp of τ_s.

We let

$$\mathcal{B}_\Omega = \bigcup_{0 \le t \le 1} \tau_t \times \{t\} \subset \mathbb{T}_{w_6}.$$

Since $\tau_1 \times \{1\} = \tau_0 \times \{0\}$ in $\mathbb{T}_{w_6}$ (see the above conditions (1)(3)), it follows that $\mathcal{B}_\Omega$ is a branched surface. Since the invariant train track τ carries the unstable foliation $\mathcal{F}$, we see that $\mathcal{B}_\Omega$ carries $\widehat{\mathcal{F}}$. It is not hard to see that $\mathcal{B}_\Omega$ is transverse to the previous branched surface $\mathcal{B}$ (up to isotopy). Let

$$\tau_{4n+8} = \varSigma_{w_{4n+8}} \cap \mathcal{B}_\Omega, \tag{3.5}$$

which is a branched 1-manifold, see Figure 19(1). Since $\varSigma_{w_{4n+8}}$ is carried by $\mathcal{B}$, we may put $(n+1)$ copies of $\varSigma_{0,6}$ which is a part of $\varSigma_{w_{4n+8}}$ (see (3.4)) into $\varSigma_{0,6} \times (2\epsilon, 1)$. We may also assume that a copy of F which is another part of $\varSigma_{w_{4n+8}}$ satisfies that $S^\epsilon = F \cap \varSigma_{0,6}^\epsilon$. Then intersections $\varSigma_{0,6}^{2\epsilon} \cap \mathcal{B}_\Omega$ and $S^\epsilon \cap \mathcal{B}_\Omega$ (see Figure 18) together with $s^{2\epsilon} = F^\epsilon \cap \varSigma_{0,6}^{2\epsilon}$ determine τ_{4n+8}. More concretely, τ_{4n+8} is constructed from a copy of $S^\epsilon \cap \mathcal{B}_\Omega$ and $(n+1)$ copies of $\varSigma_{0,6}^{2\epsilon} \cap \mathcal{B}_\Omega$, see (3.4), (3.5). We label $q_1,\ q_2,\ q_3,\ p_1^{(j)}, p_2^{(j)}, \cdots, p_6^{(j)}$ $(1 \leq j \leq n+1)$ for non-loop edges of τ_{4n+8}. Notice that edges of $q_1,\ q_2,\ q_3$ come from the edges of $S^\epsilon \cap \mathcal{B}_\Omega$ and the rest of non-loop edges come from the edges of $\varSigma_{0,6}^{2\epsilon} \cap \mathcal{B}_\Omega$. The $n+1$ edges $p_i^{(1)}, \cdots, p_i^{(n+1)}$ for each $1 \leq i \leq 6$ originate in the edges of $\varSigma_{0,6}^{2\epsilon} \cap \mathcal{B}_\Omega$. If we fix i, then the number of the labeling (j) in $p_i^{(j)}$ increases along the flow direction. We call $p_1^{(n+1)}, \cdots, p_6^{(n+1)}$ the top edges, q_1, q_2, q_3 the bottom edges, and $p_1^{(1)}, \cdots, p_6^{(1)}$ the second bottom edges etc. See Figure 19(1).

Lemma 3.9. *The branched* 1*-manifold* τ_{4n+8} *is a train track, and the unstable foliation* $\mathcal{F}_{w_{4n+8}}$ *of* $\Phi_{w_{4n+8}}$ *is carried by* τ_{4n+8}.

Proof. Since a_{4n+8} lies in the same fibered cone as a_6 (Lemma 3.6), $\mathcal{F}_{w_{4n+8}}$ is given by $\widehat{\mathcal{F}} \cap \varSigma_{w_{4n+8}}$ (Lemma 3.8) and the suspension $\widehat{\mathcal{F}}_{w_{4n+8}}$ of $\mathcal{F}_{w_{4n+8}}$ by $\Phi_{w_{4n+8}}$ is isotopic to $\widehat{\mathcal{F}}$, see [24, Corollary 3.2]. Since $\widehat{\mathcal{F}}$ is carried by $\mathcal{B}_\Omega$, so is $\widehat{\mathcal{F}}_{w_{4n+8}}$. Thus $\mathcal{F}_{w_{4n+8}}$ is carried by $\varSigma_{w_{4n+8}} \cap \mathcal{B}_\Omega (= \tau_{4n+8})$.

Observe that each component of $\varSigma_{w_{4n+8}} \setminus \tau_{4n+8}$ is either a 1-gon with one of the punctures $c_1, \cdots, c_{4n+6}$, an $(n+2)$-gon with the puncture c_{4n+7}, an $(n+1)$-gon with the puncture c_{4n+8} or a 3-gon without punctures. ("Vertical" $(n+2)$ edges of τ in Figure 19(left) bound an $(n+2)$-gon containing c_{4n+7}.) Since no bigon component is contained in $\varSigma_{w_{4n+8}} \setminus \tau_{4n+8}$, we conclude that τ_{4n+8} is a train track which carries $\mathcal{F}_{w_{4n+8}}$. □

Since $\Phi_{w_{4n+8}} : \varSigma_{w_{4n+8}} \to \varSigma_{w_{4n+8}}$ is the first return map for Φ^t, conditions (1)–(4) in the family $\{\tau_t\}_{0 \leq t \leq 1}$ ensure that the image of τ_{4n+8} under the first return map $\Phi_{w_{4n+8}}$ is carried by τ_{4n+8}, that is τ_{4n+8} is invariant under $\Gamma(w_{4n+8}) = [\Phi_{w_{4n+8}}]$. Figure 19(2) shows the image of edges of τ_{4n+8} under $\Phi_{w_{4n+8}}$. The top edges $p_1^{(n+1)}, \cdots, p_6^{(n+1)}$ map to the edge paths of the bottom and second bottom edges under the first return map. This is because these edges $p_i^{(n+1)}$'s arrive at $p_i^{(n+1)} \times \{1\} \subset \varSigma_{0,6}^1$ first along the flow lines. The identity $p_i^{(n+1)} \times \{1\} = \Phi_{w_6}(p_i^{(n+1)}) \times \{0\}$ holds in $\mathbb{T}_{w_6}$. We get the image of $p_i^{(n+1)}$ under the first return map when we push $\Phi_{w_6}(p_i^{(n+1)}) \times \{0\}$ along the flow Φ^t until it hits the fiber $\varSigma_{w_{4n+8}}$. The rest of non-loop edges except q_3 map to the *above edge* in $\mathbb{T}_{w_6}$ along the suspension flow (cf. Figure 12(1)(2)). For example, $p_1^{(n)}$ maps to $p_1^{(n+1)}$, and q_1 maps to $p_1^{(1)}$. The edge q_3 maps to $p_3^{(1)}$ and $p_4^{(1)}$.

Let $\mathfrak{p}_{4n+8} : \tau_{4n+8} \to \tau_{4n+8}$ be the train track representative under $[\Phi_{w_{4n+8}}]$. One can check that all non-loop edges of τ_{4n+8} are real edges for $\mathfrak{p}_{4n+8}$. The incident matrix $M_{\mathfrak{p}_{4n+8}}$ with respect to real edges must be Perron-Frobenius, since τ_{4n+8} carries the unstable foliation of the pseudo-Anosov homeomorphism $\Phi_{w_{4n+8}}$. Thus the largest eigenvalue of $M_{\mathfrak{p}_{4n+8}}$ gives us $\lambda(w_{4n+8})$.

Lemma 3.10. *For each* $n \geq 0$, $\lambda(w_{4n+8})$ *equals the largest root of the polynomial*

$$t^{6n+9} - 2t^{5n+8} - 2t^{5n+7} + 3t^{4n+6} + 3t^{2n+3} - 2t^{n+2} - 2t^{n+1} + 1.$$

The proof of Lemma 3.10 can be done by the computation of the characteristic polynomial of $M_{\mathfrak{p}_{4n+8}}$. Alternatively one can compute $\lambda(w_{4n+8})$ from the *clique polynomial* of the *curve complex* G_{4n+8} associated to the directed graph Γ_{4n+8} for $\mathfrak{p}_{4n+8} : \tau_{4n+8} \to \tau_{4n+8}$. In general, the curve complex G associated to a directed graph Γ is an undirected graph together with the weight on the set of vertices $V(G)$ of G. A consequence of results of McMullen in [23] tells us that $\frac{1}{\lambda(w_{4n+8})}$ equals the smallest positive root of the clique polynomial of G_{4n+8}. In this case, the topological types of the undirected graph G_{4n+8} (ignoring its weight on the set of vertices) do not depend on n. This makes the computation of the clique polynomial of G_{4n+8} straightforward. One can also prove Lemma 3.10 from the computation of the Teichmüler polynomial associated to the fibered face Ω by using the invariant train track for $\Gamma(\underline{w_6})$. For Teichmüler polynomials, see [24].

Lemmas 3.2, 3.7, 3.10 allow us to compute $\lambda(w_{2k})$ for $k \geq 3$. See Table 1.

TABLE 1. Computation of $\lambda(w_{2k})$ for small k.

	$\lambda(w_6) \approx 2.89005$
	$\lambda(w_8) \approx 2.26844$
$n \geq 1$	$\lambda(w_{4n+8}) = \lambda(w_{4n+6})$
1	≈ 1.56362
2	≈ 1.36516
3	≈ 1.27074
4	≈ 1.21532
5	≈ 1.17882
6	≈ 1.15293
7	≈ 1.13361
8	≈ 1.11863
9	≈ 1.10668
10	≈ 1.09692
11	≈ 1.08879
12	≈ 1.08193
13	≈ 1.07605
14	≈ 1.07096
15	≈ 1.06651

Remember that types of singularities of $\mathcal{F}_{w_{2n+8}}$ can be read from the shapes of the components of $\Sigma_{w_{4n+8}} \setminus \tau_{4n+8}$. From the proof of Lemma 3.9, we have the following.

Lemma 3.11. *The unstable foliation $\mathcal{F}_{w_{4n+8}}$ of $\Phi_{w_{4n+8}}$ has properties such that the last puncture c_{4n+8} has $(n+1)$ prongs and the second last puncture c_{4n+7} has $(n+2)$ prongs.*

Proof of Lemma 3.7. If $n \geq 1$, then $\mathcal{F}_{w_{4n+8}}$ has the property such that last two punctures of $\Sigma_{0,4n+8}$ have more than 1 prong (Lemma 3.11). Thus $\mathcal{F}_{w_{4n+8}}$ extends to the unstable foliation on $\Sigma_{0,4n+6}$ by filling last two punctures. This means that the pseudo-Anosov homeomorphism $\Phi_{w_{4n+8}} : \Sigma_{0,4n+8} \to \Sigma_{0,4n+8}$ extends to the pseudo-Anosov homeomorphism on $\Sigma_{0,4n+6}$ which represents $\Gamma(w_{4n+6})$ with the same dilatation as $\Phi_{w_{4n+8}}$.

The latter statement on $\mathbb{T}_{w_{4n+6}}$ in Lemma 3.7 is clear from the definition of the braid w_{4n+6}. □

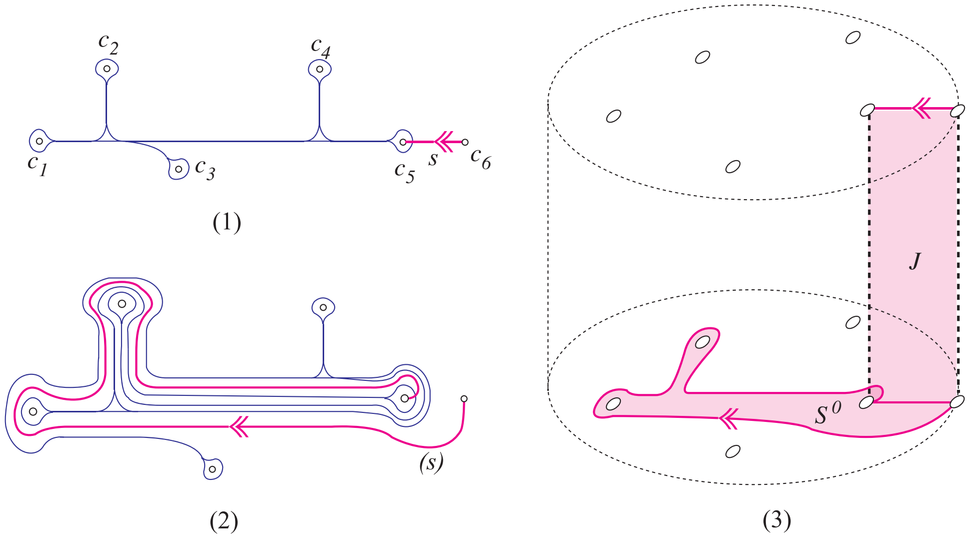


FIGURE 16. (1) Segment s, (2) $(s) := \Phi(s)$ up to isotopy relative to endpoints of $\Phi(s)$. See also Figure 12(2). (3) Surface $F = S^0 \cup J$ (shaded region) in $\mathbb{T}_{w_6}$. To get $\mathbb{T}_{w_6}$, we glue $\Sigma_{0,6} \times \{1\}$ and $\Sigma_{0,6} \times \{0\}$ by $\Phi \in \Gamma(w_6)$. Two "vertical" dotted lines are the orbits of c_5 and c_6 for Φ^t. Dotted two circles (boundaries of the disks) correspond with the last punctures of $\Sigma_{0,6} \times \{1\}$ and $\Sigma_{0,6} \times \{0\}$.

Proof of Proposition 1.3. By Lemma 3.7 together with (3.1), (3.2), we have

$$\lim_{n \to \infty} 2(2n+4)\log(\lambda(w_{4n+8})) = \lim_{n \to \infty} 2(2n+3)\log(\lambda(w_{4n+6})) = 4\log\kappa.$$

Both sides divided by 2 give us the desired claim. $\square$

Finally we ask the following question.

Question 3.12 (cf. Question 4.2 in [16])**.** *We know from the proof of Proposition 1.3 and from Lemma 3.2 that* $4\log(\frac{1+\sqrt{5}}{2} + \frac{\sqrt{2+2\sqrt{5}}}{2}) = 2(2\log\lambda(\underline{w_6}))$ *is an accumulation point of the following set of normalized entropies of pseudo-Anosov elements in* $\mathcal{H}(\mathbb{H}_g)$*:*

$$\{\mathrm{Ent}(\phi) = (2g-2)\log\lambda(\phi) \mid \phi \in \mathcal{H}(\mathbb{H}_g) \textit{ is pseudo-Anosov}, \ g \geq 2\}.$$

Is the accumulation point $4\log(\frac{1+\sqrt{5}}{2} + \frac{\sqrt{2+2\sqrt{5}}}{2})$ *the smallest one?*

APPENDIX A. A FINITE PRESENTATION OF $\mathcal{H}(\mathbb{H}_g)$

In this appendix, we will prove some claims referred in Sections 2.5 and 2.6 and determine a finite presentation for $\mathcal{H}(\mathbb{H}_g)$ (Theorem A.8).

Here we make some remarks on the spherical wicket group SW_{2n}. Let $\mathcal{SA}_n$ be the space of configurations of n disjoint smooth unknotted and unlinked arcs in D^3 with endpoints on ∂D^3.

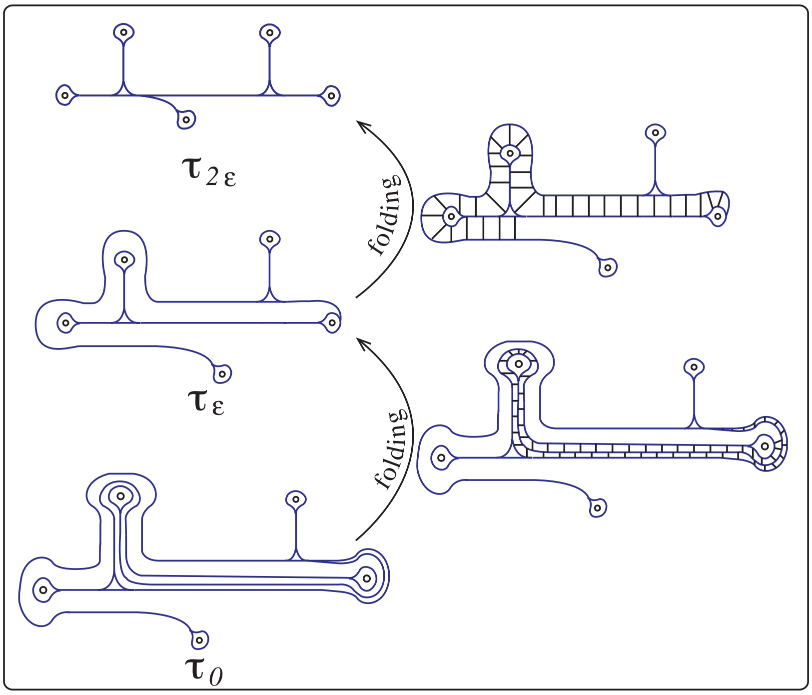


FIGURE 17. Left column shows train tracks $\tau_0 = \Phi(\tau)$ (bottom), τ_ϵ (middle), $\tau_{2\epsilon} = \tau$ (top). τ is obtained from $\Phi(\tau)$ by folding edges between a cusp. Right column explains how to fold edges from τ_0 to τ_ϵ (bottom) and from τ_ϵ to $\tau_{2\epsilon}$ (top).

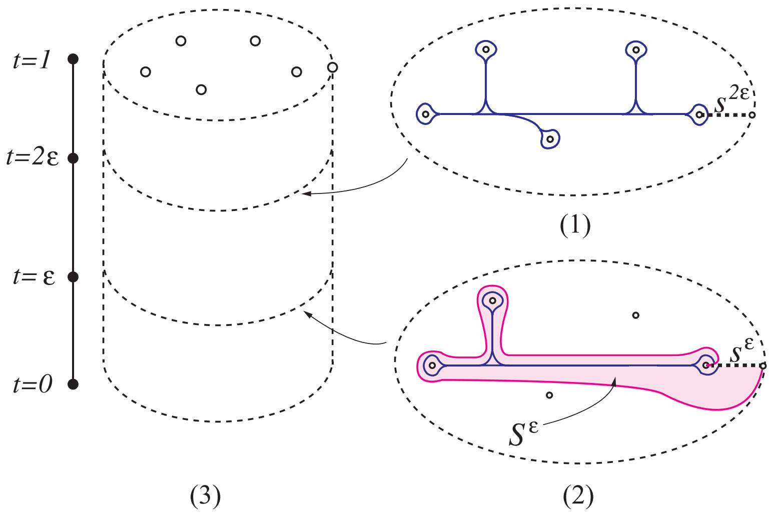
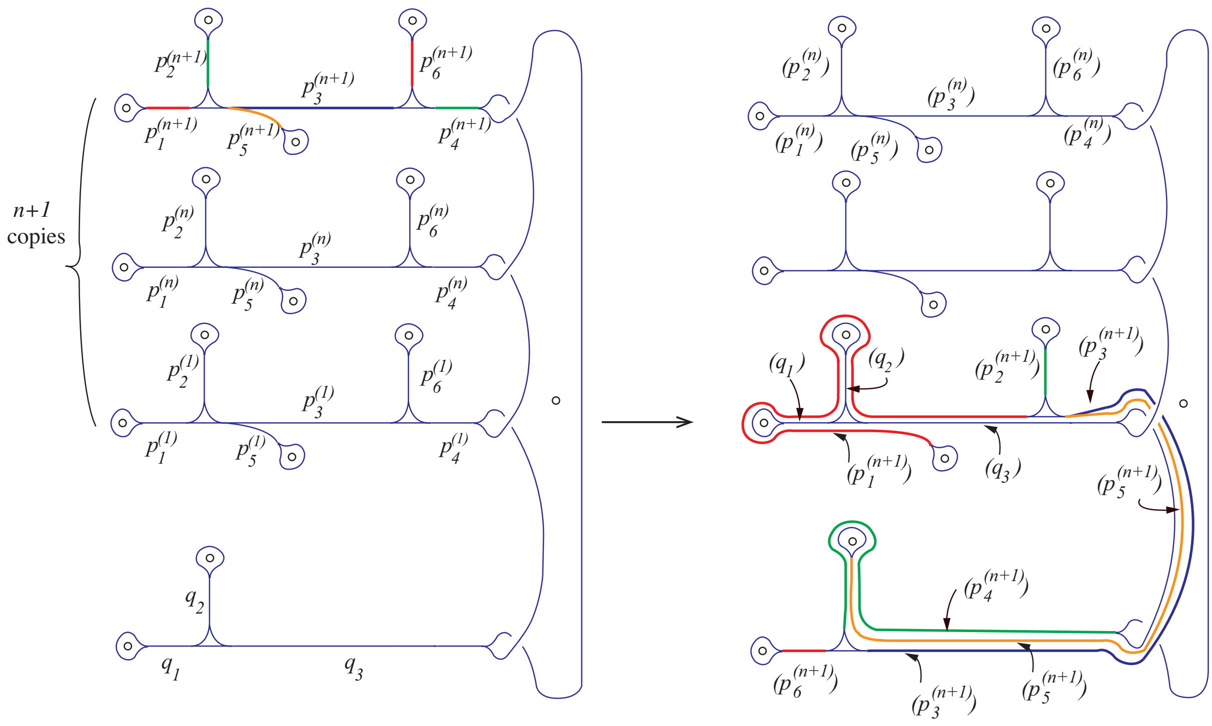
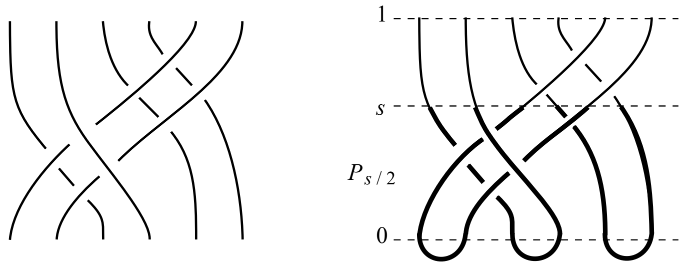


FIGURE 18. (1) $s^{2\epsilon}$ (broken line) and $\Sigma_{0,6}^{2\epsilon} \cap \mathcal{B}_\Omega$ (see also Figure 17(top of left column)). (2) s^ϵ (broken line) and $S^\epsilon \cap \mathcal{B}_\Omega \subset S^\epsilon$ (see also Figure 17(middle of left column). (3) $\Sigma_{0,6} \times [0,1] (\supset \bigcup_{0 \le t \le 1} \tau_t \times \{t\})$.



(1) (2)

FIGURE 19. (1) τ_{4n+8} and (2) its image $\Phi_{w_{4n+8}}(\tau_{4n+8})$ up to isotopy, where $n = 2$ in this figure. Small circles indicate all punctures of $\Sigma_{0,4n+8}$ but the last one. The last puncture corresponds to ∂D of a disk D such that $\tau_{4n+8} \subset D$.



(1) (2)

FIGURE 20. (1) $b \in SW_{2n}$. (2) A path in $\mathcal{SA}_n$ corresponding to b.

Brendle-Hatcher [5] defined the spherical wicket group to be $\pi_1(\mathcal{SA}_n)$. We shall see in Proposition A.1 that $\pi_1(\mathcal{SA}_n) \simeq SW_{2n}$. In [5, p.156–157], it is shown that the natural homomorphism from $\pi_1(\mathcal{SA}_n)$ to SB_{2n} induced by the map sending a configuration of n arcs to the configuration of its endpoints is injective. By this injection, we regard $\pi_1(\mathcal{SA}_n)$ as the subgroup of SB_{2n}.

The wicket group W_{2n} is defined as a subgroup of the braid group B_{2n} in the same way as the definition of SW_{2n} given in Section 2.5. Let $\mathcal{A}_n$ be the space of configurations of n disjoint smooth unknotted and unlinked arcs in $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$ with endpoints on $\partial\mathbb{R}^3_+$. In the same way as $\pi_1(\mathcal{SA}_n)$, we regard $\pi_1(\mathcal{A}_n)$ as a subgroup of B_{2n}. Brendle-Hatcher [5, Propositions 3.2, 3.6] showed that $\pi_1(\mathcal{A}_n)$ is generared by r_i, s_i ($i \in \{1, \cdots, n-1\}$), t_j ($j \in \{1, \cdots, n\}$) shown in Figure 5. In the beginning of Section 6 in [5], it is observed that $\pi_1(\mathcal{SA}_n)$ is the quotient of $\pi_1(\mathcal{A}_n)$ by the normal closure $\langle\langle\vartheta\rangle\rangle$ of $\{\vartheta\}$, where $\vartheta = t_1 s_1 s_2 \cdots s_{n-1} r_{n-1}^{-1} \cdots r_2^{-1} r_1^{-1} t_1$. Especially we see that $\pi_1(\mathcal{SA}_n)$ is generated by r_i, s_i and t_j as above.

Proposition A.1. *$SW_{2n} = \pi_1(\mathcal{SA}_n)$.*

Proof. Recall that r_i, s_i ($i \in \{1, \cdots, n-1\}$), t_j ($j \in \{1, \cdots, n\}$) are elements of SW_{2n}. Hence $\pi_1(\mathcal{SA}_n) \subset SW_{2n}$. On the other hand, for $b \in SW_{2n}$, we define a closed path $P_t (0 \leq t \leq 1)$ in $\mathcal{SA}_n$ with a base point corresponding to n trivial arcs in D^3 as follows: P_0 and P_1 are n trivial arcs in D^3, $P_{s/2} (0 \leq s \leq 1)$ is n arcs indicated by the thick arcs in Figure 20(2) and the path from $P_{1/2} = {}^b\mathbf{A}$ to P_1 is an isotopy between ${}^b\mathbf{A}$ and $\mathbf{A}$ fixing end points. Then the sequence of endpoints of the path P_t is a closed path in the configuration space of $2n$ points in S^2 whose homotopy class is the braid b. This shows $b \in \pi_1(\mathcal{SA}_n)$. Hence $SW_{2n} \subset \pi_1(\mathcal{SA}_n)$. □

In the same way as the proof of Proposition A.1, we see that $W_{2n} = \pi_1(\mathcal{A}_n)$. Under the equivalences $W_{2n} = \pi_1(\mathcal{A}_n)$ and $SW_{2n} = \pi_1(\mathcal{SA}_n)$, we have the following.

Lemma A.2. *$SW_{2n} \cong W_{2n}/\langle\langle\vartheta\rangle\rangle$.*

Remark A.3.

(1) *Brendle-Hatcher used notations W_n and SW_n for $\pi_1(\mathcal{A}_n)$ and $\pi_1(\mathcal{SA}_n)$ respectively* [5]. *In this paper, we use notations W_{2n} and SW_{2n} rather than W_n and SW_n for the same groups, because we defined W_{2n} and SW_{2n} as subgroups of B_{2n} and SB_{2n} respectively.*

(2) *In* [5], *elements of $\pi_1(\mathcal{SA}_n)$ are applied from left to right and our convention is opposed to this. Hence in our paper, we need to take the inverse of their generators and reverse the order of letters in their relations.*

As promised in Section 2.5.1, we now prove the following.

Proposition A.4. *Let ψ_1 and ψ_2 be homeomorphisms of $(D^3, \mathbf{A})$. If the restrictions of ψ_1 and ψ_2 over $S^2 = \partial D^3$ are isotopic as homeomorphisms of $(S^2, \partial\mathbf{A})$ then ψ_1 and ψ_2 are isotopic as homeomorphisms of $(D^3, \mathbf{A})$.*

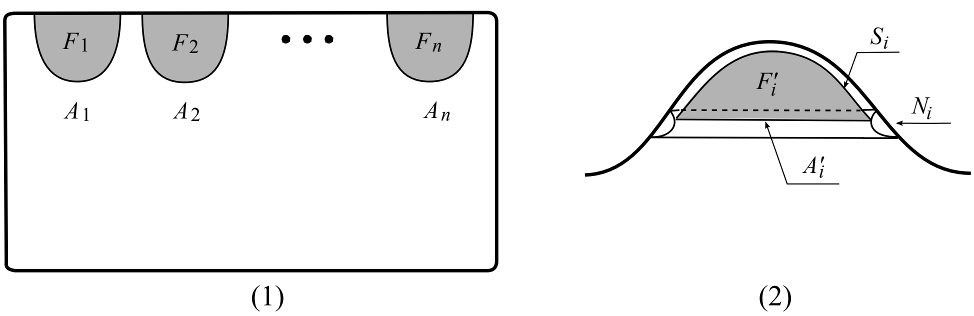


Figure 21. (1) F_i is a disk whose boundary is the union of the wicket A_i and an arc on ∂D^3. (2) N_i is a regular neighborhood of A_i, and F_i' is a meridian disk of a handlebody $D^3 \setminus (N_1 \cup \cdots \cup N_n)$.

Proof. At first, we assume that $\psi_1 = id$, $\psi_2|_{\partial D^3} = id_{\partial D^3}$. Since $\psi_2(A_i) = A_i$ and $\psi_2|_{\partial D^3} = id_{\partial D^3}$, especially, $\psi_2|_{\partial A_i} = id_{\partial A_i}$, we can isotope ψ_2 so that $\psi_2|_{A_i} = id_{A_i}$ with an isotopy preserving $\mathbf{A}$ setwise. Furthermore, we isotope ψ_2 so that $\psi_2(N_i) = N_i$ for a regular neighborhood N_i of A_i in D^3. We remark that $D^3 \setminus (N_1 \cup \cdots \cup N_n)$ is homeomorphic to a handlebody $\mathbb{H}_n$. The set $\partial D^3 \cap \partial N_i$ consists of two disks d_{2i-1} and d_{2i}, which are neighborhoods of two points ∂A_i. The boundary $\partial(D^3 \setminus (N_1 \cup \cdots \cup N_n))$ is a union of $P = \partial D^3 \setminus (d_1 \cup \cdots \cup d_{2n})$ and $U_i = \partial N_i \setminus (d_{2i-1} \cup d_{2i})$. We consider the restriction of ψ_2 on $\partial(D^3 \setminus (N_1 \cup \cdots \cup N_n))$. Then $\psi_2|_P = id$ and $\psi_2|_{U_i}$ is isotopic to the identity or a product of the Dehn twist about the core of U_i. We will show that $\psi_2|_{U_i}$ is isotopic to the identity. Let F_i be a disk in D^3 whose boundary is a union of A_i and an arc on ∂D^3 (see Figure 21 (1)). Let $F_i' = F_i \setminus N_i$, then this is a meridian disk of $D^3 \setminus (N_1 \cup \cdots \cup N_n)$ and its boundary is a union of two arcs $S_i = F_i' \cap P$, $A_i' = F_i' \cap U_i$ (see Figure 21 (2)). If we assume that $\psi_2|_{U_i}$ is not isotopic to the identity, then $\psi_2(\partial F_i') = \psi_2(S_i) \cup \psi_2(A_i') = S_i \cup \psi_2(A_i')$ is not null-homotopic in $D^3 \setminus (N_1 \cup \cdots \cup N_n)$, which contradicts the fact that $\psi_2(\partial F_i')$ bounds a disk $\psi_2(F_i')$ in $D^3 \setminus (N_1 \cup \cdots \cup N_n)$. Therefore, $\psi_2|_{U_i}$ is isotopic to the identity. Furthermore, we can isotope ψ_2 so that $\psi_2|_{N_i} = id_{N_i}$. Since the extension of a homeomorphism of $\partial(D^3 \setminus (N_1 \cup \cdots \cup N_n))$ to the 3-dimensional handlebody $D^3 \setminus (N_1 \cup \cdots \cup N_n)$ is unique up to isotopy, we have an isotopy between $\psi_2|_{D^3 \setminus (N_1 \cup \cdots \cup N_n)}$ and $id_{D^3 \setminus (N_1 \cup \cdots \cup N_n)}$. Hence ψ_2 is isotopic to id_{D^3} preserving $\mathbf{A}$ as a set.

Next, we assume that $\psi_1|_{\partial D^3} = \psi_2|_{\partial D^3}$. Then $\psi_1' = \psi_1^{-1} \circ \psi_1$, $\psi_2' = \psi_1^{-1} \circ \psi_2$ satisfy $\psi_1' = id$, $\psi_2'|_{\partial D^3} = id_{\partial D^3}$. By applying the argument of the previous paragraph to ψ_1' and ψ_2', we have an isotopy $\mathbb{G}_t' : D^3 \to D^3$ $(0 \leq t \leq 1)$ between ψ_1' and ψ_2' in $\mathrm{Homeo}_+(D^3, \mathbf{A})$. Then $\mathbb{G}_t = \psi_1 \circ \mathbb{G}_t'$ is an isotopy between ψ_1 and ψ_2 in $\mathrm{Homeo}_+(D^3, \mathbf{A})$.

Finally, we assume that $\psi_1|_{\partial D^3}$ and $\psi_2|_{\partial D^3}$ are isotopic in $\mathrm{Homeo}_+(\partial D^3, \partial \mathbf{A})$, that is to say, there is an isotopy $\mathbb{F}_t : \partial D^3 \to \partial D^3$ $(0 \leq t \leq 1)$ fixing $\partial \mathbf{A}$ such that $\mathbb{F}_0 = \psi_1|_{\partial D^3}$, $\mathbb{F}_1 = \psi_2|_{\partial D^3}$. We set the parametrization of the regular neighborhood $N(\partial D^3)$ of ∂D^3 by $\partial D^3 \times [0,1]$ so that $\partial D^3 \times \{0\} = \partial D^3$, $\partial D^3 \times \{1\} \subset \mathrm{int}(D^3)$. We define an isotopy $\mathbb{I}_t : D^3 \to D^3$ $(0 \leq t \leq 1)$ by

$$\mathbb{I}_t(x) = \begin{cases} (\mathbb{F}_{t(1-s)} \circ (\psi_1|_{\partial D^3})^{-1}(p), s) & \text{if } x = (p,s) \in \partial D^3 \times [0,1] = N(\partial D^3), \\ x & \text{if } x \notin N(\partial D^3). \end{cases}$$

Then $\mathbb{J}_t = \mathbb{I}_t \circ \psi_1$ is an isotopy in $\mathrm{Homeo}_+(D^3, \mathbf{A})$ so that $\mathbb{J}_0 = \psi_1$, $\mathbb{J}_1|_{\partial D^3} = \psi_2|_{\partial D^3}$. By the argument of the previous paragraph, there is an isotopy $\mathbb{G}_t$ between $\mathbb{J}_1$ and ψ_2 in $\mathrm{Homeo}_+(D^3, \mathbf{A})$. The concatenation of $\mathbb{J}_t$ and $\mathbb{G}_t$ is an isotopy between ψ_1 and ψ_2 in $\mathrm{Homeo}_+(D^3, \mathbf{A})$. □

We are now ready to prove Theorem 2.6 as promised in Section 2.5.1.

Proof of Theorem 2.6. By Proposition A.4, we regard $\pi_0(\mathrm{Homeo}_+(D^3, \mathbf{A}))$ as a subgroup SH_{2n} of $\mathrm{Mod}(\Sigma_{0,2n})$. The following sequence is exact (see [10, p.245] for example).

$$0 \to \langle \Delta^2 \rangle \to SB_{2n} \xrightarrow{\Gamma} \mathrm{Mod}(\Sigma_{0,2n}) \to 1.$$

As an immediate consequence of Theorem 5 in [15], we see that SH_{2n} is generated by $\Gamma(\sigma_{2i-1})$ $(i = 1, \ldots, n)$, $\Gamma(\eta_i)$ $(i = 1, \ldots, n-1)$, $\Gamma(\rho_{ij})$ $(i, j = 1, \ldots, n, j \neq i)$, and $\Gamma(\omega_{ij})$ $(i, j = 1, \ldots, n, j \neq i-1, i)$, where η_i, ρ_{ij}, ω_{ij} are as shown in Figure 22. We remark that in the case of the braid ρ_{ij}, the $(2i-1)$st and $2i$th strings pass between the $(2j-1)$st and $2j$th strings. On the other hand, in the case of the braid ω_{ij}, the $(2i-1)$st and $2i$th strings pass between the $2j$th and $(2j+1)$st

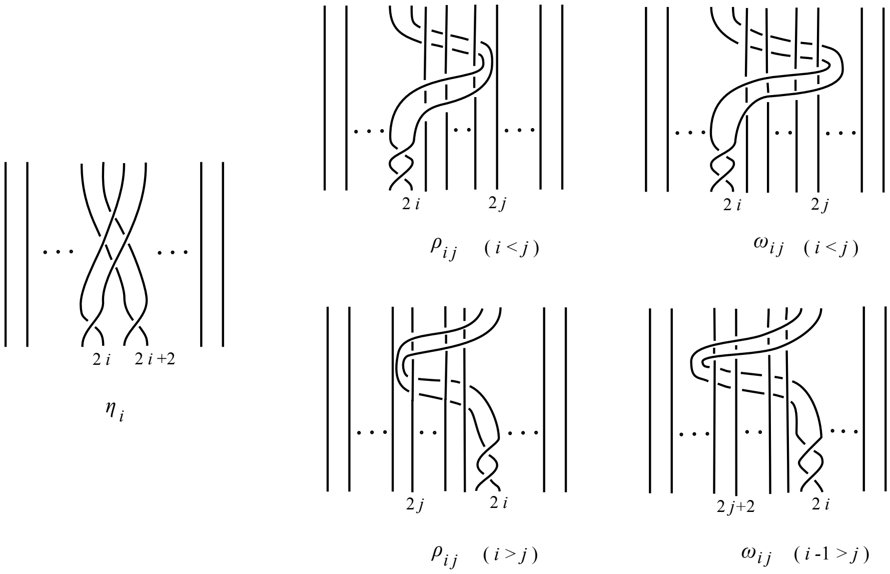


FIGURE 22. η_i, $\rho_{i,j}$, $\omega_{i,j}$ from left to right. If $i < j$, then $\rho_{i,j}$ and $\omega_{i,j}$ are the braids on the top. If $i > j$ (resp. If $i - 1 > j$), then $\rho_{i,j}$ (resp. $\omega_{i,j}$) are the braids on the bottom.

strings. As products of r_i, s_i, t_j, these braids are expressed as follows,

$$\eta_i = s_i t_i t_{i+1},$$

$$\rho_{ij} = \begin{cases} s_i s_{i+1} \cdots s_{j-2} s_{j-1} r_{j-1} s_{j-2} \cdots s_{i+1} s_i t_i^2 & \text{if } i < j, \\ s_{i-1} s_{i-2} \cdots s_{j+1} s_j r_j^{-1} s_{j+1} \cdots s_{i-2} s_{i-1} t_i^2 & \text{if } i > j, \end{cases}$$

$$\omega_{ij} = \begin{cases} s_i s_{i+1} \cdots s_{j-2} s_{j-1}^2 s_{j-2} \cdots s_{i+1} s_i t_i^2 & \text{if } i < j, \\ s_{i-1} s_{i-2} \cdots s_{j+2} s_{j+1}^2 s_{j+2} \cdots s_{i-2} s_{i-1} t_i^2 & \text{if } i - 1 > j. \end{cases}$$

On the other hand, Brendle-Hatcher [5] showed that $\pi_1(\mathcal{SA}_n)(= SW_{2n})$ is generated by r_i, s_i ($i \in \{1, \cdots, n-1\}$), t_j ($1 \in \{1, \cdots, n\}$). The images of these generators by Γ are in SH_{2n}, and $\Gamma(\eta_i)$, $\Gamma(\rho_{ij})$, $\Gamma(\omega_{ij})$ are written by products of these images. Therefore we see that $\Gamma(SW_{2n}) = SH_{2n}$. On the other hand, $\Delta^2 = (s_{n-1} \cdots s_2 s_1 t_1^2)^n$ is in SW_{2n}, and hence $SW_{2n} = \Gamma^{-1}(SH_{2n})$. As a result, Theorem 2.6 holds. □

Let $\mathrm{SHomeo}_+(\Sigma_g)$ be the subgroup of $\mathrm{Homeo}_+(\Sigma_g)$ which consists of the orientation preserving homeomorphisms on $\Sigma_g \simeq \partial \mathbb{H}_g$ that commute with $\mathcal{S} : \partial \mathbb{H}_g \to \partial \mathbb{H}_g$. In order to prove Theorem 2.8, Birman-Hilden showed the following.

Proposition A.5 (Theorem 7 in [3]). *Let ϕ_1 and $\phi_2 \in \mathrm{SHomeo}_+(\Sigma_g)$ be isotopic in $\mathrm{Homeo}_+(\Sigma_g)$. Then ϕ_1 and ϕ_2 are isotopic in $\mathrm{SHomeo}_+(\Sigma_g)$.*

By Proposition A.5, the natural surjection from $\pi_0(\mathrm{SHomeo}_+(\Sigma_g))$ to $\mathcal{H}(\Sigma_g)$ is an isomorphism. Therefore, one can define a homomorphism $q : \mathcal{H}(\Sigma_g) \to SB_{2g+2}$, see Theorem 2.8.

Recall that $\mathrm{SHomeo}_+(\mathbb{H}_g)$ is the subgroup of $\mathrm{Homeo}_+(\mathbb{H}_g)$ which consists of orientation preserving homeomorphisms on $\mathbb{H}_g$ that commute with $\mathcal{S} : \mathbb{H}_g \to \mathbb{H}_g$. We have the following which is a version of Proposition A.5.

Proposition A.6. *Let ϕ_1 and $\phi_2 \in \mathrm{SHomeo}_+(\mathbb{H}_g)$ be isotopic in $\mathrm{Homeo}_+(\mathbb{H}_g)$. Then ϕ_1 and ϕ_2 are isotopic in $\mathrm{SHomeo}_+(\mathbb{H}_g)$.*

Proof. For $\phi \in \mathrm{SHomeo}_+(\mathbb{H}_g)$, we define a homeomorphism $\underline{\phi}$ of $D^3 = \mathbb{H}_g/\iota$ by $\underline{\phi}([x]) = [\phi(x)]$, where $[x]$ is an element of $D^3 = \mathbb{H}_g/\iota$ represented by $x \in \mathbb{H}_g$. By Proposition A.5, there is an isotopy in $\mathrm{SHomeo}_+(\Sigma_g)$ between $\phi_1|_{\partial\mathbb{H}_g}$ and $\phi_2|_{\partial\mathbb{H}_g}$. This isotopy induces an isotopy between $\underline{\phi_1}|_{\partial D^3}$ and $\underline{\phi_2}|_{\partial D^3}$ in $\mathrm{Homeo}_+(\partial D^3, \partial\mathbf{A})$. By Proposition A.4, there is an isotopy between $\underline{\phi_1}$ and $\underline{\phi_2}$ in $\mathrm{Homeo}_+(D^3, \mathbf{A})$. Then the lift of this isotopy is an isotopy in $\mathrm{SHomeo}_+(\mathbb{H}_g)$ between ϕ_1 and ϕ_2. □

We are now ready to prove Theorem 2.11.

Proof of Theorem 2.11. By Proposition A.6, the natural surjection from $\pi_0(\mathrm{SHomeo}_+(\mathbb{H}_g))$ to $\mathcal{H}(\mathbb{H}_g)$ is an isomorphism. Therefore, we can define a homomorphism $Q : \mathcal{H}(\mathbb{H}_g) \to SH_{2g+2}$ so that $Q = q|_{\mathcal{H}(\mathbb{H}_g)}$, see (2.1) in Section 2.6. As a consequence of Theorem 2.8 and the fact that $\iota \in \mathcal{H}(\mathbb{H}_g)$, we see that Theorem 2.11 holds. □

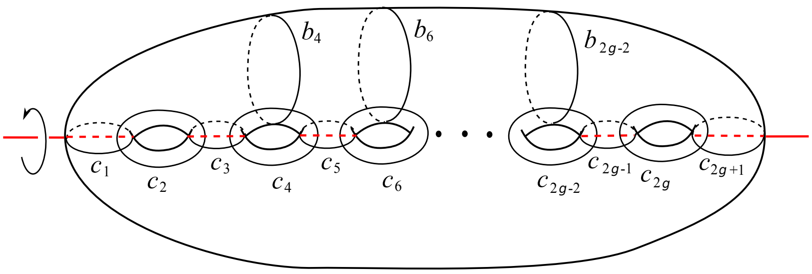


FIGURE 23. Circles on $\partial\mathbb{H}_g$.

As an application of Theorem 2.11, we determine a finite presentation for $\mathcal{H}(\mathbb{H}_g)$ (Theorem A.8). To do this, we set some circles on $\partial\mathbb{H}_g$ as in Figure 23. The circle c_{2j-1} ($j \in \{1, \cdots, g+1\}$) bounds a disk properly embedded in $\mathbb{H}_g$, and c_{2j-1} is preserved by the hyperelliptic involution $\mathcal{S}$. The circle b_{2j} ($j \in \{2, \cdots, g-1\}$) also bounds a disk properly embedded in $\mathbb{H}_g$, but b_{2j} is not preserved by $\mathcal{S}$. Let t_{c_i} and $t_{b_{2j}}$ be the left-handed Dehn twist about c_i and b_{2j} respectively.

Remark A.7. *The group $\mathrm{Mod}(\mathbb{H}_g)$ is a subgroup of the mapping class group of $\partial\mathbb{H}_g$ of infinite index whenever $g \geq 2$. This is because t_{c_2} is not an element of $\mathrm{Mod}(\mathbb{H}_g)$ and has an infinite order. The group $\mathcal{H}(\mathbb{H}_g)$ is a subgroup of $\mathrm{Mod}(\mathbb{H}_g)$ of infinite index whenever $g \geq 3$. In fact, t_{b_4} is not an element of $\mathcal{H}(\mathbb{H}_g)$ but an element of $\mathrm{Mod}(\mathbb{H}_g)$, and t_{b_4} has an infinite order.*

Theorem A.8. *$\mathcal{H}(\mathbb{H}_g)$ is generated by $\mathfrak{r}_i = t_{c_{2i}} t_{c_{2i+1}} t_{c_{2i-1}}^{-1} t_{c_{2i}}^{-1}$, $\mathfrak{s}_i = t_{c_{2i}}^{-1} t_{c_{2i+1}}^{-1} t_{c_{2i-1}}^{-1} t_{c_{2i}}^{-1}$ ($i = 1, \ldots, g$), $\mathfrak{t}_j = t_{c_{2j-1}}^{-1}$ ($j = 1, \ldots, g, g+1$) and the relations are as follows.*

(1) $\mathfrak{r}_i\mathfrak{r}_j = \mathfrak{r}_j\mathfrak{r}_i$ *for* $|i-j| > 1$, $\mathfrak{r}_i\mathfrak{r}_{i+1}\mathfrak{r}_i = \mathfrak{r}_{i+1}\mathfrak{r}_i\mathfrak{r}_{i+1}$,
(2) $\mathfrak{s}_i\mathfrak{s}_j = \mathfrak{s}_j\mathfrak{s}_i$ *for* $|i-j| > 1$, $\mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1}$,
(3) $\mathfrak{r}_i\mathfrak{s}_j = \mathfrak{s}_j\mathfrak{r}_i$ *for* $|i-j| > 1$,
(4) $\mathfrak{r}_i\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{s}_i\mathfrak{r}_{i+1}$, $\mathfrak{r}_i\mathfrak{r}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{r}_i\mathfrak{r}_{i+1}$, $\mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{r}_i = \mathfrak{r}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1}$,
(5) $\mathfrak{r}_i\mathfrak{s}_i\mathfrak{t}_i\mathfrak{r}_i = \mathfrak{t}_i\mathfrak{s}_i$,
(6) $\mathfrak{t}_i\mathfrak{t}_j = \mathfrak{t}_j\mathfrak{t}_i$,
(7) $\mathfrak{r}_i\mathfrak{t}_j = \mathfrak{t}_j\mathfrak{r}_i$ *for* $j \neq i, i+1$, $\mathfrak{t}_{i+1}\mathfrak{r}_i = \mathfrak{r}_i\mathfrak{t}_i$,
(8) $\mathfrak{s}_i\mathfrak{t}_j = \mathfrak{t}_j\mathfrak{s}_i$ *for* $j \neq i, i+1$, $\mathfrak{t}_j\mathfrak{s}_i = \mathfrak{s}_i\mathfrak{t}_k$, *for* $\{i, i+1\} = \{j, k\}$,

(9) $(\mathfrak{s}_g \cdots \mathfrak{s}_2\mathfrak{s}_1\mathfrak{t}_1^2)^{g+1} = 1$,
(10) $(\mathfrak{t}_1\mathfrak{s}_1\mathfrak{s}_2 \cdots \mathfrak{s}_g\mathfrak{r}_g^{-1} \cdots \mathfrak{r}_2^{-1}\mathfrak{r}_1^{-1}\mathfrak{t}_1)^2 = 1$ *and* $\mathfrak{t}_1\mathfrak{s}_1\mathfrak{s}_2 \cdots \mathfrak{s}_g\mathfrak{r}_g^{-1} \cdots \mathfrak{r}_2^{-1}\mathfrak{r}_1^{-1}\mathfrak{t}_1$ *commutes with* $\mathfrak{r}_i, \mathfrak{s}_i, \mathfrak{t}_i$.

Proof. We use Theorems 2.6 and 2.11. Brendle-Hatcher expressed a finite presentation for $\pi_1(\mathcal{A}_{g+1})(= W_{2g+2})$ in [5, Propositions 3.2, 3.6]. The relations (1)–(4) come from [5, Proposition 3.2] and (5)–(8) come from [5, Proposition 3.6]. The relation (9) means that Δ^2 is trivial in SH_{2g+2}. In the relation (10), $\mathfrak{t}_1\mathfrak{s}_1\mathfrak{s}_2 \cdots \mathfrak{s}_g\mathfrak{r}_g^{-1} \cdots \mathfrak{r}_2^{-1}\mathfrak{r}_1^{-1}\mathfrak{t}_1$ equals ι, and the relation means $\iota^2 = 1$ and any element of $\mathcal{H}(\mathbb{H}_g)$ commutes with ι. □

By a straightforward computation together with Theorem A.8, we have the following.

Corollary A.9. *The abelianization* $\mathcal{H}(\mathbb{H}_g)^{ab} = \mathcal{H}(\mathbb{H}_g)/[\mathcal{H}(\mathbb{H}_g), \mathcal{H}(\mathbb{H}_g)]$ *is isomorphic to* $\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ *for any* $g \geq 2$.

Corollary A.9 is in contrast with the abelianizations of other groups which contain $\mathcal{H}(\mathbb{H}_g)$ as a subgroup. In fact, $\mathrm{Mod}(\Sigma_1)^{ab} = \mathbb{Z}/12\mathbb{Z}$, $\mathrm{Mod}(\Sigma_2)^{ab} = \mathbb{Z}/10\mathbb{Z}$, and $\mathrm{Mod}(\Sigma_g)^{ab}$ is trivial when $g \geq 3$ (see [10, §5.1] for example). In the case of the hyperelliptic mapping class groups, $\mathcal{H}(\Sigma_g)^{ab} = \mathbb{Z}/2(2g+1)\mathbb{Z}$ when g is even and $\mathcal{H}(\Sigma_g)^{ab} = \mathbb{Z}/4(2g+1)\mathbb{Z}$ when g is odd. They are proved straightforwardly from the presentation of $\mathcal{H}(\Sigma_g)$ by Birman-Hilden [3, Theorem 8]. For the handlebody groups, $\mathrm{Mod}(\mathbb{H}_g)^{ab}$ is a finite abelian group when $g \geq 3$, see [35, 19].

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