BRAIDS, ORDERINGS AND MINIMAL VOLUME CUSPED HYPERBOLIC 3-MANIFOLDS

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Abstract. It is well-known that there is a faithful representation of braid groups on automorphism groups of free groups, and it is also well-known that free groups are bi-orderable. We investigate which n-strand braids give rise to automorphisms which preserve some bi-ordering of the free group F_n of rank n. As a consequence of our work we find that of the two minimal volume hyperbolic 2-cusped orientable 3-manifolds, one has bi-orderable fundamental group whereas the other does not. We prove a similar result for the 1-cusped case, and have further results for more cusps. In addition, we study pseudo-Anosov braids and find that typically those with minimal dilatation are not order-preserving.

1. Introduction

If $<$ is a strict total ordering of the elements of a group G such that $g < h$ implies $fg < fh$ for all $f, g, h \in G$, we call $(G, <)$ a *left-ordered* group. If the left-ordering $<$ is also invariant under right-multiplication, we call $(G, <)$ a *bi-ordered* group (sometimes known in the literature simply as "ordered" group). If a group admits such an ordering it is said to be *left-* or *bi-orderable*. It is easy to see that a group is left-orderable if and only if it is right-orderable. If $(G, <)$ is a bi-ordered group, then $<$ is invariant under conjugation: $g < h$ if and only if $fgf^{-1} < fhf^{-1}$ for all $f, g, h \in G$. Nontrivial examples of bi-orderable groups are the free groups F_n of rank n, as discussed in Appendix A.

An automorphism ϕ of a group G is said to *preserve an ordering* $<$ of G if for every $f, g \in G$ we have $f < g \implies \phi(f) < \phi(g)$; we also say $<$ is *ϕ-invariant*. If $\phi : G \to G$ preserves an ordering $<$ of G, then the nth power ϕ^n preserves the ordering $<$ of G for each integer n. We note that if ϕ^n preserves an ordering it does not necessarily follow that ϕ does.

E. Artin [2, 3] observed that each n-strand braid corresponds to an automorphism of F_n. This paper concerns the question of which braids give rise to automorphisms which preserve some bi-ordering of F_n. In turn this is related to the orderability of the fundamental group of the complement of certain links in S^3, namely the braid closure together with its axis, which we call a braided link. We pay special attention to pseudo-Anosov mapping classes and their stretch factors (dilatations), and cusped hyperbolic 3-manifolds of small volume.

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This paper is organized as follows: Section 2 reviews the relation among braids, mapping class groups, free group automorphisms and certain links in the 3-sphere S^3. In Section 3 we recall basic properties of orderable groups, with explicit bi-ordering of free groups further described in Appendix A. The study of order-preserving braids and their relation to bi-ordering the group of the corresponding braided links is initiated in Section 4. Applications to cusped hyperbolic 3-manifolds of minimal volume are considered in Section 5. In Section 6 we give many examples of non-order-preserving braids, including pseudo-Anosov braids with minimal dilation as well as large dilatations. We also find a family of pretzel links whose fundamental groups can not be bi-orderable. Appendix B is devoted to a proof that the fundamental group of the Whitehead link complement is bi-orderable.

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2. Braids and $\mathrm{Aut}(F_n)$

Let B_n be the n-strand braid group, which has the well-known presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$ and $\sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i$ if $|i-j| = 1$. See Figure 1(1)(2).

2.1. **Mapping classes.** Let D_n denote the disk with n punctures, which we may picture as equally spaced along a diameter of the disk and labelled 1 to n. $\mathrm{Mod}(D_n)$ denotes the mapping class group of D_n and $\mathrm{Mod}(D_n, \partial D)$ the mapping class group of homeomorphisms fixed on the boundary pointwise. There is a well-known isomorphism

$$\bar{\Gamma} : B_n \to \mathrm{Mod}(D_n, \partial D)$$

which sends σ_i to a half twist h_i which interchanges the punctures labelled i and $i+1$, see Figure 1(3). The kernel of the obvious map $\mathrm{Mod}(D_n, \partial D) \to \mathrm{Mod}(D_n)$ is infinite cyclic, generated by a Dehn twist along a simple closed curve parallel to the boundary of the disk. Using the isomorphism $\bar{\Gamma} : B_n \to \mathrm{Mod}(D_n, \partial D)$ together with this obvious map, we have the surjective homomorphism

$$\Gamma : B_n \to \mathrm{Mod}(D_n)$$

whose kernel is generated by the full twist $\Delta_n^2 \in B_n$, where $\Delta_n \in B_n$ is the half twist.

Elements of $\mathrm{Mod}(D_n)$ are classified into three types: periodic, reducible and pseudo-Anosov, called Nielsen-Thurston types [38]. If two given mapping classes are conjugate to each other, then their Nielsen-Thurston types are the same. We say that $\beta \in B_n$ is *periodic* (resp. *reducible*, *pseudo-Anosov*) if its mapping class $\Gamma(\beta) \in \mathrm{Mod}(D_n)$ is of the corresponding type.

2.2. **Free group automorphisms.** Let β be an n-strand braid. Let $\phi : D_n \to D_n$ be a representative of the mapping class $\bar{\Gamma}(\beta) \in \mathrm{Mod}(D_n, \partial D)$. Obviously ϕ represents a mapping class $\Gamma(\beta) \in \mathrm{Mod}(D_n)$. If one passes to the induced map $\phi_* = \phi_{*p} : \pi_1(D_n, p) \to \pi_1(D_n, p)$ of the fundamental group of D_n, using a point p on the boundary as basepoint, this defines the Artin representation

$$B_n \to \mathrm{Aut}(F_n)$$

which can be defined on the generators as follows, where $x_1, x_2 \ldots, x_n$ are the free generators of the free group F_n of rank n and $\mathrm{Aut}(F_n)$ is the group of automorphisms of F_n. The generator σ_i induces the automorphism

$$x_i \mapsto x_i x_{i+1} x_i^{-1}, \quad x_{i+1} \mapsto x_i, \quad x_j \mapsto x_j \text{ if } j \neq i, i+1. \tag{2.1}$$

See Figure 2. It is known that the Artin representation is faithful. Its image is the subgroup of automorphisms of F_n that take each x_i to a conjugation of some x_j and which take the product $x_1 x_2 \cdots x_n$ to itself.

We note that if $\Gamma(\beta) = \Gamma(\beta') \in \mathrm{Mod}(D_n)$ for n-strand braids β and β', then $\beta' = \beta\Delta_n^{2k}$ for some integer k. The images of β and $\beta\Delta_n^{2k}$ under the Artin representation are the same up to an inner automorphism

$$x \to (x_1 x_2 \cdots x_n)^k x (x_1 x_2 \cdots x_n)^{-k}.$$

By abuse of notation, from now on we will use the same symbol β for the braid, the mapping class $\Gamma(\beta) \in \mathrm{Mod}(D_n)$ and the corresponding automorphism of F_n.

An automorphism ϕ of $F_n = \langle x_1, \ldots, x_n \rangle$ is said to be *symmetric* if for each generator x_j, the image $\phi(x_j)$ is a conjugate of some x_k. Every automorphism on F_n corresponding to the action of a braid on F_n is symmetric. A symmetric automorphism $\phi : F_n \to F_n$ is *pure* if ϕ sends each x_i to a conjugate of itself. Pure braids (see Section 4.2) induce symmetric and pure automorphisms.

Since braid words, like paths, are typically read from left to right, we adopt the convention that braids act on F_n on the right. If $x \in F_n$, we denote the action of $\beta \in B_n$ by $x \to x^\beta$, and if $\beta, \gamma \in B_n$ we have the identity $x^{\beta\gamma} = (x^\beta)^\gamma$.

Definition 2.1. *An n-strand braid β is said to be order-preserving if there exists some bi-ordering $<$ of F_n preserved by the automorphism $x \to x^\beta$ of F_n.*

One sees that $\beta \in B_n$ is order-preserving if and only if $\beta\Delta_n^{2k}$ is order-preserving for some (hence all) $k \in \mathbb{Z}$ (Corollary 4.2).

As discussed below, there is some ambiguity in defining the action of B_n on F_n, depending on choices of a representative of the mapping class $\beta \in \mathrm{Mod}(D_n)$, basepoint in D_n and generators of $\pi_1(D_n)$. As we'll see, this ambiguity is irrelevant in the question of whether a braid β is order-preserving.

2.3. **Basepoints.** It is sometimes convenient to use a basepoint and generators different from that used in the Artin representation. Specifically, we consider a representative $\phi : D_n \to D_n$ of a mapping class in $\mathrm{Mod}(D_n, \partial D)$ and we may assume that $p, q \in D_n$ are two different points (not necessarily on the boundary of the disk), each fixed by ϕ. Then we have induced maps $\phi_{*p} : \pi_1(D_n, p) \to \pi_1(D_n, p)$ and $\phi_{*q} : \pi_1(D_n, q) \to \pi_1(D_n, q)$. Of course, $\pi_1(D_n, p)$ and $\pi_1(D_n, q)$ are isomorphic, but not canonically. We can construct an isomorphism by choosing a path ℓ in D_n from q to p which then defines an isomorphism $h : \pi_1(D_n, p) \to \pi_1(D_n, q)$ sending the class of a loop α in D_n based at p to the class of the loop $\ell\alpha\ell^{-1}$ based at q. Consider the diagram

$$\begin{array}{ccc} \pi_1(D_n, p) & \xrightarrow{\phi_{*p}} & \pi_1(D_n, p) \\ h\downarrow & & h\downarrow \\ \pi_1(D_n, q) & \xrightarrow{\phi_{*q}} & \pi_1(D_n, q) \end{array}$$

which is not necessarily commutative. However, we leave it to the reader to check the following

Proposition 2.2. *The above diagram commutes up to conjugation. Specifically, $h \circ \phi_{*p}$ equals $\phi_{*q} \circ h$ followed by a conjugation in $\pi_1(D_n, q)$, the conjugating element being the class of the loop $\ell(\phi \circ \ell)^{-1}$ in $\pi_1(D_n, q)$.*

Corollary 2.3. *The map ϕ_{*p} preserves a bi-ordering of $\pi_1(D_n, p)$ if and only if ϕ_{*q} preserves a bi-ordering of $\pi_1(D_n, q)$.*

Proof. If ϕ_{*p} preserves the bi-ordering $<_p$ of $\pi_1(D_n, p)$, define a bi-ordering $<_q$ of $\pi_1(D_n, q)$ by the formula $f <_q g \iff h^{-1}(f) <_p h^{-1}(g)$ for $f, g \in \pi_1(D_n, q)$. Then one checks that $f <_q g \implies \phi_{*q}(f) <_q \phi_{*q}(g)$ using conjugation invariance of bi-orderings. The converse is proved similarly. □

If one passes from $\mathrm{Mod}(D_n, \partial D)$ to $\mathrm{Mod}(D_n)$, there is a further ambiguity regarding the action of a braid on $\pi_1(D_n)$. However, this ambiguity corresponds to conjugation by a power of Δ_n^2, so again it is irrelevant to the question of preserving a bi-ordering. More concretely, given an n-strand braid β, let $\phi : D_n \to D_n$ be any representative of the mapping class $\beta \in \mathrm{Mod}(D_n)$. We take any basepoint q of D_n possibly $\phi(q) \neq q$. By choosing a path ℓ in D_n from q to $\phi(q)$, we have the induced map

$$\phi_{*q} : \pi_1(D_n, q) \to \pi_1(D_n, q) \tag{2.2}$$

which sends the class of a loop α in D_n based at q to the class of the loop $\ell\alpha\ell^{-1}$ based at the same point. By using Corollary 2.3, one sees that β is order-preserving if and only if $\phi_{*q} : \pi_1(D_n, q) \to \pi(D_n, q)$ preserves a bi-ordering of $\pi_1(D_n, q)$.

By abuse of notation again, we denote the induced map ϕ_{*q} by β, when q is specified.

These remarks show that if one allows different choices of basepoint, say a basepoint q, the action of B_n on F_n should really be regarded as a representation

$$B_n \to \mathrm{Out}(\pi_1(D_n, q)) \cong \mathrm{Out}(F_n),$$

the group of outer automorphisms. Recall that $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ is the (normal) subgroup of inner automorphisms of a group G.

2.4. **Mapping tori and braided links.** For a braid $\beta \in B_n$, we denote the mapping torus by

$$\mathbb{T}_\beta = D_n \times [0,1]/(y,1) \sim (\phi(y), 0),$$

where $\phi : D_n \to D_n$ is a representative of $\beta \in \mathrm{Mod}(D_n)$. By the hyperbolization theorem of Thurston [37], $\mathbb{T}_\beta$ is hyperbolic if and only if β is pseudo-Anosov.

The closure $\widehat{\beta}$ of a braid β is a knot or link in the 3-sphere S^3 and the *braided link*, denote by $\mathrm{br}(\beta)$, is the closure $\widehat{\beta}$, together with the braid axis A, which is an unknotted curve that $\widehat{\beta}$ runs around in a monotone manner: $\mathrm{br}(\beta) = \widehat{\beta} \cup A$. See Figure 3(1)(2). Whereas all links can be realized as $\widehat{\beta}$, this is not true for $\mathrm{br}(\beta)$, as each component of $\widehat{\beta}$ has nonzero linking number with the braid axis A. As an example the Whitehead link considered in Appendix B is not a braided link.

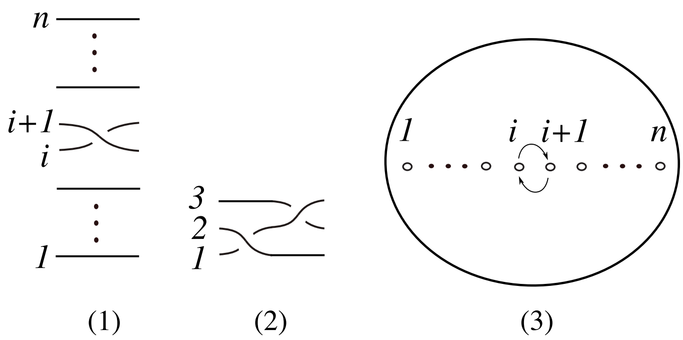


FIGURE 1. (1) $\sigma_i \in B_n$. (2) $\sigma_1\sigma_2^{-1} \in B_3$. (3) $h_i \in \mathrm{Mod}(D_n, \partial D)$.

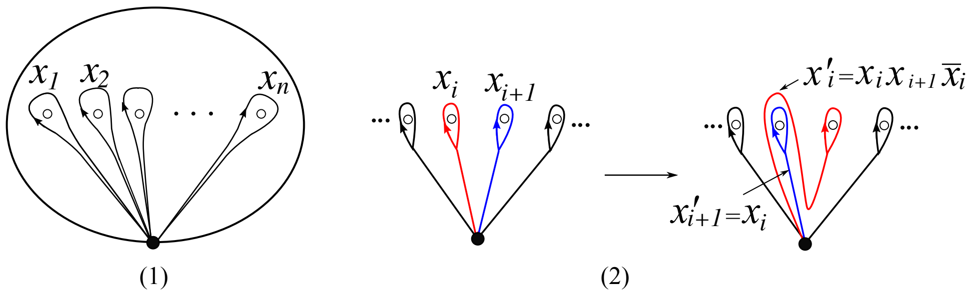


FIGURE 2. A basepoint $\bullet$ of $\pi_1(D_n)$ lies on ∂D. (1) Generators x_i's of F_n. (2) $\sigma_i : F_n \to F_n$, where x' := the image of x under σ_i and $\overline{x} := x^{-1}$.

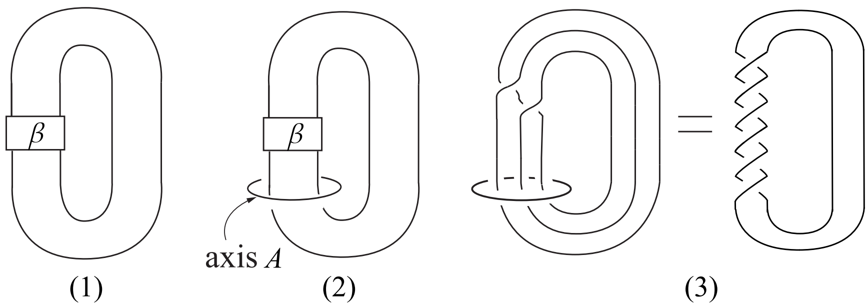


FIGURE 3. (1) Closure $\widehat{\beta}$. (2) $\mathrm{br}(\beta) = \widehat{\beta} \cup A$. (3) $\mathrm{br}(\sigma_1\sigma_2)$ is equivalent to the $(6,2)$-torus link.

We see that $\mathbb{T}_\beta$ is homeomorphic with the complement of the braid closure $\widehat{\beta}$ in the solid torus $D^2 \times S^1$. The interior $Int(\mathbb{T}_\beta)$ can be identified with the complement of $\widehat{\beta} \cup A$ in S^3, so we have the following.

Lemma 2.4. *$Int(\mathbb{T}_\beta)$ is homeomorphic to $S^3 \setminus \mathrm{br}(\beta)$.*

Of course the fundamental group of $Int(\mathbb{T}_\beta)$ is isomorphic with that of $\mathbb{T}_\beta$, which in turn is the semidirect product

$$\pi_1(S^3 \setminus br(\beta)) \cong \pi_1(\mathbb{T}_\beta) \cong F_n \rtimes_\beta \mathbb{Z}.$$

There is a (split) exact sequence

$$1 \to F_n \to \pi_1(\mathbb{T}_\beta) \to \mathbb{Z} = \langle t \rangle \to 1.$$

One may consider $F_n \rtimes_\beta \mathbb{Z}$ as the set of ordered pairs

$$F_n \rtimes_\beta \mathbb{Z} = \{(f, t^p) \mid f \in F_n,\ p \in \mathbb{Z}\},$$

with the multiplication given by

$$(f, t^p)(g, t^q) = (ft^p g t^{-p}, t^{p+q}), \quad \text{where} \quad tgt^{-1} = g^\beta.$$

3. Orderable groups

In this section we outline a few well-known facts about orderable groups. For more details, see [11, 6]. Further details regarding bi-ordering of F_n may be found in Appendix A.

Proposition 3.1. *A group G is left-orderable if and only if there exists a sub-semigroup $\mathcal{P} \subset G$ such that for every $g \in G$ exactly one of $g = 1$, $g \in \mathcal{P}$ or $g^{-1} \in \mathcal{P}$ holds.*

Indeed such a $\mathcal{P}$ defines a left-ordering $< \, = \, <_{\mathcal{P}}$ by the rule $g < h \iff g^{-1}h \in \mathcal{P}$. Conversely, a left-ordering $<$ defines a *positive cone* $\mathcal{P} = \mathcal{P}_< := \{g \in G \mid 1 < g\}$ satisfying the conditions of Proposition 3.1.

Proposition 3.2. *A group G is bi-orderable if and only if it posesses $\mathcal{P} \subset G$ as in Proposition 3.1, and in addition $g^{-1}\mathcal{P}g = \mathcal{P}$ for all $g \in G$.*

We note that a left- or bi-ordering of G is preserved by an automorphism $\phi : G \to G$ if and only if $\phi(\mathcal{P}) = \mathcal{P}$, where $\mathcal{P}$ is the positive cone of the ordering.

Proposition 3.3. *Suppose $1 \to K \stackrel{i}{\hookrightarrow} G \stackrel{p}{\to} H \to 1$ is an exact sequence of groups. If K and H are left-ordered with positive cones $\mathcal{P}_K$ and $\mathcal{P}_H$ respectively, then G is left-orderable using the positive cone $\mathcal{P}_G := i(\mathcal{P}_K) \cup p^{-1}(\mathcal{P}_H)$.*

This is sometimes called the *lexicographic* order of G, considered as an extension.

Proposition 3.4. *In Proposition 3.3, if K and H are bi-ordered, then the formula $\mathcal{P}_G = i(\mathcal{P}_K) \cup p^{-1}(\mathcal{P}_H)$ defines a bi-ordering of G if and only if the bi-ordering of K is respected under conjugation by elements of G; equivalently $g^{-1}\mathcal{P}_K g = \mathcal{P}_K$ for all $g \in G$.*

A subset S of a left-ordered group $(G, <)$ is said to be *convex* if for all $s, s' \in S$ and $g \in G$ satisfying $s < g < s'$, we have $g \in S$.

Proposition 3.5. *If $1 \to K \stackrel{i}{\to} G \stackrel{p}{\to} H \to 1$ is an exact sequence of groups with $(G, <)$ a left-ordered group and K a convex subgroup of $(G, <)$, then there is a left-ordering $\prec$ of H in which $1 \prec h$ if and only if some (and hence every) element $g \in p^{-1}(h)$ satisfies $1 < g$. If $(G, <)$ is a bi-ordered group, so is $(H, \prec)$.*

Being bi-orderable is a much stronger property than being left-orderable. An intermediate property of groups is to be *locally-indicable*, which means that every finitely-generated nontrivial subgroup has the infinite cyclic group $\mathbb{Z}$ as a quotient.

Proposition 3.6. *For a group, the following implications hold: bi-orderable $\Longrightarrow$ locally-indicable $\Longrightarrow$ left-orderable $\Longrightarrow$ torsion-free. None of these implications is reversible.*

Proposition 3.7 ([7])**.** *If L is a knot or link in S^3, then its group $\pi_1(S^3 \setminus L)$ is locally indicable and therefore left-orderable.*

Certain knot groups and link groups are bi-orderable, and this is a particular focus of this paper. For example, torus knot groups are not bi-orderable, but the figure-eight knot 4_1 has bi-orderable group [33], and we shall see that the same is true of the Whitehead link and many links constructed from braids as above. One of the reasons bi-orderability of knot groups is of interest has to do with surgery and L-spaces, which were introduced by Ozsváth and Szabó [32] and include all 3-manifolds with finite fundamental group.

Theorem 3.8 ([10])**.** *If K is a knot in S^3 for which $\pi_1(S^3 \setminus K)$ is bi-orderable, then surgery on K cannot produce an L-space.*

We note that this is not true in general for links. As we shall see in Theorem B.1, the Whitehead link has bi-orderable fundamental group, but one can construct lens spaces by certain surgeries on the link.

Consider a group K, an automorphism $\phi : K \to K$ and the semidirect product $G = K \rtimes_\phi \mathbb{Z}$. Recall $G \cong \{(k, t^p) \mid k \in K,\ p \in \mathbb{Z}\}$ with multiplication given by

$$(k_1, t^p)(k_2, t^q) = (k_1 t^p k_2 t^{-p}, t^{p+q}) = (k_1 \phi^p(k_2), t^{p+q}), \quad \text{where} \quad tkt^{-1} = \phi(k).$$

Proposition 3.9. *Suppose K is bi-orderable and $\phi : K \to K$ is an automorphism. Then $G = K \rtimes_\phi \mathbb{Z}$ is bi-orderable if and only if there exists a bi-ordering of K which is preserved by ϕ.*

Proof. Suppose that G is bi-ordered by $<$. Since bi-orderings are invariant under conjugation, the equation $\phi(k) = tkt^{-1}$ implies that $<$, restricted to K, is ϕ-invariant. For the converse, suppose ϕ preserves a bi-ordering $\prec$ of K. Then we use the lexicographic ordering defined by $(k_1, t^p) < (k_2, t^q)$ if and only if $p < q$ as integers or $p = q$ and $k_1 \prec k_2$, to order G. It is easily checked that this is a bi-ordering using the identity $t^p k t^{-p} = \phi^p(k)$ and the assumption that ϕ preserves the ordering $\prec$. □

Let Σ be an orientable surface. Then $\pi_1(\Sigma)$ is bi-orderable, see [35]. Let $\mathrm{Mod}(\Sigma)$ be the mapping class group of Σ, let $f = [\phi] \in \mathrm{Mod}(\Sigma)$ and assume $\phi(p) = p$ for some $p \in \Sigma$. Since the fundamental group of the mapping torus $\mathbb{T}_f$ of f is the semidirect product $\pi_1(\Sigma) \rtimes_{\phi_*} \mathbb{Z}$, where $\phi_* : \pi_1(\Sigma, p) \to \pi_1(\Sigma, p)$ is an automorphism induced from f, we have the following from Proposition 3.9.

Proposition 3.10. *$\pi_1(\mathbb{T}_f)$ is bi-orderable if and only if there exists a bi-ordering of $\pi_1(\Sigma, p)$ which is preserved by $\phi_* : \pi_1(\Sigma, p) \to \pi_1(\Sigma, p)$, where ϕ is a representative of the mapping class f.*

If $\phi(x) = x$ for $x \in G$, then we say that the orbit of x (under ϕ) is *trivial.*

Proposition 3.11. *Let G be a left-orderable group. If an automorphism $\phi : G \to G$ preserves a left-ordering $<$ of G, then ϕ cannot have any nontrivial finite orbits.*

Proof. If the orbit of $x \in G$ is nontrivial, we may assume $x < \phi(x)$. Then $\phi(x) < \phi^2(x)$, and by transitivity and induction $x < \phi^m(x)$ for all $m \geq 1$ and therefore the orbit of x, $\{\phi^n(x) \mid n \in \mathbb{Z}\}$, is infinite. □

Proposition 3.11 says that if $\phi : G \to G$ has a nontrivial finite orbit, then ϕ does not preserve any left-ordering of G.

Consider an automorphism $\phi : F_n \to F_n$ of a free group. The following two criteria for ϕ being order-preserving (or not) will be useful; they involve the abelianization $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ and its eigenvalues, which are, *a priori*, n complex numbers, possibly with multiplicity.

Theorem 3.12 ([33])**.** *Let $\phi : F_n \to F_n$ be an automorphism. If every eigenvalue of $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ is real and positive, then there is a bi-ordering of F_n which is ϕ-invariant.*

Theorem 3.13 ([10])**.** *If there exists a bi-ordering of F_n which is ϕ-invariant, then ϕ_{ab} has at least one real and positive eigenvalue.*

This is useful in showing that certain fibred 3-manifolds have fundamental groups which are *not* bi-orderable. However, we note that in the case of braids it does not apply in that way, for if ϕ is an automorphism of F_n induced by a braid $\beta \in B_n$, or more generally a symmetric automorphism, then ϕ_{ab} is simply a permutation of the generators of $\mathbb{Z}^n$ and therefore has at least one eigenvalue equal to one.

We note that Theorem 3.12 cannot have a full converse. It has been observed in [28] that for $n \geq 3$ there exist automorphisms of F_n which preserve a bi-ordering of F_n, but whose eigenvalues are precisely the nth roots of unity in $\mathbb{C}$. Such examples appear in the discussion in Section 4.6.

In Appendix A a certain class of bi-orderings of the free group F_n, called *standard* orderings is defined, using the lower central series.

Proposition 3.14. *If $\phi : F_n \to F_n$ is a non-pure symmetric automorphism, then ϕ cannot preserve any standard bi-ordering of F_n.*

Proof. Assume $\phi : F_n \to F_n$ is not pure, but preserves a standard bi-ordering $<$ of F_n. Then ϕ_{ab} is a nontrivial permutation of the generators of the abelianization $\mathbb{Z}^n$. By Proposition A.1, the commutator subgroup $[F_n, F_n]$ is convex relative to $<$, and therefore $F_n/[F_n, F_n] \cong \mathbb{Z}^n$ inherits a bi-ordering $<_1$ according to Proposition 3.5. Since ϕ preserves $<$, one easily checks that ϕ_{ab} preserves the order $<_1$ of $\mathbb{Z}^n$. But, ϕ being non-pure symmetric implies that ϕ_{ab} is a nontrivial permutation of the generators of $\mathbb{Z}^n$ which are the images of the x_i. By Proposition 3.11, ϕ_{ab} cannot preserve any ordering of $\mathbb{Z}^n$, a contradiction. □

Proposition 3.15. *If $\phi : F_n \to F_n$ is a pure symmetric automorphism, then ϕ is order-preserving. In fact, it preserves every standard ordering of F_n.*

Proof. A pure symmetric automorphism $\phi : F_n \to F_n$ induces the identity map $\phi_{ab} = id : \mathbb{Z}^n \to \mathbb{Z}^n$, so Proposition A.3 applies. □

For any symmetric automorphism $\phi : F_n \to F_n$, there exists $k \geq 1$ so that ϕ^k is pure symmetric. By Proposition 3.15, we have the following.

Corollary 3.16. *If $\phi : F_n \to F_n$ is a symmetric automorphism, then there exists $k \geq 1$ such that ϕ^k is order-preserving.*

4. Order-preserving braids

4.1. **Basic properties.** From Lemma 2.4 and Proposition 3.10 we have the following.

Proposition 4.1. *A braid $\beta \in B_n$ is order-preserving if and only if $\pi_1(S^3 \setminus \mathrm{br}(\beta))$ is bi-orderable.*

Let δ_n be the n-strand braid

$$\delta_n = \sigma_1\sigma_2 \cdots \sigma_{n-1}$$

and let $\Delta = \Delta_n \in B_n$ be the half twist

$$\Delta = (\sigma_1\sigma_2 \cdots \sigma_{n-1})(\sigma_1\sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1.$$

The full twist Δ^2 is written by $\Delta^2 = \delta_n^n = (\delta_n\sigma_1)^{n-1}$, which means that δ_n and $\delta_n\sigma_1$ are roots of Δ^2. We note that Δ^2 commutes with all n-strand braids and in fact generates the centre of B_n when $n \geq 3$, see [24, Theorem 1.24].

Corollary 4.2. *A braid $\beta \in B_n$ is order-preserving if and only if $\beta\Delta^{2k}$ is order-preserving for some (hence all) $k \in \mathbb{Z}$. Moreover, they preserve exactly the same bi-orderings of F_n.*

Proof. This follows since $S^3 \setminus \mathrm{br}(\beta)$ and $S^3 \setminus \mathrm{br}(\beta\Delta^{2k})$ are homeomorphic: By using the *disk twist* as we shall define in Section 4.4, we see that kth power of the disk twist $\mathfrak{t}^k$ about the disk bounded by the braid axis A of β sends the exterior $\mathcal{E}(\mathrm{br}(\beta))$ of the link $\mathrm{br}(\beta)$ to the exterior $\mathcal{E}(\mathrm{br}(\beta\Delta^{2k}))$. An alternative argument is that Δ^2 acts by conjugation of F_n, so it preserves *every* bi-order of F_n. □

As noted by Garside [16] every n-strand braid has an expression $\beta\Delta^{2k}$ where β is a *Garside positive* braid word, meaning that β can be written as a word in the σ_i generators without negative exponents. Thus a question of a braid being order-preserving can always be reduced to the case of positive braids. Notice that changing a braid β by conjugation does not change the link $\mathrm{br}(\beta)$ up to isotopy, so we have:

Corollary 4.3. *Let $\alpha, \beta \in B_n$. Then β is order-preserving if and only if $\alpha\beta\alpha^{-1}$ is order-preserving.*

The very simplest of nontrivial braids are the generators σ_i.

Proposition 4.4. *The generators $\sigma_i \in B_n$ are not order-preserving.*

Proof. Suppose that σ_i preserves a bi-order $<$ of F_n. We may assume $x_i < x_{i+1}$. Then $x_i^{\sigma_i} < x_{i+1}^{\sigma_i}$, or in other words $x_i x_{i+1} x_i^{-1} < x_i$, see (2.1). But conjugation invariance of the ordering then yields $x_{i+1} < x_i$, which is a contradiction.

An alternative argument is as follows. We take a basepoint p_1 in the interior of D_n. There exists a homeomorphism $\phi : D_n \to D_n$ which represents $\sigma_i \in \mathrm{Mod}(D_n)$ such that the induced map $\phi_* : \pi_1(D_n, p_1) \to \pi_1(D_n, p_1)$ gives rise to the following automorphism on $F_n = \langle x_1, \cdots, x_n \rangle$:

$$\sigma_i : x_i \mapsto x_{i+1}, \quad x_{i+1} \mapsto x_{i+1} x_i x_{i+1}^{-1}, \quad x_j \mapsto x_j \text{ if } j \neq i, i+1, \tag{4.1}$$

see Figure 4(2). (In the figures, we denote by x', the image of $x \in F_n$ under the automorphism of F_n, and denote by $\overline{x}$, the inverse x^{-1} of x.) Suppose that σ_i

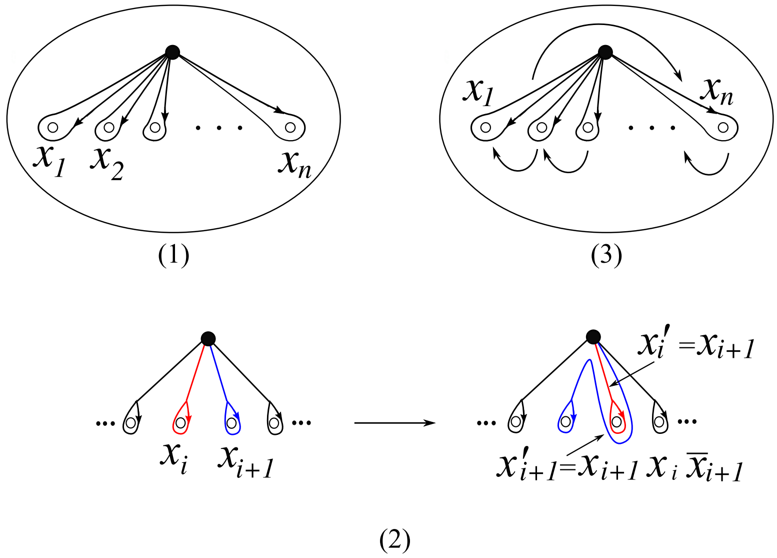


FIGURE 4. A basepoint • of $\pi_1(D_n)$ is taken to be in the interior of D_n. (1) Generators x_i's of F_n. (2) $\sigma_i : F_n \to F_n$. (3) $\delta_n : x_n \mapsto x_{n-1} \mapsto \cdots \mapsto x_1 \mapsto x_n$.

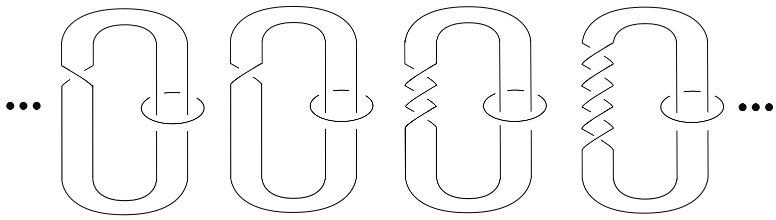


FIGURE 5. Links whose groups are not bi-orderable.

preserves a bi-order $<$ of F_n. We may assume $x_i < x_{i+1}$. Then $x_{i+1} < x_{i+1}x_ix_{i+1}^{-1}$ since σ_i preserves $<$. This implies that $x_{i+1} < x_i$ by conjugation invariance of the bi-ordering. This is a contradiction. □

Example 4.5. *Proposition 4.4 together with Corollary 4.2 implies the links of Figure 5 have complements whose fundamental groups are not bi-orderable. Those complements are homeomorphic to each other.*

4.2. **Pure braids.** As is well-known, there is a homomorphism $B_n \to \mathcal{S}_n$ of the n-strand braid group onto the permutation group of n letters, and the kernel is the group of pure braids P_n.

Proposition 4.6. *Every pure braid $\beta \in P_n$ is order-preserving, and in fact preserves every standard bi-order of F_n.*

Proof. For a pure braid β, the image of the Artin representatation is pure symmetric. This completes the proof by Proposition 3.15. □

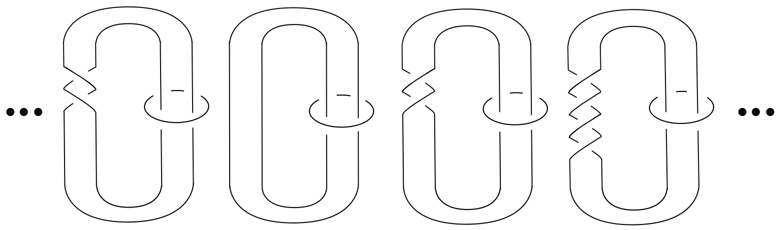


FIGURE 6. Links whose groups are bi-orderable.

The 2nd power σ_i^2 of each generator σ_i is a pure braid. Thus we have the following.

Corollary 4.7. *The braids $\sigma_i^2 \in P_n$ are order-preserving.*

Example 4.8. *The 2-strand braid σ_1^2 gives rise to the examples of links in Figure 6 whose complements have bi-orderable fundamental groups. All the link complements are homeomorphic to one another. It is clear that they are homeomorphic with $D_2 \times S^1$, whose fundamental group is isomorphic with $F_2 \times \mathbb{Z}$.*

By Corollary 3.16, we immediately obtain the following.

Corollary 4.9. *For every braid β some power β^k is order-preserving. The fundamental group $\pi_1(\mathbb{T}_{\beta^k})$ is bi-orderable and may be regarded as a normal subgroup of index k in $\pi_1(\mathbb{T}_\beta)$.*

4.3. **Periodic braids.** Suppose that $\beta \in B_n$ is a periodic braid. It is known (see for example [17, page 30]) that there exists an integer $k \in \mathbb{Z}$ such that β is conjugate to either

Type 1. $(\delta_n \sigma_1)^k = (\sigma_1 \sigma_2 \cdots \sigma_{n-1} \sigma_1)^k$ or
Type 2. $\delta_n^k = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^k$.

Theorem 4.10. *Let $\beta \in B_n$ be a periodic braid.*

1. *If β is conjugate to $(\delta_n \sigma_1)^k$ for some k, then β is order-preserving and $\pi_1(\mathbb{T}_\beta)$ is bi-orderable.*
2. *If β is conjugate to δ_n^k for some k with $k \not\equiv 0 \pmod n$, then β is not order-preserving and $\pi_1(\mathbb{T}_\beta)$ can not be bi-orderable.*

Note that δ_n^k is order-preserving when $k \equiv 0 \pmod n$ since $\delta_n^n = \Delta^2$. Theorem 4.10 is a consequence of [7, Theorem 1.5]: If β is a periodic braid, then $\mathbb{T}_\beta$ is a Seifert fibered 3-manifold. If β is of type 1, then $\mathbb{T}_\beta$ has no exceptional fibres. If β is of type 2 and if k is not a multiple of n, then $\mathbb{T}_\beta$ has an exceptional fibre.

We will give an alternative, more explicit proof of Theorem 4.10 in Section 4.5.

4.4. **Disk twists.** We review a method to construct links in S^3 whose complements are homeomorphic to each other. Let L be a link in S^3. We denote a tubular neighborhood of L by $\mathcal{N}(L)$, and the exterior of L, that is $S^3 \setminus \mathrm{int}(\mathcal{N}(L))$ by $\mathcal{E}(L)$. Suppose that L contains an unknot K as a sublink. Then $\mathcal{E}(K)$ (resp. $\partial\mathcal{E}(K)$) is homeomorphic to a solid torus (resp. torus). We denote the link $L \setminus K$ by L_K. Taking a disk D bounded by the longitude of $\mathcal{N}(K)$, we define two homeomorphisms

$$T_D : \mathcal{E}(K) \to \mathcal{E}(K)$$

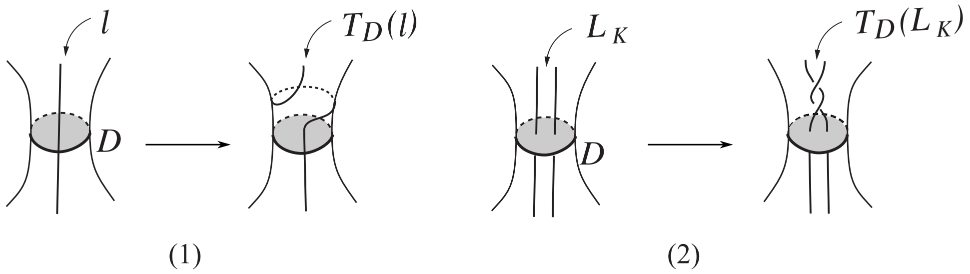


FIGURE 7. (1) Image of ℓ under T_D, where ℓ is an arc on $\partial\mathcal{E}(K)$ which passes through ∂D. (2) Local picture of L_K and its image $T_D(L_K)$.

and

$$\mathfrak{t}_D : \mathcal{E}(L)(= \mathcal{E}(K \cup L_K)) \to \mathcal{E}(K \cup T_D(L_K))$$

as follows. We cut $\mathcal{E}(K)$ along D. We have resulting two sides obtained from D. Then we reglue the two sides by rotating either of the sides 360 degrees so that the mapping class of the restriction $T_D|_{\partial\mathcal{E}(K)} : \partial\mathcal{E}(K) \to \partial\mathcal{E}(K)$ defines the right-handed Dehn twist about ∂D, see Figure 7(1). Such an operation defines the homeomorphism $T_D : \mathcal{E}(K) \to \mathcal{E}(K)$. If m segments of L_K pass through D, then $T_D(L_K)$ is obtained from L_K by adding a full twist braid Δ_m^2 near D. In the case $m = 2$, see Figure 7(2). For example, if L is equivalent to $\mathrm{br}(\beta)$ for some $\beta \in B_n$ and if K is taken to be the braid axis A of β, then $T_D(L_K)$ is equivalent to the closure of $\beta\Delta_n^2$. Notice that $T_D : \mathcal{E}(K) \to \mathcal{E}(K)$ determines the latter homeomorphism

$$\mathfrak{t}_D : \mathcal{E}(L)(= \mathcal{E}(K \cup L_K)) \to \mathcal{E}(K \cup T_D(L_K)).$$

We call $\mathfrak{t}_D$ the *(right-handed) disk twist about* D.

For any integer $m \neq 0$, we have a homeomorphism of the mth power T_D^m : $\mathcal{E}(K) \to \mathcal{E}(K)$ so that $T_D^m|_{\partial\mathcal{E}(K)} : \partial\mathcal{E}(K) \to \partial\mathcal{E}(K)$ is the mth power of the right-handed Dehn twist about ∂D. Observe that T_D^m converts $L = K \cup L_K$ into a link $K \cup T_D^m(L_K)$ in S^3 such that $S^3 \setminus L$ is homeomorphic to $S^3 \setminus (K \cup T^m(L_K))$. We denote by $\mathfrak{t}_D^m$, a homeomorphism: $\mathcal{E}(L)(= \mathcal{E}(K \cup L_K)) \to \mathcal{E}(K \cup T_D^m(L_K))$ and call $\mathfrak{t}_D^m$ the mth power of *(right-handed) disk twist* $\mathfrak{t}_D$ about D.

4.5. **Alternative proof of Theorem 4.10.**

Proof of Theorem 4.10. We prove the claim (1). Consider the pure 2-strand braid σ_1^2. Then $\mathrm{br}(\sigma_1^2)$ is a link consisting of three unknotted components, including the axis. Performing a disk twist n times on one of the components of the closed braid $\widehat{\sigma_1^2}$ converts the braided link of the 2-strand braid σ_1^2 to the braided link of the $(n+2)$-strand braid β', which is conjugate to the type 1 braid $\sigma_1\sigma_2\cdots\sigma_{n+1}\sigma_1$, see Figure 8. But the disk twist being a homeomorphism of the complement of the link, we see that $\mathbb{T}_{\sigma_1^2} \simeq \mathbb{T}_{\beta'}$. But since σ_1^2 is pure, $\mathbb{T}_{\sigma_1^2}$ has bi-orderable fundamental group, in fact isomorphic with $F_2 \times \mathbb{Z}$. Hence the fundamental group of $\mathbb{T}_{\beta'}$ is bi-orderable and $\sigma_1\sigma_2\cdots\sigma_{n+1}\sigma_1$ is order-preserving. Thus the kth power $(\sigma_1\sigma_2\cdots\sigma_{n+1}\sigma_1)^k$ is also order-preserving.

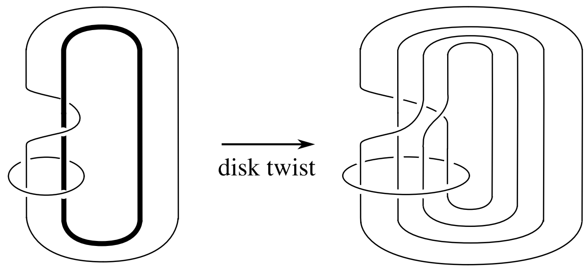


FIGURE 8. nth power of the disk twist converts the braided link of σ_1^2 to that of $\sigma_1\sigma_2\cdots\sigma_{n+1}\sigma_1$. ($n=2$ in this case.)

We turn to the claim (2). We use the basepoint p_1 in the interior of D_n. There exists a homeomorphism $\phi : D_n \to D_n$ which represents $\delta_n \in \mathrm{Mod}(D_n)$ such that the induced map $\phi_* : \pi_1(D_n, p_1) \to \pi_1(D_n, p_1)$ gives rise to the following automorphism[1] on $F_n = \langle x_1, \cdots, x_n\rangle$:

$$\delta_n : x_1 \mapsto x_n \mapsto x_{n-1} \mapsto \cdots \mapsto x_2 \mapsto x_1, \tag{4.2}$$

see Figure 4(2). The orbit of x_1 is nontrivial (since $x_1 \neq x_n$) and finite. By Proposition 3.11, δ_n is not order-preserving. Then $k \not\equiv 0 \pmod n$ if and only if δ_n^k has a nontrivial finite orbit of x_1. Thus δ_n^k is not order-preserving. □

Proposition 4.11. *The half twist $\Delta_n \in B_n$ is order-preserving if and only if n is odd.*

Proof. If $n = 2m+1$, then Δ_n is conjugate to $(\sigma_1\sigma_2\cdots\sigma_{2m}\sigma_1)^m$, and if $n = 2m$, then Δ_n is conjugate to $(\sigma_1\sigma_2\cdots\sigma_{2m-1})^m$ with $m \not\equiv 0 \pmod{2m}$. By Theorem 4.10, we finish the proof. □

As a special case of Theorem 4.10(2), we see that the periodic 3-strand braid $\sigma_1\sigma_2$ is not order-preserving. Another way to see this is to observe that the 3-strand braid $\sigma_1\sigma_2$ gives the following automorphism on $F_3 = \langle x, y, z\rangle$ by using the Artin representation $B_3 \to \mathrm{Aut}(F_3)$

$$z \mapsto y \mapsto x \mapsto xyzy^{-1}x^{-1}$$

One can show this doesn't preserve a bi-order of F_3 as follows. Assume some bi-order $<$ were preserved by the map, then supposing without loss of generality that $x < y$, we would have $xyzy^{-1}x^{-1} < x$ (applying $\sigma_1\sigma_2$), hence $yzy^{-1} < x$ (conjugation invariance), so $xyx^{-1} < xyzy^{-1}x^{-1}$ (applying $\sigma_1\sigma_2$) and then $xyx^{-1} < x$ (transitivity) and finally the contradiction $y < x$ (conjugation again).

On the other hand $\sigma_1\sigma_2^{-1} \in B_3$ is pseudo-Anosov, and in fact it is the simplest pseudo-Anosov 3-strand braid; see Section 6.1. We thank George Bergman for pointing out the following argument. It will also follow from more general result in Section 6.2; see Corollary 6.4.

Theorem 4.12. *The braid $\sigma_1\sigma_2^{-1} \in B_3$ is not order-preserving.*

[1]The product $\sigma_1\sigma_2\cdots\sigma_{n-1} \in \mathrm{Aut}(\pi_1(D_n, p_1))$ of σ_i's in (4.1) is given by $x_1 \mapsto x_n$, $x_j \to x_n x_{j-1} x_n^{-1}$ if $j \neq 1$. This is equal to $\delta_n \in \mathrm{Aut}(\pi_1(\mathrm{D_n}, \mathrm{p_1}))$ in (4.2) up to an inner automorphism.

Proof. Let x, y, z be the free generators of F_3. Using the Artin representation $B_3 \to \mathrm{Aut}(F_3)$, one sees that the action of $\sigma_1\sigma_2^{-1}$ is

$$x \mapsto xzx^{-1}, \quad y \mapsto x, \quad z \mapsto z^{-1}yz.$$

Consider the orbit of the element $w = y^{-1}x$ under this action. Assuming there is a bi-order $<$ of F_3 invariant under this action, we may assume without loss of generality that $1 < w$, and therefore all elements of the orbit of w are positive. Moreover $1 < w$ implies $y < x$ and since $w \mapsto zx^{-1}$ we have $y < x < z$.

Now the calculation

$$w \mapsto zx^{-1} \mapsto z^{-1}yzxz^{-1}x^{-1} = (z^{-1}y)zx^{-1}x(xz^{-1})x^{-1}$$

and the facts that $z^{-1}y$ and $x(xz^{-1})x^{-1}$ are negative show that the action is decreasing on the orbit of zx^{-1}, which is also the orbit of w.

Calculating the preimages of the generators, we have the action of $\sigma_1\sigma_2^{-1}$ expressed as

$$y \mapsto x, \quad y^{-1}xyzy^{-1}x^{-1}y \mapsto y, \quad y^{-1}xy \mapsto z,$$

and therefore

$$y^{-1}xyz^{-1}y^{-1}x^{-1}yy \mapsto y^{-1}x = w.$$

But notice that $y^{-1}xyz^{-1}y^{-1}x^{-1}yy = w(yz^{-1})y^{-1}(x^{-1}y)y$ and since the expressions in the parentheses are < 1 (i.e, $yz^{-1} < 1$ and $y^{-1}(x^{-1}y)y < 1$), we conclude that the action is increasing on the orbit of w. This contradiction shows that an invariant bi-order of F_3 cannot exist. □

4.6. **Explicit orderings preserved by periodic braids of type 1.** We have seen that for $n \geq 3$ the root $\delta_n\sigma_1$ of the full twist $\Delta^2 \in B_n$ preserves an ordering of F_n. In this section we will explicitly construct uncountably many such orderings.

Recall that $\delta_n \in B_n$ induces the following automorphism on F_n, using the basepoint in the interior of D_n

$$\delta_n : x_1 \mapsto x_n \mapsto x_{n-1} \mapsto \cdots \mapsto x_2 \mapsto x_1,$$

see (4.2). By (4.1) $\sigma_1 \in B_n$ induces the following automorphism on F_n, using the same basepoint:

$$\sigma_1 : x_1 \mapsto x_2, \ x_2 \mapsto x_2x_1x_2^{-1}, \ x_j \mapsto x_j \text{ if } j \neq 1, 2.$$

Thus the automorphism $\delta_n\sigma_1$ on F_n is given by

$$\delta_n\sigma_1 : x_1 \to x_n, \quad x_2 \to x_2, \quad x_3 \to x_2x_1x_2^{-1}, \quad x_4 \to x_3, \quad \cdots, \quad x_n \to x_{n-1}.$$

(For the case $n = 3$, just take the first three terms above.)

Here is another way to realize this automorphism of F_n, elaborating on Example 3.6 in [28]. Fix $n \geq 3$ and consider the free group $F_2 = \langle u, v\rangle$ of rank 2 and the homomorphism of F_2 onto the cyclic group $G = \langle t \mid t^{n-1} = 1\rangle$ given by $u \to t$ and $v \to 1$. The kernel $\mathcal{K} = \mathcal{K}_n$ of this map is a normal subgroup of F_2, of index $n - 1$. If we realize F_2 as the fundamental group of a bouquet of two circles labelled u and v, the covering space corresponding to $\mathcal{K}$ is a finite planar graph as pictured in Figure 9 whose fundamental group is free of rank n. Using the basepoint in the

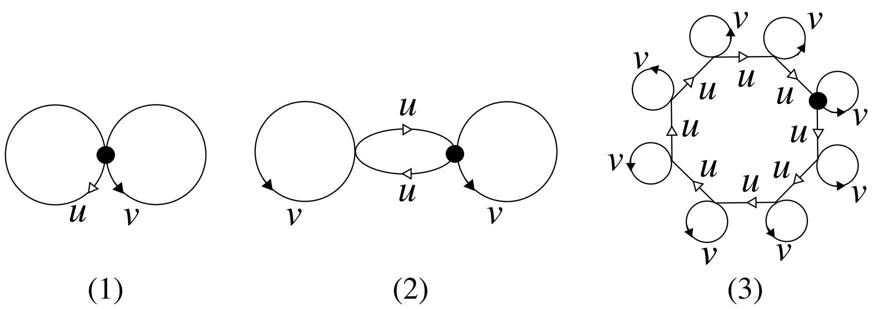


FIGURE 9. (1) A bouquet of two circles u and v. (2) Covering space corresponding to $\mathcal{K}_n$ when $n = 3$. (3) Covering space corresponding to $\mathcal{K}_n$ when $n = 9$.

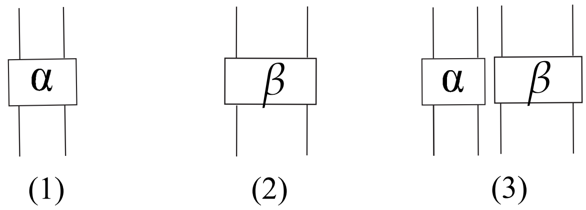


FIGURE 10. (1) $\alpha \in B_m$. (2) $\beta \in B_n$. (3) $\alpha \otimes \beta \in B_{m+n}$.

covering space depicted in Figure 9, we see that $\mathcal{K}$ has the free generators $z_1, \ldots, z_n$, where

$$\begin{aligned} z_1 = v, \quad z_2 &= u^{n-1}, \quad z_3 = u^{n-2}vu^{2-n}, \quad z_4 = u^{n-3}vu^{3-n}, \\ \ldots \quad , z_{n-1} &= u^2vu^{-2}, \quad z_n = uvu^{-1}. \end{aligned}$$

Now consider the automorphism of the normal subgroup $\mathcal{K}$ of F_2 given by $\phi(x) = uxu^{-1}$. This is not an inner automorphism of $\mathcal{K}$ but it is an inner automorphism of the larger group F_2. Therefore, if we take *any* bi-ordering of F_2, then its restriction to $\mathcal{K}$ will be preserved by ϕ. By inspection, the action of ϕ on $\mathcal{K}$ is given by

$$z_1 \to z_n, \quad z_2 \to z_2, \quad z_3 \to z_2z_1z_2^{-1}, \quad z_4 \to z_3, \quad \cdots, \quad z_n \to z_{n-1}.$$

Under the isomorphism $\mathcal{K} \cong F_n$ given by $z_i \mapsto x_i$ for each i, we see that ϕ corresponds to $\delta_n\sigma_1$. One can use this isomorphism to see that any bi-ordering of F_2 restricted to $\mathcal{K}$ provides an ordering of F_n invariant under $\delta_n\sigma_1$. Finally note that any ordering of F_n respected by $\delta_n\sigma_1$ will also be respected by $(\delta_n\sigma_1)^k$ for any integer k.

Note that we can use, for example, any standard ordering of F_2. However, this ordering restricted to $\mathcal{K}$ cannot be a standard ordering (defined using the lower central series of $\mathcal{K}$) by Proposition 3.14.

4.7. **Tensor product of braids.** Given braids $\alpha \in B_m$ and $\beta \in B_n$ one can form the $(m+n)$-strand braid $\alpha \otimes \beta \in B_{m+n}$ with α on the first m strings and β on the last n strings, but no crossings between any of the first m strings with any of the last n strings (Figure 10); see for example [24, page 69]. The action of $\alpha \otimes \beta$ on $F_{m+n} \cong F_m \star F_n$ is just the free product $\alpha \star \beta : F_m \star F_n \to F_m \star F_n$.

The following Lemma and Proposition are proved in [34, Corollary 4].

Lemma 4.13. *Suppose $(G, <_G)$ and $(H, <_H)$ are bi-ordered groups. Then there is a bi-ordering of $G \star H$ which extends the orderings of the factors and such that whenever $\phi : G \to G$ and $\psi : H \to H$ are order-preserving automorphisms, the ordering of $G \star H$ is preserved by the automorphism $\phi \star \psi : G \star H \to G \star H$.*

Proposition 4.14. *The braids $\alpha \in B_m$ and $\beta \in B_n$ are order-preserving if and only if $\alpha \otimes \beta \in B_{m+n}$ is order-preserving.*

There is a natural inclusion $B_m \subset B_{m+n}$ $(n \geq 1)$ given by $\beta \mapsto \beta \otimes 1_n$, where 1_n is the identity braid of B_n.

Corollary 4.15. *A braid $\beta \in B_m$ is order-preserving if and only if $\beta \otimes 1_n \in B_{m+n}$ is order-preserving.*

4.8. **Do order-preserving braids form a subgroup?** Let $OP_n \subset B_n$ denote the set of n-strand braids which are order-preserving. It is clear that $\beta \in OP_n$ if and only if $\beta^{-1} \in OP_n$ and that the identity braid belongs to OP_n. Moreover, $P_n \subset OP_n$ by Proposition 4.6. It is natural to ask whether OP_n forms a subgroup of B_n, in other words whether OP_n is closed under multiplication. For $n = 2$ the answer is affirmative. Noting that $\Delta_2^2 = \sigma_1^2$ we conclude from Propositions 4.4 and 4.6 and Corollary 4.2 that OP_2 consists of exactly the 2-strand braids σ_1^k with k even. Therefore OP_2 is exactly the subgroup P_2.

Proposition 4.16. *For $n > 2$ the set OP_n is not a subgroup of B_n.*

Proof. Consider the n-strand braids $\alpha = \sigma_1\sigma_2\sigma_1$ and $\beta = \sigma_1^{-2}$. Then α is order-preserving, being an extension in B_n of type 1 periodic braid $\sigma_1\sigma_2\sigma_1 \in B_3$ (see Corollary 4.15), and β is order-preserving, being a pure braid. But $\alpha\beta = \sigma_1\sigma_2\sigma_1^{-1}$ is not order-preserving, as it is conjugate to σ_2 which is not order-preserving by Proposition 4.4. □

Although not a subgroup for $n > 2$, OP_n is a large subset of B_n:

Proposition 4.17. *For $n > 2$ the set OP_n of order-preserving n-braids generates B_n.*

Proof. We saw above that $\sigma_1\sigma_2\sigma_1^{-1}$ is a product of braids $\alpha, \beta \in OP_n$; therefore σ_2 is also a product of appropriate conjugates of α and β; these conjugates are also in OP_n. But all the generators σ_i of B_n are conjugate to each other, and therefore are also products of elements of OP_n. It follows that all braids are products of elements of OP_n. □

5. Small volume cusped hyperbolic 3-manifolds

It is known by Gabai-Meyerhoff-Milley [15] that the Weeks manifold is the unique closed orientable hyperbolic 3-manifold of smallest volume. Its fundamental group is not left-orderable; see Calegari-Dunfield [8]. In this section we will see that certain minimum volume orientable n-cusped 3-manifolds can be distinguished by orderability properties of their fundamental groups. We also prove that some orientable hyperbolic n-cusped 3-manifolds with the smallest known volumes have bi-orderable fundamental groups.

Let C_3 and C_4 be the chain links with 3 and 4 components as in Figure 11(1) and (2). For $n \geq 5$, let C_n be *the minimally twisted n-chain link*; see [23, Section 1] for the definition of such a link.

Figure 11(3) and (4) show C_5 and C_6. Let $\mathbb{W}_n$ be the n-fold cyclic cover over one component of the Whitehead link complement $\mathbb{W}$. (See Figure 22(2) for $\mathbb{W}_3$.) It seems that $S^3 \setminus C_n$ and $\mathbb{W}_{n-1}$ play an important role for the study of the minimal volume hyperbolic 3-manifolds with n cusps, where $n \geq 3$, as we shall see below.

5.1. **One cusp.** Cao-Meyerhoff [9] proved that a minimal volume orientable hyperbolic 3-manifold with 1 cusp is homeomorphic to either the complement of figure-eight knot in S^3 or its sibling manifold (which can be described as 5/1 Dehn surgery on one component of the Whitehead link). We note that the sibling manifold cannot be described as a knot complement, as its first homology group is $\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.

Like the figure-eight knot complement, it can be described as a punctured torus bundle over S^1.

Theorem 5.1. *The figure-eight knot complement has bi-orderable fundamental group. The fundamental group of its sibling is not bi-orderable.*

Proof. The first assertion is proved in [33], using the monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ associated with the fibration, which has two positive eigenvalues, and Theorem 3.12. The sibling has the monodromy $\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$, see [22, Proposition 3 and Note]. This has the two negative eigenvalues $(-3 \pm \sqrt{5})/2$. By Theorem 3.13, the sibling has non-bi-orderable fundamental group. □

5.2. **Two cusps.** Agol [1] proved that the minimal volume orientable hyperbolic 3-manifold with 2 cusps is homeomorphic to either the Whitehead link complement $\mathbb{W}$ or the $(-2, 3, 8)$-pretzel link complement $\mathbb{W}'$.

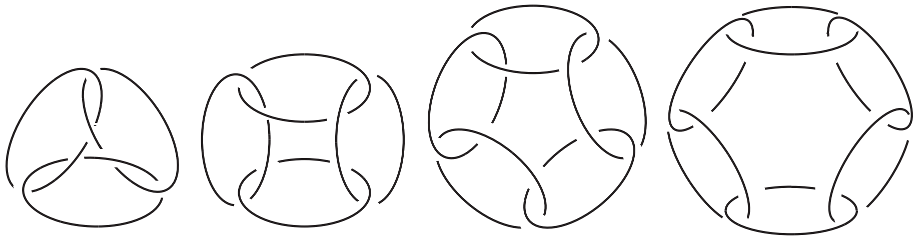
Theorem 5.2. *The fundamental group of $\mathbb{W}$ is bi-orderable. The fundamental group of $\mathbb{W}'$ is not bi-orderable.*

Proof. The first assertion follows from Theorem B.1. We shall prove in Theorem 6.1 that $\delta_5\sigma_1^2 \in B_5$ is not order-preserving. This together with Proposition 4.1 implies the second assertion, since $\mathrm{br}(\delta_5\sigma_1^2)$ is equivalent to the $(-2, 3, 8)$-pretzel link; see Figure 19. □

5.3. **Four cusps.** It is proved by Ken'ichi Yoshida [39] that the minimal volume orientable hyperbolic 3-manifold with 4 cusps is homeomorphic to $S^3 \setminus C_4$.

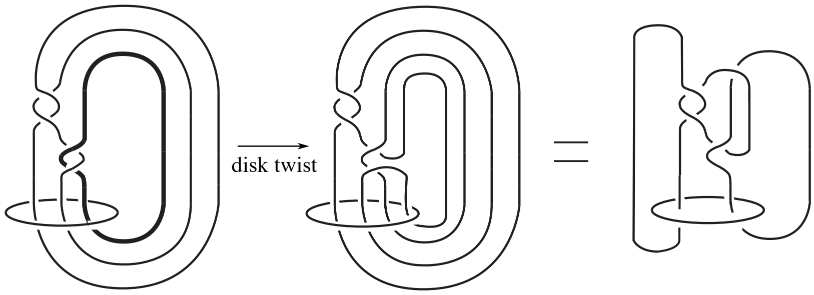
Theorem 5.3. *The complement $S^3 \setminus C_4$ has bi-orderable fundamental group.*

Proof. Consider $\sigma_1^{-2}\sigma_2^2 \in P_3$. Then $\pi_1(S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^2))$ is bi-orderable by Proposition 4.6. We now see that $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^2)$ is homeomorphic to $S^3 \setminus C_4$. Take a disk D bounded by the thickened unknot $K \subset \mathrm{br}(\sigma_1^{-2}\sigma_2^2)$, see Figure 12(1). Then the left-handed disk twist $\mathfrak{t}_D^{-1}$ sends $\mathcal{E}(\mathrm{br}(\sigma_1^{-2}\sigma_2^2))$ to the exterior of a link L as shown



(1) (2) (3) (4)

FIGURE 11. (1) C_3. (2) C_4. (3) C_5. (4) C_6.



(1) (2) (3)

FIGURE 12. $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^2)$ is homeomorphic to $S^3 \setminus C_4$. (1) $\mathrm{br}(\sigma_1^{-2}\sigma_2^2)$. (2)(3) Links which are equivalent to C_4.

in Figure 12(2). We observe that L is equivalent to C_4, see Figure 12(2)(3). Thus $\pi_1(S^3 \setminus C_4)$ is bi-orderable. □

Remark 5.4. *In the proof of Theorem 5.3, if we replace* $\mathfrak{t}_D^-$ *with* $\mathfrak{t}_D^+$*, then* $\mathfrak{t}_D^+$ *sends* $\mathcal{E}(\mathrm{br}(\sigma_1^{-2}\sigma_2^2))$ *to* $\mathcal{E}(\mathrm{br}(\sigma_1^{-2}\sigma_3\sigma_2\sigma_3))$*. It follows that* $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^2)$ *is homeomorphic to* $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_3\sigma_2\sigma_3)$*. By Proposition 4.1,* $\sigma_1^{-2}\sigma_3\sigma_2\sigma_3 (= \sigma_3\sigma_1^{-2}\sigma_2\sigma_3)$ *which is conjugate to* $\sigma_1^{-2}\sigma_2\sigma_3^2 \in B_4$ *is order-preserving.*

5.4. **Five cusps.** It has been conjectured [1] that $S^3 \setminus C_5$ has the smallest volume among orientable hyperbolic 3-manifolds with 5 cusps.

Theorem 5.5. *The complement* $S^3 \setminus C_5$ *has bi-orderable fundamental group.*

Proof. We take $\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2} \in P_4$. Then $\pi_1(S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2}))$ is bi-orderable. To see $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$ is homeomorphic to $S^3 \setminus C_5$, we first take a disk D bounded by the thickened unknot $K \subset \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$ as in Figure 13(1). The disk twist $\mathfrak{t}_D$ sends $\mathcal{E}(\mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2}))$ to the exterior of a link L as shown in Figure 13(2). Next we take a disk D' bounded by the thickened unknot $K' \subset L$, see Figure 13(2). Then $\mathfrak{t}_{D'}$ sends $\mathcal{E}(L)$ to the exterior of a link L' as in Figure 13(2). We see that L' is equivalent to C_5. Thus $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2}) \simeq S^3 \setminus C_5$, and $\pi_1(S^3 \setminus C_5)$ is bi-orderable. □

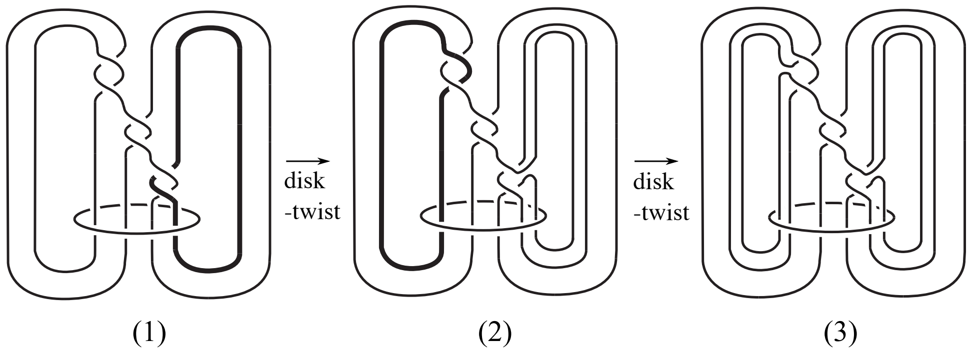


FIGURE 13. $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$ is homeomorphic to $S^3 \setminus C_5$. (1) $\mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$. (3) Link which is equivalent to C_5.

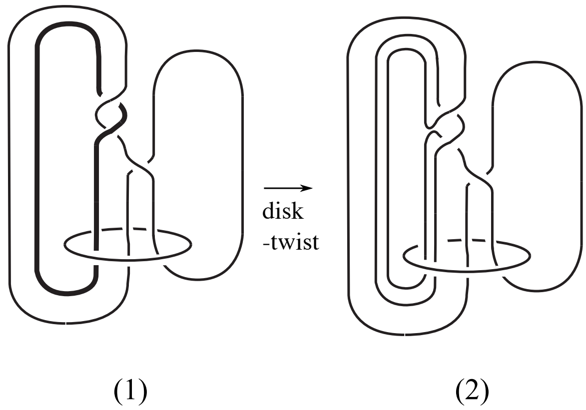


FIGURE 14. $S^3 \setminus \mathrm{br}(\sigma_1^2\sigma_2^{-1})$ is homeomorphic to $S^3 \setminus C_3$. (1) $\mathrm{br}(\sigma_1^2\sigma_2^{-1})$. (2) Link which is equivalent to C_3.

5.5. **Three cusps.** The 3-chain link complement $N = S^3 \setminus C_3$, which Gordon and Wu [18] named the *magic manifold*, has the smallest known volume among orientable hyperbolic 3-manifolds with 3 cusps. The magic manifold N is homeomorphic to $S^3 \setminus \mathrm{br}(\sigma_1^2\sigma_2^{-1})$, where $\sigma_1^2\sigma_2^{-1} \in B_3$. To see this, we take a disk D bounded by the thickened unknot $K \subset \mathrm{br}(\sigma_1^2\sigma_2^{-1})$ as shown in Figure 14(1). Then $\mathfrak{t}_D^{-1}$ sends $\mathcal{E}(\mathrm{br}(\sigma_1^2\sigma_2^{-1}))$ to the exterior of a link L as shown in Figure 14(2). Since L is equivalent to C_3, it follows that $S^3 \setminus \mathrm{br}(\sigma_1^2\sigma_2^{-1})$ is homeomorphic to N. We note that the 2nd power $(\sigma_1^2\sigma_2^{-1})^2$ is a pure braid. Hence $\pi_1(S^3 \setminus \mathrm{br}((\sigma_1^2\sigma_2^{-1})^2))$ is bi-orderable, and it has index 2 in $\pi_1(N)$.

Question 5.6. *Is $\pi_1(N)$ bi-orderable? In other words is $\sigma_1^2\sigma_2^{-1} \in B_3$ order-preserving?*

5.6. n **cusps for** $n \geq 6$. It is a conjecture by Agol [1] that for $n \leq 10$, the minimally twisted n chain link complement $S^3 \setminus C_n$ has the minimal volume among orientable hyperbolic 3-manifold with n cusps.

The following theorem follows from Lemmas 5.8 and 5.9 below.

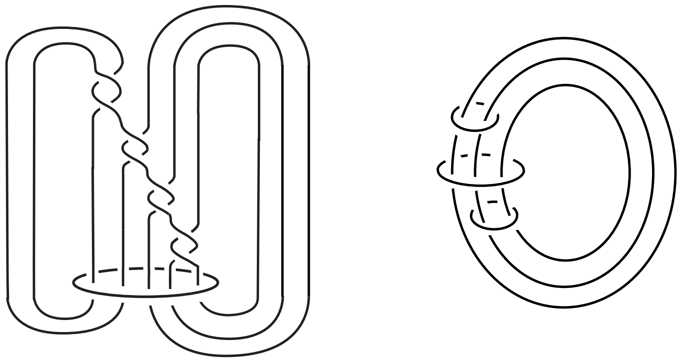


Figure 15. (1) $\mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2}\sigma_4^{-2})$. (2) The 6-circled link L_1. The complements of these links are homeomorphic to each other.

Theorem 5.7. *The complement $S^3 \setminus C_6$ has bi-orderable fundamental group.*

Let L_1 be the 6-circle link as shown in Figure 15(2).

Lemma 5.8. *Let $\beta = \sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2}\sigma_4^{-2} \in P_5$. Then $S^3 \setminus L_1$ is homeomorphic to $S^3 \setminus \mathrm{br}(\beta)$, and $\pi_1(S^3 \setminus L_1)$ is bi-orderable.*

Proof. We consider the two trivial components K, $K' \subset \mathrm{br}(\beta)$ bounded by the closure of the first string and the closure of the last string. Let D and D' be the disk bounded by K and K'. We do the disk twists $\mathfrak{t}_D$ and $\mathfrak{t}_{D'}$. Then the resulting link is equivalent to L_1. Thus $S^3 \setminus \mathrm{br}(\beta)$ is homeomorphic to $S^3 \setminus L_1$. Since β is a pure braid, this implies that $\pi_1(S^3 \setminus L_1)$ is bi-orderable. □

We thank Ken'ichi Yoshida who conveys the following argument to the authors.

Lemma 5.9. *$S^3 \setminus L_1$ is homeomorphic to $S^3 \setminus C_6$.*

Proof. We take four 3-punctured spheres embedded in the exterior $\mathcal{E}(L_1)$ as shown in Figure 16(2). The four 3-punctured spheres are also embedded in $\mathcal{E}(C_6)$ as shown in Figure 18(2). Let U (resp. U') be a subset of $\mathcal{E}(L_1)$ (resp. a subset of $\mathcal{E}(C_6)$) bounded by these 3-punctured spheres. Note that $\mathcal{E}(L_1)$ (resp. $\mathcal{E}(C_6)$) are the double of U (resp. U') with respect to the four 3-punctured spheres, see Figure 16 and Figure 18(3). We now see that U is homeomorphic to U'. This is enough to prove the lemma since the double of a manifold is uniquely determined.

We push the bottom shaded 3-punctured sphere in U as shown in Figure 16(3), and deform it into the shaded 3-punctured sphere as in Figure 16(6). (The middle red colored annulus in (3) is modified into the bottom red colored annulus in (6).) As a result, we get U'. □

Kaiser-Purcell-Rollins proved that the volume of $\mathbb{W}_{n-1}$ is strictly smaller than that of $S^3 \setminus C_n$ if $12 \leq n \leq 25$ or $n \geq 60$; see [23, Theorems 1.1, 4.1]. At the time of

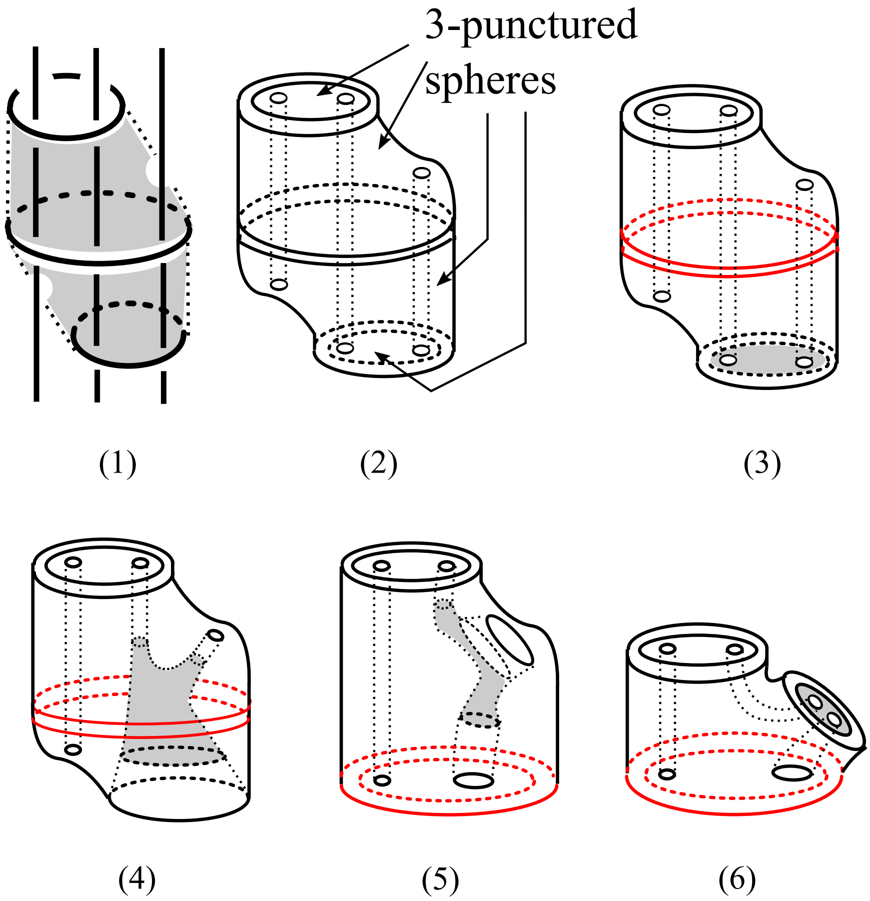


FIGURE 16. (1) (A part of) L_1. (2) A subset $U \subset \mathcal{E}(L_1)$ bounded by four 3-punctured spheres. Figures (3)–(6) show a modification of U into U'. (cf. Figure 18(2).)

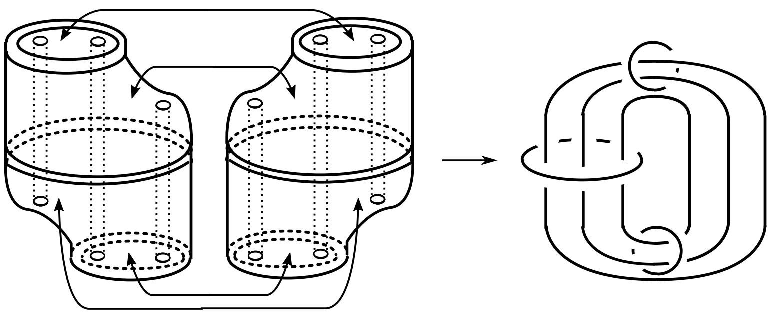


FIGURE 17. The double of U with respect to the four 3-punctured spheres is homeomorphic to $\mathcal{E}(L_1)$.

this writing, it seems that $\mathbb{W}_{n-1}$ has the smallest known volume among orientable hyperbolic 3-manifolds with n cusps if $n \geq 11$. (See also Table 1 in [23].)

Proposition 5.10. *The fundamental group of* $\mathbb{W}_n$ *is bi-orderable for each* $n \geq 2$.

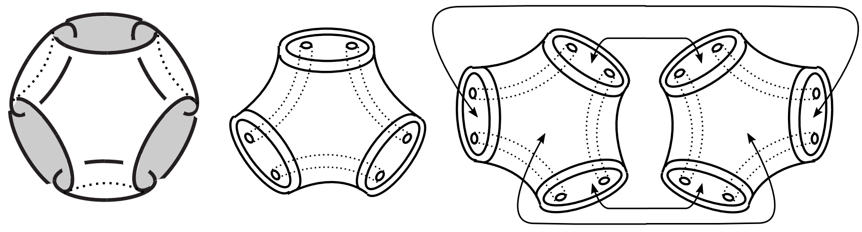


FIGURE 18. (1) C_6. (2) A subset $U' \subset \mathcal{E}(C_6)$ bounded by four 3-punctured spheres. (cf. Figure 16(6).) (3) The double of U' with respect to the four 3-punctured spheres is homeomorphic to $\mathcal{E}(C_6)$.

Proof. Let $p : \mathbb{W}_n \to \mathbb{W}$ be an n-fold cyclic covering map. Then $p_*(\pi_1(\mathbb{W}_n))$ has index n in $\pi_1(\mathbb{W})$. Since $\pi_1(\mathbb{W})$ is bi-orderable, so does $\pi_1(\mathbb{W}_n)$. □

5.7. **A result by M. Baker and a question.** We learned the following theorem from Hidetoshi Masai.

Theorem 5.11 (Baker [4])**.** *The* 6*-circle link* L_1 *is an arithmetic link. Every link* L *in* S^3 *occurs as a sublink of a link* J *such that* $S^3 \setminus J$ *is a covering space of* $S^3 \setminus L_1$*. In particular* L *is a sublink of an arithmetic link in* S^3*.*

Theorem 5.11 and Lemma 5.8 immediately give us the following.

Theorem 5.12. *Let* L *be a link in* S^3*. Then* L *is a sublink of a link whose complement has bi-orderable fundamental group.*

The link with 4 components shown in Figure 22(2) is obtained from C_3 adding a trivial knot. Its complement which is homeomorphic to $\mathbb{W}_3$ has bi-orderable fundamental group (Proposition 5.10), although we have not decided yet $N = S^3 \setminus C_3$ has bi-orderable fundamental group. We ask the following.

Question 5.13. *Let* M *be a* 3*-manifold. Does there exist a knot* K *in* M *such that* $\pi_1(M \setminus K)$ *is bi-orderable?*

6. Non-order-preserving braids

In this section, we give some sequences of non-order-preserving braids. Our sequences include examples of pseudo-Anosov braids with small dilatations. We also provide examples of non-order-preserving pseudo-Anosov braids with arbitrary large dilatations.

6.1. **Pseudo-Anosov braids with smallest dilatations.** Let $\beta \in B_n$ be a pseudo-Anosov braid. Let $\lambda(\beta) > 1$ be the dilatation (i.e, stretch factor) of the corresponding pseudo-Anosov mapping class $\beta \in \mathrm{Mod}(D_n)$. (See [12] for the definition of dilatations, for example.) It is known that there exists a minimum, denoted by $\delta(D_n)$ among dilatations for all pseudo-Anosov n-strand braids. The explicit values of minimal dilatations $\delta(D_n)$'s are determined for $3 \leq n \leq 8$ by Ko-Los-Song, Ham-Song and Lanneau-Thiffeault. The following n-strand braids realize $\delta(D_n)$.

- $n = 3$; $\sigma_1^{-1}\sigma_2 \in B_3$, see [29, 19] for example.
- $n = 4$; $\sigma_1^{-1}\sigma_2\sigma_3 \in B_4$, see [36]. (See also [20].)

- $n = 5$; $\delta_5 \sigma_4^{-1} \sigma_3^{-1} \in B_5$, see [20].
- $n = 6$; $\sigma_2 \sigma_1 \sigma_2 \sigma_1 \delta_5^2 \in B_6$, see [27].
- $n = 7$; $\sigma_4^{-2} \delta_7^2 \sim \delta_7^2 \sigma_6^{-1} \sigma_5^{-1} \in B_7$, see [27].
- $n = 8$; $\sigma_2^{-1} \sigma_1^{-1} \delta_8^5 \sim \delta_8^5 \sigma_7^{-1} \sigma_6^{-1} \in B_8$, see [27].

Here $\beta \sim \beta'$ means that β is conjugate to β'. We will prove in Section 6.2 that for $3 \leq n \leq 8$ except $n = 6$, the above n-strand braids with the smallest dilatations are not order-preserving (Corollary 6.4, Theorem 6.7, Lemma 6.8). At the time of this writing we do not know whether the above 6-strand braid $\sigma_2 \sigma_1 \sigma_2 \sigma_1 \delta_5^2$ is order-preserving or not. We remark that Question 5.6 is equivalent to the one asking whether $\sigma_2 \sigma_1 \sigma_2 \sigma_1 \delta_5^2$ is order-preserving or not, since $S^3 \setminus \mathrm{br}((\sigma_2 \sigma_1 \sigma_2 \sigma_1 \delta_5^2)^{-1} \Delta^2)$ is homeomorphic to the magic manifold N, see [26, Page 39 and Corollary 3.2].

6.2. **Sequences of pseudo-Anosov braids.**

Theorem 6.1. *For $n \geq 3$ and $k \geq 1$, $\delta_n \sigma_1^{2k} = (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) \sigma_1^{2k} \in B_n$ is not order-preserving.*

If $n \geq 5$, then $\delta_n \sigma_1 \sigma_2 \in B_n$ is pseudo-Anosov, see [21, Theorem 3.11]. This claim can be proved by using the criterion by Bestvina-Handel [5]. (See also [21, Section 2.4].) Since $\delta_n \sigma_1^2 \sim \delta_n \sigma_1 \sigma_2$, the braid $\delta_n \sigma_1^2$ is pseudo-Anosov. By using the same criterion, it is not hard to see that $\sigma_1^{2k-2} \delta_n \sigma_1 \sigma_2 \sim \delta_n \sigma_1^{2k}$ is pseudo-Anosov for $n \geq 5$ and $k \geq 1$. Viewing the transition matrix associated to the pseudo-Anosov braid $\delta_n \sigma_1^{2k}$, we see that the largest eigenvalue of the transition matrix, which is equal to $\lambda(\delta_n \sigma_1^{2k})$, goes to ∞ as n goes to ∞.

Proof of Theorem 6.1. We fix a basepoint of $\pi_1(D_n)$ in the interior of D_n. By (4.2) we have the automorphism

$$\delta_n : x_1 \mapsto x_n \mapsto x_{n-1} \mapsto \cdots \mapsto x_3 \mapsto x_2 \mapsto x_1.$$

We turn to braid $\sigma_1^{2k} \in B_n$. By (4.1) we have the following automorphism induced by $\sigma_1 \in B_n$ using the same basepoint of $\pi_1(D_n)$:

$$\sigma_1 : x_1 \mapsto x_2, \quad x_2 \mapsto x_2 x_1 x_2^{-1}, \quad x_j \mapsto x_j \text{ for } j \neq 1, 2.$$

Computing the automorphism σ_1^2 on F_n, we get

$$\sigma_1^2 : x_1 \mapsto x_2 x_1 x_2^{-1}, \quad x_2 \mapsto x_2 x_1 x_2 x_1^{-1} x_2^{-1}, \quad x_j \mapsto x_j \text{ for } j \neq 1, 2.$$

In the same manner, the automorphism σ_1^{2k} for $k \geq 1$ is given by

$$\begin{aligned} \sigma_1^{2k} : x_1 &\mapsto (x_2 x_1)^{k-1} x_2 x_1 x_2^{-1} (x_1^{-1} x_2^{-1})^{k-1}, \quad x_2 \mapsto (x_2 x_1)^k x_2 (x_1^{-1} x_2^{-1})^k, \\ x_j &\mapsto x_j \quad \text{for } j \neq 1, 2. \end{aligned}$$

Thus $\delta_n \sigma_1^{2k}$ induces the following automorphism on F_n.

$$\begin{aligned} \phi := \delta_n \sigma_1^{2k} : \ x_1 \mapsto x_n \mapsto x_{n-1} \mapsto \cdots \mapsto x_3 &\mapsto (x_2 x_1)^k x_2 (x_1^{-1} x_2^{-1})^k, \\ x_2 &\mapsto (x_2 x_1)^{k-1} x_2 x_1 x_2^{-1} (x_1^{-1} x_2^{-1})^{k-1}. \end{aligned}$$

We assume that ϕ preserves some bi-ordering $<$ on F_n. Without loss of generality, we may assume that $x_1 < x_2$. Then we have

$$(x_2 x_1)^{k-1} x_2 x_1 x_2^{-1} (x_1^{-1} x_2^{-1})^{k-1} = (x_2 x_1)^k x_1 (x_1^{-1} x_2^{-1})^k < (x_2 x_1)^k x_2 (x_1^{-1} x_2^{-1})^k$$

by conjugation invariance of the bi-ordering. This implies that

$$x_2 = \phi^{-1}((x_2x_1)^{k-1}x_2x_1x_2^{-1}(x_1^{-1}x_2^{-1})^{k-1}) < \phi^{-1}((x_2x_1)^k x_2(x_1^{-1}x_2^{-1})^k) = x_3,$$

since ϕ^{-1} also preserves the same bi-ordering $<$. Hence we have $x_1 < x_2 < x_3$. Notice that x_3 is contained in the orbit $\mathcal{O}(x_1)$ under ϕ, i.e,

$$x_1 \mapsto \phi(x_1) = x_n \mapsto \cdots \mapsto x_3 \mapsto (x_2x_1)^k x_2(x_1^{-1}x_2^{-1})^k \mapsto \cdots$$

Since $x_1 < x_3$, this implies that the ordering $<$ is increasing on $\mathcal{O}(x_1)$, i.e,

$$x_1 < \phi(x_1) = x_n < \phi^2(x_1) = x_{n-1} < \cdots < x_3 < (x_2x_1)^k x_2(x_1^{-1}x_2^{-1})^k < \cdots$$

Notice that $x_1 < x_2$ implies that

$$\phi(x_1) = x_n < \phi(x_2) = (x_2x_1)^{k-1}x_2x_1x_2^{-1}(x_1^{-1}x_2^{-1})^{k-1}.$$

This together with $x_1 < x_n$ tells us that

$$x_1 < (x_2x_1)^{k-1}x_2x_1x_2^{-1}(x_1^{-1}x_2^{-1})^{k-1}. \tag{6.1}$$

We multiply the both sides of (6.1) by $(x_2x_1)^{k-1}x_2$ on the right.

$$x_1(x_2x_1)^{k-1}x_2 < (x_2x_1)^{k-1}x_2x_1x_2^{-1}(x_1^{-1}x_2^{-1})^{k-1}(x_2x_1)^{k-1}x_2 = (x_2x_1)^{k-1}x_2x_1.$$

Thus

$$(x_1x_2)^k = x_1(x_2x_1)^{k-1}x_2 < (x_2x_1)^{k-1}x_2x_1 = (x_2x_1)^k.$$

On the other hand, since $x_2 < x_3$ and $x_3 < (x_2x_1)^k x_2(x_1^{-1}x_2^{-1})^k$, we have

$$x_2 < (x_2x_1)^k x_2(x_1^{-1}x_2^{-1})^k. \tag{6.2}$$

Multiply the both sides of (6.2) by $(x_2x_1)^k$ on the right.

$$x_2(x_2x_1)^k < (x_2x_1)^k x_2(x_1^{-1}x_2^{-1})^k(x_2x_1)^k = x_2(x_1x_2)^k.$$

This implies that $(x_2x_1)^k < (x_1x_2)^k (< (x_2x_1)^k)$, a contradiction. Thus $\delta_n\sigma_1^{2k}$ is not order-preserving. □

We have $\mathrm{br}(\delta_n\sigma_1^{2k}) = \mathrm{br}(\sigma_1^{2k}\delta_n)$, and it is equivalent to the $(-2, 2k+1, 2n-2)$-pretzel link; see Figure 19 in the case $n = 5$ and $k = 1$. By Theorem 6.1 and Propsition 4.1, we have the following.

Corollary 6.2. *For each $n \geq 3$ and $k \geq 1$, the fundamental group of the $(-2, 2k+1, 2n-2)$-pretzel link complement is not bi-orderable.*

We turn to another sequence of braids.

Theorem 6.3. *Let $n \geq 3$ and $k \geq 1$. Then $\delta_n^{n-1}\sigma_1^{2k} = (\sigma_1\sigma_2\cdots\sigma_{n-1})^{n-1}\sigma_1^{2k} \in B_n$ is not order-preserving.*

Proof. Fix a basepoint of $\pi_1(D_n)$ in the interior of D_n. By using (4.2), we calculate the automorphism $\delta_n^{n-1} = (\sigma_1\sigma_2\cdots\sigma_{n-1})^{n-1}$.

$$\delta_n^{n-1} : x_1 \mapsto x_2 \mapsto \cdots \mapsto x_{n-1} \mapsto x_n \mapsto x_1.$$

In the same manner as in the proof of Theorem 6.1, we have the following automorphism on F_n obtained from the braid $\delta_n^{n-1}\sigma_1^{2k}$ using the same basepoint of $\pi_1(D_n)$:

$$\begin{aligned} \delta_n^{n-1}\sigma_1^{2k} : x_2 \mapsto x_3 \mapsto \cdots \mapsto x_{n-1} \mapsto x_n &\mapsto (x_2x_1)^{k-1}x_2x_1x_2^{-1}(x_1^{-1}x_2^{-1})^{k-1}, \\ x_2 &\mapsto (x_2x_1)^k x_2(x_1^{-1}x_2^{-1})^k. \end{aligned}$$

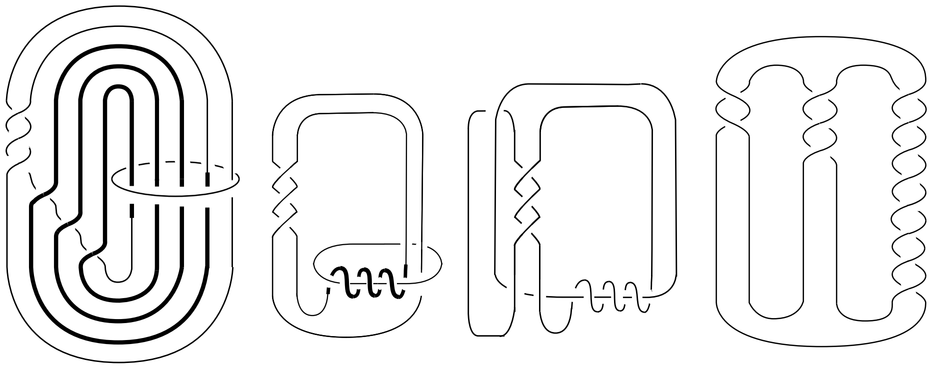


FIGURE 19. $\mathrm{br}(\sigma_1^{2k}\delta_n)$ is equivalent to the $(-2, 2k+1, 2n-2)$-pretzel link ($n = 5$, $k = 1$ in this case), see (1) $\to$ (2) $\to$ (3) $\to$ (4). (1) $\mathrm{br}(\sigma_1^2\delta_5)$. (4) $(-2, 3, 8)$-pretzel link.

Assume that $\delta_n^{n-1}\sigma_1^{2k}$ preserves some bi-ordering on F_n. In the same manner as in the proof of Theorem 6.1, one arrives a contradiction. □

Corollary 6.4. *Let $n \geq 3$ and $k \geq 1$. Then $\sigma_1^{-2k}\delta_n = \sigma_1^{-2k+1}\sigma_2\sigma_3\cdots\sigma_{n-1} \in B_n$ is not order-preserving. In particular $\sigma_1^{-1}\sigma_2\sigma_3\cdots\sigma_{n-1} \in B_n$ is not order-preserving for each $n \geq 3$.*

Proof. By Theorem 6.3, the inverse $(\delta_n^{n-1}\sigma_1^{2k})^{-1}$ is not order-preserving. Hence $(\delta_n^{n-1}\sigma_1^{2k})^{-1}\Delta^{2\ell}$ is not order-preserving for each $\ell \in \mathbb{Z}$. We have

$$\begin{aligned}(\delta_n^{n-1}\sigma_1^{2k})^{-1}\Delta^2 &= ((\sigma_1\sigma_2\cdots\sigma_{n-1})^{n-1}\sigma_1^{2k})^{-1}(\sigma_1\sigma_2\cdots\sigma_{n-1})^n \\ &= \sigma_1^{-2k}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1})^{n-1}(\sigma_1\sigma_2\cdots\sigma_{n-1})^n \\ &= \sigma_1^{-2k}\sigma_1\sigma_2\cdots\sigma_{n-1}(= \sigma_1^{-2k}\delta_n).\end{aligned}$$

This completes the proof. □

Remark 6.5. *For each $n \geq 3$, $\sigma_1^{-2}\delta_n = \sigma_1^{-1}\sigma_2\sigma_3\cdots\sigma_{n-1} \in B_n$ is pseudo-Anosov, see* [21, Theorem 3.9]. *It is not hard to see that*

$$\sigma_1^{-2k}\delta_n = \sigma_1^{-2k+1}\sigma_2\cdots\sigma_{n-1}$$

is pseudo-Anosov for $n \geq 3$ and $k \geq 1$ by using the Bestvina-Handel criterion. The Nielsen-Thurston types of a mapping class and its inverse are the same. Thus $\delta_n^{n-1}\sigma_1^{2k} \in B_n$ in Theorem 6.3 is pseudo-Anosov for $n \geq 3$ and $k \geq 1$.

Using similar modifications of links as in Figure 19, we verify that $\mathrm{br}(\sigma_1^{-2k}\delta_n)$ is equivalent to the $(-2, -2k+1, 2n-2)$-pretzel link. Thus we have the following.

Corollary 6.6. *For each $n \geq 3$ and $k \geq 1$, the fundamental group of the $(-2, -2k+1, 2n-2)$-pretzel link complement is not bi-orderable.*

We turn to the last sequence of braids which plays an important role for the study of pseudo-Anosov minimal dilatations, see [21, 26].

Theorem 6.7. *For each $n \geq 5$,*

$$\delta_n^2 \sigma_{n-1}^{-1} \sigma_{n-2}^{-1} = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})(\sigma_1 \sigma_2 \cdots \sigma_{n-3}) \in B_n$$

is not order-preserving.

The braid $\delta_n^2 \sigma_{n-1}^{-1} \sigma_{n-2}^{-1}$ is reducible if n is even. (In fact it is easy to find an essential simple closed curve on D_n containing the punctures labelled $2, 4, \cdots, n-2$ which is invariant under the corresponding mapping class of $\mathrm{Mod}(D_n)$.) If n is odd, the braid $\delta_n^2 \sigma_{n-1}^{-1} \sigma_{n-2}^{-1}$ is pseudo-Anosov, see [21, Theorem 3.11, Figure 18(c)].

Proof of Theorem 6.7. We fix a basepoint of $\pi_1(D_n)$ in the interior of D_n. From the automorphism σ_i of F_n for each $i = 1, \cdots, n-1$ (see (4.1)), we calculate the automorphism σ_i^{-1} on F_n:

$$\sigma_i^{-1} : x_i \mapsto x_i^{-1} x_{i+1} x_i, \quad x_{i+1} \mapsto x_i, \quad x_j \mapsto x_j \text{ if } j \neq i, i+1.$$

By (4.2) together with automorphisms σ_i^{-1} for $i = n-1, n-2$, we see that the following automorphism is arisen from the braid $\delta_n \sigma_{n-1}^{-1} \sigma_{n-2}^{-1} = \sigma_1 \sigma_2 \cdots \sigma_{n-3}$.

$$\begin{aligned} \delta_n \sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \quad : \quad & x_1 \mapsto x_{n-2} \mapsto x_{n-3} \mapsto \cdots \mapsto x_2 \mapsto x_1, \\ & x_{n-1} \mapsto x_{n-2}^{-1} x_{n-1} x_{n-2}, \quad x_n \mapsto x_{n-2}^{-1} x_n x_{n-2}, \end{aligned}$$

see Figure 20. Thus the automorphism $\psi := \delta_n(\delta_n \sigma_{n-1}^{-1} \sigma_{n-2}^{-1}) : F_n \to F_n$ is given by

$$\begin{aligned} \psi = \delta_n^2 \sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \quad : \quad & x_1 \mapsto x_{n-2}^{-1} x_n x_{n-2}, \\ & x_2 \mapsto x_{n-2},\ x_3 \mapsto x_1, \cdots, x_{n-2} \mapsto x_{n-4},\ x_{n-1} \mapsto x_{n-3}, \\ & x_n \mapsto x_{n-2}^{-1} x_{n-1} x_{n-2}. \end{aligned}$$

Suppose that n is even. Then the orbit of x_2 under ψ is nontrivial and finite:

$$x_2 \mapsto x_{n-2} \mapsto x_{n-4} \mapsto \cdots \mapsto x_4 \mapsto x_2.$$

Thus ψ does not preserve any left-ordering on F_n by Proposition 3.11.

Suppose that n is odd. Then the above automorphism $\psi : F_n \to F_n$ is described as follows.

$$\begin{aligned} x_{n-1} \mapsto x_{n-3} \mapsto \cdots \mapsto x_2 \mapsto x_{n-2} \mapsto x_{n-4} \mapsto x_1 \quad & \mapsto \quad x_{n-2}^{-1} x_n x_{n-2}, \\ x_n \quad & \mapsto \quad x_{n-2}^{-1} x_{n-1} x_{n-2}. \end{aligned}$$

We suppose that ψ preserves a bi-ordering $<$ of F_n. Without of loss of generality, we may assume that $x_{n-1} < x_n$. By the conjugation invariance of the bi-ordering, $x_{n-2}^{-1} x_{n-1} x_{n-2} < x_{n-2}^{-1} x_n x_{n-2}$ holds. Since ψ^{-1} preserves the same ordering $<$, this implies that $x_n < x_1$, see the above definition of ψ. Hence we have $x_{n-1} < x_n < x_1$. This tells us that the ordering $<$ is increasing on the orbit $\mathcal{O}(x_{n-1})$, i.e,

$$x_{n-1} < x_{n-3} < \cdots < x_2 < x_{n-2} < x_{n-4} < x_1 < x_{n-2}^{-1} x_n x_{n-2} < \cdots$$

In particular $x_{n-2} < x_{n-2}^{-1} x_n x_{n-2}$ which implies that $x_{n-2} < x_n$ by the conjugation invariance of the bi-ordering. This together with the ψ-invariance of $<$ gives us that $x_{n-4} < x_{n-2}^{-1} x_{n-1} x_{n-2}$.

On the other hand, by $x_{n-1} < x_{n-2}$, we have $x_{n-2}^{-1} x_{n-1} x_{n-2} < x_{n-2}$ by the conjugation invariance again. However we have $x_{n-2} < x_{n-4} < x_{n-2}^{-1} x_{n-1} x_{n-2}$, in particular

$$x_{n-2} < x_{n-2}^{-1} x_{n-1} x_{n-2} \ (< x_{n-2}),$$

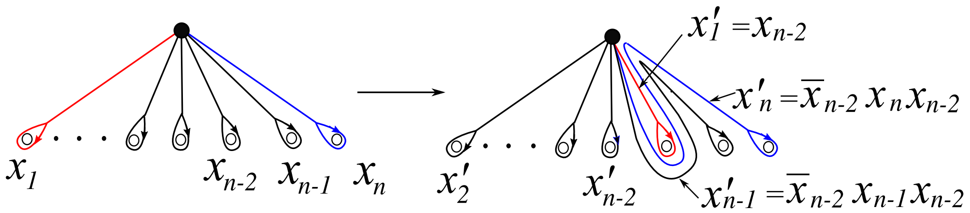


FIGURE 20. $\delta_n \sigma_{n-1}^{-1} \sigma_{n-2}^{-1} = \sigma_1 \sigma_2 \cdots \sigma_{n-3} : F_n \to F_n$.

which is a contradiction. This completes the proof. □

Finally we claim that the following pseudo-Anosov braid with the minimal dilatation $\delta(D_8)$ is not order-preserving.

Lemma 6.8. *The braid* $\delta_8^5 \sigma_7^{-1} \sigma_6^{-1} = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7)^4 \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 \in B_8$ *is not order-preserving.*

Proof. Fix a basepoint of $\pi_1(D_8)$ in the interior of D_8. The following automorphism corresponds to the braid $\delta_8^5 \sigma_7^{-1} \sigma_6^{-1}$.

$$\begin{aligned} \phi := \delta_8^5 \sigma_7^{-1} \sigma_6^{-1} : x_7 \mapsto x_2 \mapsto x_5 \mapsto x_6 \mapsto x_1 \mapsto x_4 \mapsto & \quad x_6^{-1} x_8 x_6, \\ x_8 \mapsto x_3 \mapsto & \quad x_6^{-1} x_7 x_6. \end{aligned}$$

Assume that ϕ preserves a bi-ordering $<$. Without loss of generality, we may suppose that $x_7 < x_8$. Then $x_6^{-1} x_7 x_6 < x_6^{-1} x_8 x_6$ by the conjugation invariance. Since ϕ^{-1} preserves the same ordering $<$, we have

$$x_8 = \phi^{-2}(x_6^{-1} x_7 x_6) < x_1 = \phi^{-2}(x_6^{-1} x_8 x_6).$$

From $x_7 < x_8$ and $x_8 < x_1$, we get $x_7 < x_1$. Hence the ordering is increasing on the orbit of x_7:

$$x_7 < x_2 < x_5 < x_6 < x_1 < x_4 < x_6^{-1} x_8 x_6 < \phi(x_6^{-1} x_8 x_6) < \cdots$$

In particular $x_6 < x_6^{-1} x_8 x_6$, and this implies that $x_6 < x_8$ by the conjugation invariance. Then we get $\phi^2(x_6) = x_4 < \phi^2(x_8) = x_6^{-1} x_7 x_6$. On the other hand, $x_7 < x_6$ implies that $x_6^{-1} x_7 x_6 < x_6$ by the conjugation invariance again. But this is a contradiction, since

$$x_6 < x_1 < x_4 < x_6^{-1} x_7 x_6 (< x_6).$$

Thus we conclude that ϕ does not preserve any bi-ordering of F_8. □

6.3. **Questions.** Consider the braid $\beta_{m,n} \in B_{m+n+1}$ for $m, n \geq 1$ given by

$$\beta_{m,n} = \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_m^{-1} \sigma_{m+1} \sigma_{m+2} \cdots \sigma_{m+n},$$

see Figure 21(1). For example $\beta_{1,2} = \sigma_1^{-1} \sigma_2 \sigma_3$. The link $\mathrm{br}(\beta_{m,n})$ is equivalent to a 2-bridge link as shown in Figure 21(2). For any $m, n \geq 1$, the braid $\beta_{m,n}$ is pseudo-Anosov, see [21, Theorem 3.9]. We proved in Corollary 6.4 that $\beta_{1,n} = \sigma_1^{-1} \sigma_2 \cdots \sigma_{n+1}$ is not order-preserving for each $n \geq 1$. We conjecture that $\beta_{m,n}$ is not order-preserving for any $m, n \geq 1$.

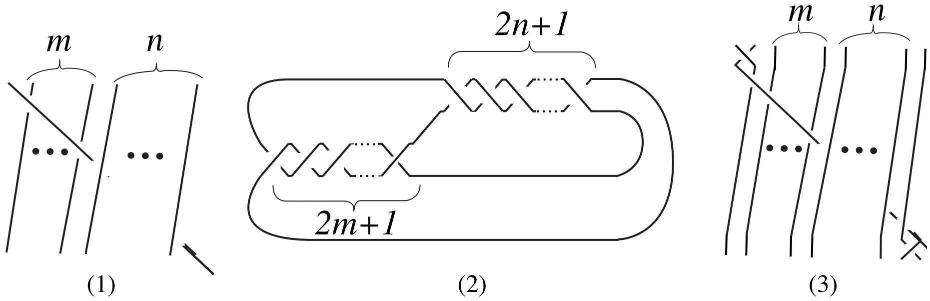


FIGURE 21. (1) $\beta_{m,n} \in B_{m+n+1}$. (2) 2-bridge link which is equivalent to $\mathrm{br}(\beta_{m,n})$. (3) $\gamma_{m,n} \in B_{m+n+3}$.

Let us introduce the braid $\gamma_{m,n} \in B_{m+n+3}$ for $m, n \geq 0$. If $m, n \geq 1$, $\gamma_{m,n}$ is of the form

$$\gamma_{m,n} = \sigma_1^{-2}\sigma_2^{-1} \cdots \sigma_{m+1}^{-1}\sigma_{m+2} \cdots \sigma_{m+n+1}\sigma_{m+n+2}^2.$$

For example $\gamma_{1,1} = \sigma_1^{-2}\sigma_2^{-1}\sigma_3\sigma_4^2 \in B_5$. If we ignore the 1st string and the last string of $\gamma_{m,n}$, the resulting braid is an $(m+n+1)$-strand braid which is equal to $\beta_{m,n}$. We let $\gamma_{0,0} = \sigma_1^{-2}\sigma_2^2 \in P_3$, $\gamma_{0,1} = \sigma_1^{-2}\sigma_2\sigma_3^2 \in B_4$ (see Remark 5.4), and $\gamma_{1,0} = \sigma_1^{-2}\sigma_2^{-1}\sigma_3^2 \in B_4$.

One can prove in the same manner as in Theroem 5.3 and Remark 5.4 that for each $m, n \geq 0$

$$S^3 \setminus \mathrm{br}(\gamma_{m,n}) \simeq S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^2) \simeq S^3 \setminus C_4.$$

Since $\sigma_1^{-2}\sigma_2^2$ is a pseudo-Anosov, pure 3-strand braid (in other words, $S^3 \setminus C_4$ is a hyperbolic 3-manifolds whose fundamental group if bi-orderable), we have the following.

Lemma 6.9. *The braid $\gamma_{m,n}$ is pseudo-Anosov and order-preserving for $m, n \geq 0$.*

If $(m,n) \neq (0,0)$, braids $\gamma_{m,n}$'s are non-pure and order-preserving. By Proposition 3.14, the braid $\gamma_{m,n}$ preserves some bi-ordering which is not standard ordering of the free group. Notice that for $m, n \geq 0$, the permutation of $\mathcal{S}_{m+n+3}$ associated to $\gamma_{m,n}$ has more than 1 cycle, i.e, the closure of the braid $\gamma_{m,n}$ is a link with more than 1 component. We ask the following.

Question 6.10. *Does there exist an order-preserving braid $\beta \in B_n$ $(n \geq 3)$ whose permutation is cyclic (i.e, the closure $\widehat{\beta}$ is a knot)?*

APPENDIX A. ORDERING FREE GROUPS

For any group G the lower central series $G = \gamma_1 G \supset \gamma_2 G \supset \cdots$ is defined inductively by the formula $\gamma_{k+1}G := [G, \gamma_k G]$. It is well known (see for example [30]) that if F is a free group (or hyperbolic surface group) then the lower central quotients $\gamma_k F/\gamma_{k+1}F$ are free abelian of finite rank and also that $\cap_{k=1}^{\infty}\gamma_k F = \{1\}$. Following [31] one can bi-order F as we will now describe. Choose an arbitrary bi-order $<_k$ of the (free abelian) group $\gamma_k F/\gamma_{k+1}F$ and define the positive cone for an ordering $<$ of F as follows. For $1 \neq g \in F$, declare g to be positive if $1 <_k [g]$ in $\gamma_k F/\gamma_{k+1}F$, where k is the unique integer for which $g \in \gamma_k F \setminus \gamma_{k+1}F$. It is routine to check that this defines a bi-ordering $<$ of F. We shall say that an ordering of F defined in this way is a *standard* ordering of F. If the rank of F is greater than one, there are uncountably many standard orderings. However there are uncountably many non-standard orderings of F as well, for example as in the end of Section 4.6.

Proposition A.1. *For any standard ordering of the free group F, all the lower central subgroups $\gamma_k F$ are convex.*

Proof. Using invariance under multiplication it is sufficient to suppose $1 < f < g$ and $g \in \gamma_k F$, and show that $f \in \gamma_k F$. Now $f \in \gamma_j F \setminus \gamma_{j+1} F$ for some unique positive integer j. Assume for contradiction that $j < k$. By the definition of $<$ we have that $1 <_j [f]$ in $\gamma_j F/\gamma_{j+1}F$. But since $j < k$ we also have $g \in \gamma_j F$ and we see that $f^{-1}g \in \gamma_j F \setminus \gamma_{j+1} F$. But since $1 < f^{-1}g$ we have $1 <_j [f^{-1}g]$ in $\gamma_j F/\gamma_{j+1}F$. However, $g \in \gamma_{j+1}$ and so $[f^{-1}g] = [f^{-1}]$, which implies the contradiction that $1 <_j [f^{-1}]$. Therefore we conclude that $j \geq k$ and therefore $f \in \gamma_j F \subset \gamma_k F$. □

Note that for any automorphism of a group $\phi : G \to G$ we have $\phi(\gamma_k G) = \gamma_k G$ and so there are induced automorphisms $\phi_k : \gamma_k G/\gamma_{k+1}G \to \gamma_k G/\gamma_{k+1}G$. In particular, ϕ_1 is the same as the abelianization $\phi_{ab} : G/[G, G] \to G/[G, G]$. The following is a well-known group theoretic result, but we include a proof, suggested by Thomas Koberda, for the reader's convenience.

Lemma A.2. *Suppose $\phi : G \to G$ is an automorphism of a group G such that the abelianization $\phi_{ab} : G/[G, G] \to G/[G, G]$ is the identity map. Then for each $k \geq 1$, the homomorphism $\phi_k : \gamma_k G/\gamma_{k+1}G \to \gamma_k G/\gamma_{k+1}G$ is also the identity map.*

Proof. We prove $\phi_k = id$ by induction on k, the hypothesis being the base case. Note that the hypothesis implies that, for every $g \in G$ we have $\phi(g) = gc$ where $c \in \gamma_2 G$ may depend upon g. Now, suppose that $\phi_k : \gamma_k G/\gamma_{k+1}G \to \gamma_k G/\gamma_{k+1}G$ is the identity. This implies that for $h \in \gamma_k G$ we have that $\phi(h) = hd$, where $d \in \gamma_{k+1}G$, also dependent upon h.

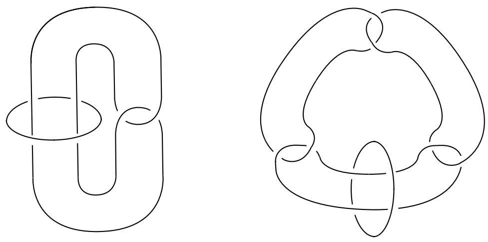
To show that $\phi_{k+1} = id$, consider any nonidentity element of $\gamma_{k+1}G$. Such an element is a product of commutators of the form $[g, h]$ with $g \in G, h \in \gamma_k G$, so it suffices to show that

$$\phi_{k+1}([g, h]) \equiv [g, h] \mod \gamma_{n+2}G.$$

We calculate $\phi_{k+1}([g, h]) \equiv [\phi(g), \phi(h)] = [gc, hd] = gchdc^{-1}g^{-1}d^{-1}h^{-1} \equiv gchc^{-1}g^{-1}h^{-1}$, the latter equivalence following because $d \in \gamma_{k+1}G$ commutes with every element of G, modulo $\gamma_{k+2}G$. But note that $gchc^{-1}g^{-1}h^{-1} = [g, h]x$, where $x = hg(h^{-1}chc^{-1})g^{-1}h^{-1}$. Observe that the expression in parentheses is a commutator of $h^{-1} \in \gamma_k G$ and $c \in \gamma_2 G$. It is well-known (see for example [30, page 293]) that $[\gamma_m G, \gamma_n G] \subset \gamma_{m+n}G$. Therefore, $h^{-1}chc^{-1} \in \gamma_{k+2}G$. Being a conjugate, x also belongs to $\gamma_{k+2}G$, and therefore $\phi([g, h]) \equiv [g, h]x \equiv [g, h] \mod \gamma_{k+2}G$. □

Proposition A.3. *Suppose $\phi : F_n \to F_n$ is such that $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ is the identity map $\phi_{ab} = id : \mathbb{Z}^n \to \mathbb{Z}^n$. Then ϕ preserves* every *standard bi-ordering of F_n.*

Proof. By Lemma A.2, with $G = F_n$ we see that ϕ_k is the identity automorphism of $\gamma_k F_n/\gamma_{k+1}F_n$ and therefore preserves *every* ordering of $\gamma_k F_n/\gamma_{k+1}F_n$. It follows that ϕ preserves the positive cone of every standard ordering of F_n. □



(1) (2)

Figure 22. (1) Whitehead link W. (2) $\mathbb{W}_3$ is homeomorphic to the complement of this link.

Appendix B. The Whitehead link

Let W be the Whitehead link in S^3, see Figure 22(1). Our goal is to prove:

Theorem B.1. *The fundamental group of the Whitehead link complement* $\mathbb{W} = S^3 \setminus W$ *is bi-orderable.*

We first recall the Murasugi sum of surfaces. See [25, Section 4.2] for more details. Let R_1, R_2 and R be compact oriented surface embedded in S^3. We say that R is a $(2n$-)*Murasugi sum* of R_1 and R_2 if we have the following.

(1) $R = R_1 \cup R_2$, and $R_1 \cap R_2$ is a disk D which satisfies the following.
  (1.1) ∂D is a $2n$-gon with edges $a_1, b_1, a_2, b_2, \cdots, a_n, b_n$ enumerated in this order.
  (1.2) $a_i \subset \partial R_1$, and a_i is a proper arc in R_2 for all i.
  (1.3) $b_i \subset \partial R_2$, and b_i is a proper arc in R_1 for all i.
(2) There exist 3-balls $\mathbb{B}_1$ and $\mathbb{B}_2$ in S^3 such that
  (2.1) $\mathbb{B}_1 \cup \mathbb{B}_2 = S^3$, $\mathbb{B}_1 \cap \mathbb{B}_2 = \partial \mathbb{B}_1 = \partial \mathbb{B}_2 = S^2$.
  (2.2) $\mathbb{B}_i \supset R_i$ for $i = 1, 2$.
  (2.3) $\partial \mathbb{B}_1 \cap R_1 = \partial \mathbb{B}_2 \cap R_2 = D$.

In this paper we only use 2-, 4-Murasugi sums, see Figures 24 and 26. A 2-Murasugi sum corresponds to a connected sum of links. A 4-Murasugi sum is a so-called *plumbing*. To state the next theorem, we let $L_i = \partial R_i$ for $i = 1, 2$ and $L = \partial R$ which are oriented links.

Theorem B.2 (Theorem 1.3 and Corollary 1.4 in Gabai [13])**.** *Suppose that R is a Murasugi sum of R_1 and R_2. Then L is a fibered link with a fiber R if and only if L_i is a fibered link with a fiber R_i for $i = 1, 2$.*

Let R_i, R and L_i be as in Theorem B.2, and let $f_i : R_i \to R_i$ denote the monodromy. We may assume that $f_i|_{\partial R_i}$ equals the identity map *id*. Let R be a Murasugi sum of R_1 and R_2. By Theorem B.2, $L = \partial R$ is a fibered link with a fiber R if L_i is a fibered link with a fiber R_i for $i = 1, 2$. The following theorem tells us what the monodromy $f : R \to R$ looks like.

Theorem B.3 (Corollary 1.4 in [13])**.** *The monodromy $f : R \to R$ is given by the product (i.e, the composition) $f = f_2' f_1' : R \to R$, where $f_i'|_{R_i}$ equals f_i and $f_i'|_{R \setminus R_i}$ equals the identity map for $i = 1, 2$.*

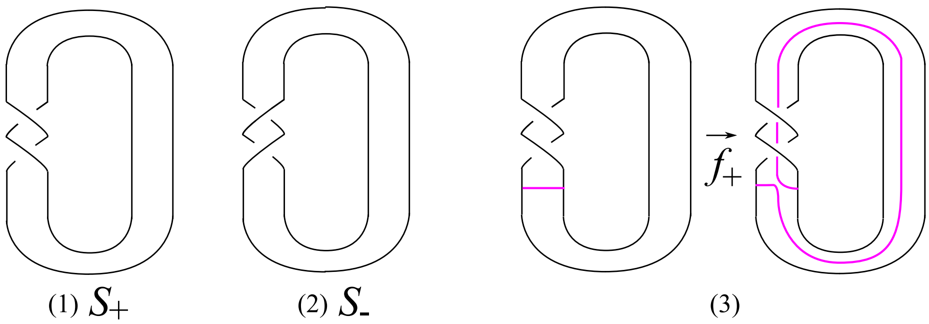


FIGURE 23. Hopf bands (1) S_+ and (2) S_-. (3) Monodromy $f_+ : S_+ \to S_+$ is the right handed Dehn twist about the core circle of S_+. The figure illustrates the image of a proper arc (as shown in the left) under f_+.

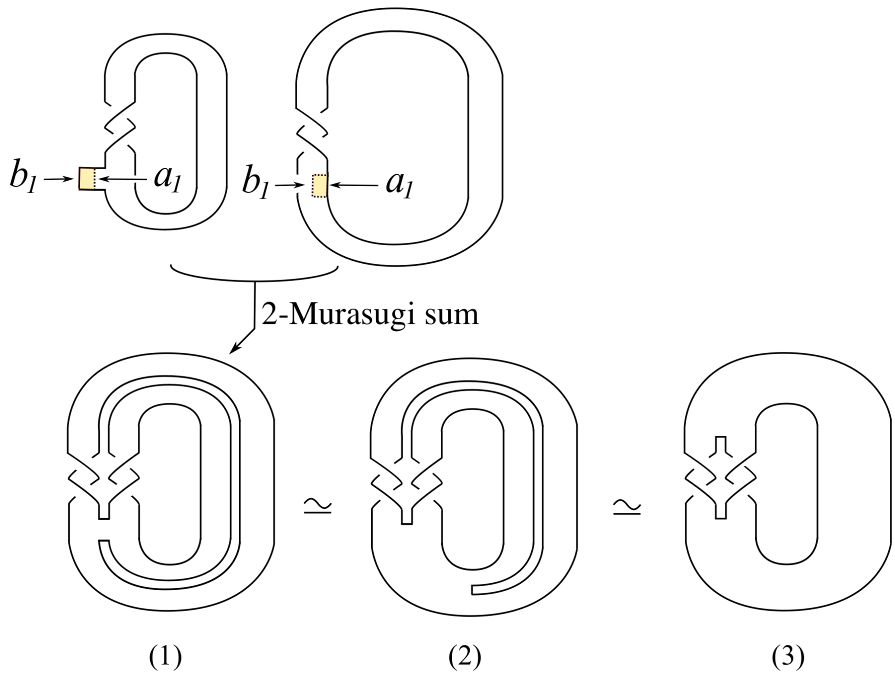


FIGURE 24. (1) 2-Murasugi sum $\hat{S}$ of S_+ and S_-. (The 2-gon with edges a_1, b_1 is shaded.) (2)(3) Surfaces which are isotopic to $\hat{S}$.

Convention B.4. *The product $f_2' f_1'$ means that we first apply f_2', then apply f_1'.*

A *Hopf band* is an unknotted annulus in S^3. Two kinds of hopf bands S_+ and S_- as in Figure 23(1) and (2) are called *positive* and *negative* respectively. The links $L_+ = \partial S_+$ and $L_- = \partial S_-$ are called the *Hopf links.* It is known that $L_\pm$ is a fibered link with a fiber $S_\pm$. The monodromy $f_+ : S_+ \to S_+$ (resp. $f_- : S_- \to S_-$) is the right handed Dehn twist (resp. left handed Dehn twist) about the core circle of the annulus, see [14, Figure 1].

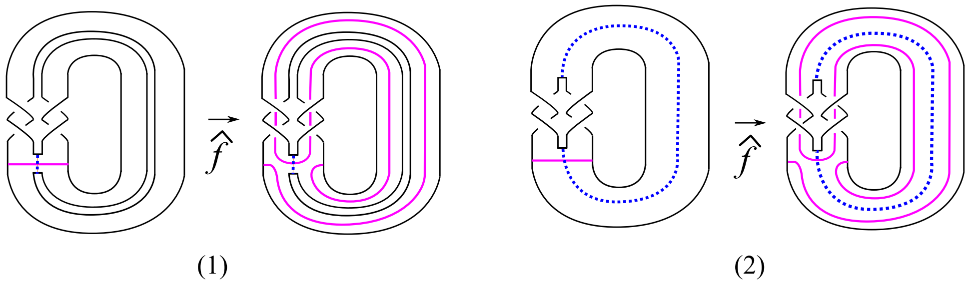


FIGURE 25. (1) $\hat{f} = f'_+ f'_- : \hat{S} \to \hat{S}$ using $\hat{S}$ in Figure 24(1). (2) $\hat{f} = f'_+ f'_- : \hat{S} \to \hat{S}$ using $\hat{S}$ in Figure 24(3). Both (1) and (2) illustrate the images of solid and broken arcs (as shown in the left) under $\hat{f}$.

Proof of Theorem B.1. We will find a fibered surface S of the Whitehead link W and its monodromy $f : S \to S$. Let $\hat{S}$ be the 2-Murasugi sum of hopf bands S_+ and S_-, see Figure 24(1). Figure 24(3) is a surface which is isotopic to $\hat{S}$. By Theorem B.2, $\partial\hat{S}$ is a fibered link with a fiber $\hat{S}$. Theorem B.3 tells us that $f'_+ f'_- : \hat{S} \to \hat{S}$ serves the monodromy $\hat{f} : \hat{S} \to \hat{S}$. Figure 25(1)(2) shows the images of two proper arcs (solid and broken arcs) under $\hat{f} : \hat{S} \to \hat{S}$. This is the so-called *point-pushing map*, see [12, Section 4.2].

A 4-Murasugi sum of $\hat{S}$ and S_+ gives rise to a fibered surface S of W, which is a torus with 2 boundary components, see Figure 26. By Theorem B.3, the product $(\hat{f})' f'_+ : S \to S$ serves the monodromy $f : S \to S$. Note that $(\hat{f})' : S \to S$ is a pushing map along the arc m, and $f'_+ : S \to S$ is the right handed Dehn twist about a simple closed curve ℓ, see Figure 27(1).

Shrinking one of the boundary components to a puncture, one can take a simple model as a representative of S, which is a torus with one boundary component and with a puncture as in Figure 27(2). Abusing the notation, we denote such a simple model by the same notation S. We also denote the corresponding loop based at the puncture and the corresponding closed loop in the simple model by the same notations m and ℓ, see Figure 27(2).

We choose a basepoint p on ∂S, and take oriented loops c_0, c_1 and c_2 based at this point so that c_0 is a loop surrounding the puncture, and c_1, c_2 are the meridian and the longitude of the torus, see Figure 28(1). Abusing the notations, we denote the equivalence class of c_i in $\pi_1(S,p) = F_3$ by the same notation c_i. (Then $\{c_0, c_1, c_2\}$ is a generating set of $\pi_1(S,p)$.) The images of c_1, c_2 and c_3 under

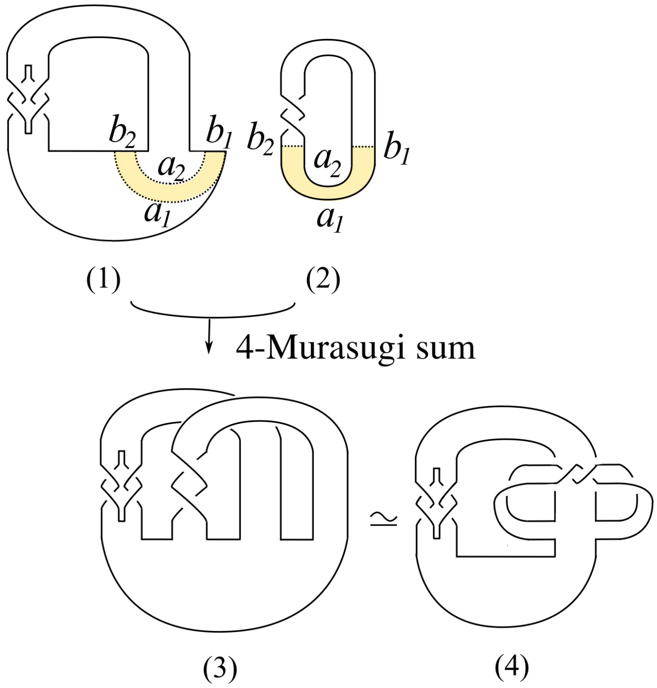


FIGURE 26. (1) $\hat{S}$. (2) S_+. (3)(4) 4-Murasugi sum of $\hat{S}$ and S_+ is a fibered surface S of W. (The 4-gon with edges a_1, b_1, a_2, b_2 is shaded in the figures (1)(2).)

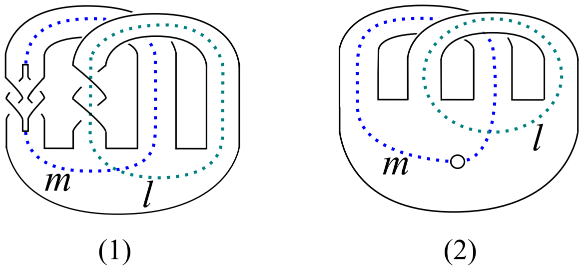


FIGURE 27. (1) A fibered surface S of W. (2) A simple model of S.

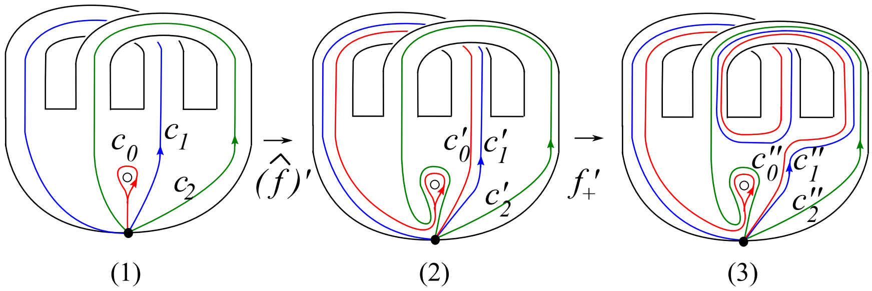


FIGURE 28. (1) Loops c_0, c_1, c_2. (2) Images of c_i's under $(\hat{f})'$, where $c'_i := (\hat{f})'(c_i)$. (3) Images of c_i's under $f = (\hat{f})'f'_+$, where $c''_i := f(c_i)$.

$f_* = ((\hat{f})' f'_+)_* : \pi_1(S,p) \to \pi_1(S,p)$ are given as follows. See Convention B.4.

$$\begin{array}{rcl} f_*(c_0) & = & c_2 c_0^{-1} c_1 c_0 c_1^{-1} c_0 c_2^{-1}, \\ f_*(c_1) & = & c_2 c_0^{-1} c_1, \\ f_*(c_2) & = & c_2 c_0^{-1}, \end{array}$$

see Figure 28(3) for f_*. Let us consider the abelianization $(f_*)_{\mathrm{ab}} : \mathbb{Z}^3 \to \mathbb{Z}^3$. From the above computation of $f_*(c_i)$, we have

$$\begin{array}{rcl} (f_*)_{\mathrm{ab}}[c_0] & = & [c_0], \\ (f_*)_{\mathrm{ab}}[c_1] & = & -[c_0] + [c_1] + [c_2], \\ (f_*)_{\mathrm{ab}}[c_2] & = & -[c_0] \qquad\quad + [c_2]. \end{array}$$

By calculation one sees that the characteristic polynomial of $(f_*)_{\mathrm{ab}}$ equals $(t-1)^3$, and all eigenvalues of $(f_*)_{\mathrm{ab}}$ are 1. By Theorem 3.12, it follows that f_* is order-preserving. Note that $\pi_1(\mathbb{W}) = F_3 \rtimes_{f_*} \mathbb{Z}$. By Proposition 3.10, we conclude that $\pi_1(\mathbb{W})$ is bi-orderable. □

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