POLYNOMIAL CONJUGACY INVARIANTS OF PSEUDO-ANOSOV MAPS DERIVED FROM TRAIN TRACK REPRESENTATIVES

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Abstract. We investigate two kinds of polynomials associated with pseudo-Anosov maps. Each kind consists of three polynomials that are conjugacy invariants of pseudo-Anosov mapping classes and these polynomials contain more information than pseudo-Anosov stretch factors. The first kind is defined by the train track representatives. The second kind is obtained from the first kind by incorporating additional data that encode the transitions among the unpunctured singularities of the invariant foliations. The Birman-Brinkmann-Kawamuro polynomials coincide with those of the second kind. As an application, we prove that the Birman-Brinkmann-Kawamuro polynomials are conjugacy invariants of pseudo-Anosov mapping classes which is also proved by Dodge-Kawamuro, independently.

1. Introduction

Let $\Sigma = \Sigma_{g,s}$ be an orientable surface of genus g with s punctures. The *mapping class group* $\mathrm{MCG}(\Sigma)$ is the group of isotopy classes of orientation preserving self-homeomorphisms on Σ preserving punctures set-wise. Assume that $3g-3+s \geq 1$. The Nielsen-Thurston classification of surface homeomorphisms states that any homeomorphism on Σ is isotopic to one of the three types, periodic, reducible, and pseudo-Anosov [Thu88]. Here, a homeomorphism $f : \Sigma \to \Sigma$ is *periodic* if some power of f is the identity map on Σ, and *reducible* if there exists a collection of mutually and non-homotopic essential simple closed curves $C_1, \ldots, C_n$ in Σ, called *reducing curves* for f, such that $f(C_1 \cup \cdots \cup C_n) = C_1 \cup \cdots \cup C_n$. A homeomorphism $f : \Sigma \to \Sigma$ is *pseudo-Anosov* if there exist a constant $\lambda > 1$ and a pair of transverse measured foliations $(\mathcal{F}^u, \mu^u)$ and $(\mathcal{F}^s, \mu^s)$ such that

$$f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u) \quad \text{and} \quad f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s).$$

We call λ the *stretch factor* of the pseudo-Anosov mapping class $[f]$. The constant λ is an algebraic integer that is a conjugacy invariant of $[f]$ in $\mathrm{MCG}(\Sigma)$ (see [FLP12]), and $\mathcal{F}^u$ and $\mathcal{F}^s$ are called *unstable* and *stable foliations*. They are singular foliations in Σ and admit a common finite set of singularities that include punctures of Σ (Figure 1). The set of singularities consists of periodic points of f. We define a *singularity orbit* of f as the periodic orbit of a singularity.

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Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map and $\mathcal{S}$ the set of all unpunctured singularities. We say that f is *fully punctured* if $\mathcal{S} = \emptyset$, that is, the set of singularities equals the set of punctures of Σ. We have a fully punctured pseudo-Anosov map $f^\circ : \Sigma^\circ \to \Sigma^\circ$ obtained from $f : \Sigma \to \Sigma$ by removing all unpunctured singularities, that is, $f^\circ := f|_{\Sigma^\circ} : \Sigma^\circ \to \Sigma^\circ$ where $\Sigma^\circ := \Sigma \setminus \mathcal{S}$.

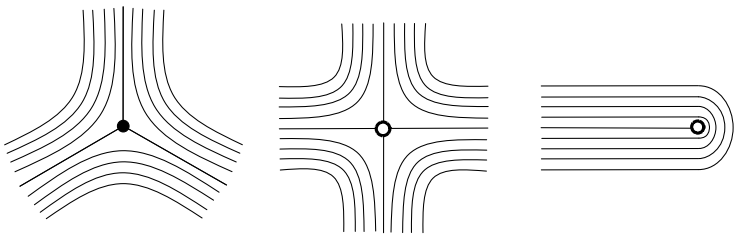


FIGURE 1. The unstable/stable foliations near singularities. ($\bullet$ and $\circ$ denote unpunctured and punctured singularities, respectively.) (1) unpunctured 3-pronged, (2) punctured 4-pronged and (3) punctured 1-pronged singularities.

In this paper, we investigate two kinds of polynomials associated with pseudo-Anosov maps. Each kind consists of three polynomials that are conjugacy invariants of pseudo-Anosov mapping classes. See the polynomials (1)(2)(3) in Table 1. To introduce the first kind, recall that for the computation of the stretch factor of $[f]$, one uses a train track τ which is a finite C^1 graph embedded in Σ. If τ is *f-invariant*, then f determines a graph map $\mathfrak{t} : \tau \to \tau$ that takes vertices to vertices, edges to edge paths. See Section 2.3. This graph map $\mathfrak{t} : \tau \to \tau$ gives a combinatorial description of the pseudo-Anosov map and is called a *train track representative* of $[f]$. Let $\mathcal{E}(\tau)$ be the set of all edges of τ, and let $|\mathcal{E}(\tau)|$ be its cardinality. The graph map $\mathfrak{t}$ naturally induces a linear map

$$\mathfrak{t}_* : \mathbb{R}^{|\mathcal{E}(\tau)|} \to \mathbb{R}^{|\mathcal{E}(\tau)|}$$

using a correspondence between an ith edge of τ and a standard unit vector of $\mathbb{R}^{|\mathcal{E}(\tau)|}$ whose ith component is 1. Then the largest real root of the characteristic polynomial of $\mathfrak{t}_*$ on $\mathbb{R}^{|\mathcal{E}(\tau)|}$ is the stretch factor of $[f]$. Note that this polynomial is not a conjugacy invariant of $[f]$, since it depends on the choice of the train track. Instead of the vector space $\mathbb{R}^{|\mathcal{E}(\tau)|}$, we consider the *weight space* $W(\tau)$. This is a subspace of $\mathbb{R}^{|\mathcal{E}(\tau)|}$ which consists of vectors that satisfy the switch conditions. Then $\mathfrak{t}_*(W(\tau)) \subset W(\tau)$ and we have the restriction map $\mathfrak{t}_* : W(\tau) \to W(\tau)$. The first kind of polynomials consists of the following three characteristic polynomials given in Theorem 1.1.

Theorem 1.1. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map. The characteristic polynomial $\chi(\mathfrak{t}_*)$ of $\mathfrak{t}_* : W(\tau) \to W(\tau)$ factors as $\chi(\mathfrak{t}_*) = \chi(\mathfrak{t}_*|_{Z(\tau)}) \cdot \chi(\overline{\mathfrak{t}_*})$. Polynomials $\chi(\mathfrak{t}_*)$, $\chi(\mathfrak{t}_*|_{Z(\tau)})$ and $\chi(\overline{\mathfrak{t}_*})$ are conjugacy invariants of $[f] \in \mathrm{MCG}(\Sigma)$.*

We call the three polynomials in Theorem 1.1 the *train track polynomials* of $[f]$. In Section 2.4, we define the Thurston symplectic form ω on $W(\tau)$ which

is not a symplectic form in general. We denote by $Z(\tau)$, the radical of ω, and consider a direct sum decomposition $W(\tau) \simeq Z(\tau) \oplus W(\tau)/Z(\tau)$. The map $\mathfrak{t}_* : W(\tau) \to W(\tau)$ preserves the form ω, and the quotient space $W(\tau)/Z(\tau)$ inherits the form ω. Hence, we have the restriction map $\mathfrak{t}_*|_{Z(\tau)} : Z(\tau) \to Z(\tau)$ and the induced map $\overline{\mathfrak{t}_*} : W(\tau)/Z(\tau) \to W(\tau)/Z(\tau)$. The characteristic polynomials of $\mathfrak{t}_*|_{Z(\tau)}$ and $\overline{\mathfrak{t}_*}$ occur as factors of $\chi(\mathfrak{t}_*)$ in Theorem 1.1.

In the proof of Theorem 1.1, we need the notion of *unstable/stable stars* introduced by Hironaka-Tsang [HT25]. We also need an explicit description of the radical of the Thurston symplectic form given by [HT25].

TABLE 1. Two kinds of polynomial conjugacy invariants: Train track polynomials and BBK polynomials.

Train track polynomials	**BBK polynomials**
Train track representative $\mathfrak{t} : \tau \to \tau$	Train track map $\mathfrak{g} : G \to G$
$\mathfrak{t}_* : W(\tau) \to W(\tau)$	$\mathfrak{g}_* : W(G) \to W(G)$
(1) $\chi(\mathfrak{t}_*)$ of $\mathfrak{t}_* : W(\tau) \to W(\tau)$	(1) $\chi(\mathfrak{g}_*)$ of $\mathfrak{g}_* : W(G) \to W(G)$
(2) Radical polynomial $\chi(\mathfrak{t}_*\|_{Z(\tau)})$	(2) Puncture polynomial $\chi(\mathfrak{g}_*\|_{Z(G)})$
(3) Symplectic polynomial $\chi(\overline{\mathfrak{t}_*}) = \chi(\overline{\mathfrak{g}_*})$	
$\chi(\mathfrak{t}_*) = \chi(\mathfrak{t}_*\|_{Z(\tau)})\chi(\overline{\mathfrak{t}_*})$	$\chi(\mathfrak{g}_*) = \chi(\mathfrak{g}_*\|_{Z(G)})\chi(\overline{\mathfrak{g}_*})$

The second kind of polynomials is based on the Bestvina-Handel algorithm [BH95, Bri00]. See Section 4. Let $f : \Sigma \to \Sigma$ be a homeomorphism on a punctured surface $\Sigma = \Sigma_{g,s}$ for $s \geq 1$. Then one can find an *induced graph map* $\mathfrak{g}' : G' \to G'$ of $[f]$, where G' is embedded in Σ and is homotopy equivalent to Σ. From the graph map $\mathfrak{g}' : G' \to G'$, the algorithm determines the Nielsen-Thurston type of f. In case $[f]$ is pseudo-Anosov, the algorithm finds an *efficient graph map* $\mathfrak{g} : G \to G$ that is an induced graph map of $[f]$, where G is embedded in Σ and is homotopy equivalent to Σ. We call the resulting graph map $\mathfrak{g} : G \to G$ a *train track map* of $[f]$. In [BH95] Bestvina-Handel construct a train track $\tau = \tau(G)$ together with a surjection $\pi : \tau \to G$. They also construct a train track representative $\mathfrak{t} : \tau \to \tau$ that is an extension of $\mathfrak{g} : G \to G$. Then $\tau = \tau(G)$ has a property such that for each *even type* (resp. *odd type*) vertex v of G, $\pi^{-1}(v)$ forms an infinitesimal even-gon (resp. odd-gon) corresponding to an unpunctured even-pronged (resp. odd-pronged) singularity of $\mathcal{F}^u$ and $\mathcal{F}^s$. See Definition 4.6.

Let $\mathfrak{g}_* : \mathbb{R}^{|\mathcal{E}(G)|} \to \mathbb{R}^{|\mathcal{E}(G)|}$ be a linear map induced by the train track map $\mathfrak{g} : G \to G$. Then $\mathfrak{g}_*$ restricts to a subspace $W(G) := \pi_*(W(\tau))$, where $\pi_* : \mathbb{R}^{|\mathcal{E}(\tau)|} \to \mathbb{R}^{|\mathcal{E}(G)|}$ is a surjection induced by $\pi : \tau \to G$. Denoting $\pi_*|_{W(\tau)} : W(\tau) \to W(G)$ by $\pi_* : W(\tau) \to W(G)$, we have the exact sequence

$$0 \longrightarrow \ker \pi_* \longrightarrow W(\tau) \xrightarrow{\pi_*} W(G) \longrightarrow 0$$

The Thurston symplectic form ω on $W(\tau)$ gives rise to a skew-symmetric form $\overline{\omega}$ on $W(G)$. Birman-Brinkmann-Kawamuro [BBK12] studied $\mathfrak{g}_* : W(G) \to W(G)$, the restriction map $\mathfrak{g}_*|_{Z(G)} : Z(G) \to Z(G)$ and the induced map $\overline{\mathfrak{g}_*} : W(G)/Z(G) \to W(G)/Z(G)$ that are linear maps parallel to those defining the train track polynomials. We call their characteristic polynomials the *BBK polynomials* that are of the second kind in this paper.

In Section 5.1, we prove that the dimension of the subspace $\ker \pi_*$ is equal to the number of even type vertices of G. The map $\mathfrak{t}_* : W(\tau) \to W(\tau)$ restricts to $\ker \pi_*$. The restriction $\mathfrak{t}_*|_{\ker \pi_*} : \ker \pi_* \to \ker \pi_*$ encodes the transitions among unpunctured even-pronged singularities of the pseudo-Anosov map. We prove in Theorem 5.4 that the characteristic polynomial $\chi(\mathfrak{t}_*|_{\ker \pi_*})$ is a conjugacy invariant of $[f] \in \mathrm{MCG}(\Sigma)$.

The following result states a relation between the two kinds of polynomials.

Theorem 1.2. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map, where $\Sigma = \Sigma_{g,s}$ with $s \geq 1$. Let $\mathfrak{g}\colon G \to G$ be a train track map and $\mathfrak{t}\colon \tau \to \tau$ the train track representative of $[f]$. Then we have the following.*

(1) $\chi(\mathfrak{t}_*) = \chi(\mathfrak{g}_*)\chi(\mathfrak{t}_*|_{\ker \pi_*})$.
(2) $\chi(\mathfrak{t}_*|_{Z(\tau)}) = \chi(\mathfrak{g}_*|_{Z(G)})\chi(\mathfrak{t}_*|_{\ker \pi_*})$.
(3) $\chi(\overline{\mathfrak{t}_*}) = \chi(\overline{\mathfrak{g}_*})$

In particular, if f is a fully punctured pseudo-Anosov map, then $\chi(\mathfrak{t}_*) = \chi(\mathfrak{g}_*)$, $\chi(\mathfrak{t}_*|_{Z(\tau)}) = \chi(\mathfrak{g}_*|_{Z(G)})$ and $\chi(\overline{\mathfrak{t}_*}) = \chi(\overline{\mathfrak{g}_*})$.

By Theorem 1.2, the BBK polynomials can be described by the train track polynomials and the polynomial $\chi(\mathfrak{t}_*|_{\ker \pi_*})$. Since these polynomials are conjugacy invariants (Theorems 1.1 and 5.4), Theorem 1.2 implies that the BBK polynomials are conjugacy invariants of $[f] \in \mathrm{MCG}(\Sigma)$.

Corollary 1.3. *With the same notation as in Theorem 1.2, the polynomials $\chi(\mathfrak{g}_*)$, $\chi(\mathfrak{g}_*|_{Z(G)})$ and $\chi(\overline{\mathfrak{g}_*})$ are conjugacy invariants of $[f] \in \mathrm{MCG}(\Sigma)$.*

Remark 1.4. In 2012, Birman-Brinkmann-Kawamuro [BBK12] stated the invariance of the BBK polynomials. Dodge-Kawamuro [DK26] provide more details on the proof that the BBK polynomials are invariants. They also prove that the BBK polynomials are conjugacy invariants of pseudo-Anosov mapping classes. However, the relation between the train track polynomials and the BBK polynomials is not addressed in [DK26].

The following result is proved in Section 5 and is essentially contained in [HT25, BBK12]. See Definition 5.12 for the notion of (anti-)palindromic polynomials.

Proposition 1.5. *Polynomials $\chi(\overline{\mathfrak{t}_*})$, $\chi(\mathfrak{t}_*|_{Z(\tau)})$ and $\chi(\mathfrak{t}_*) = \chi(\mathfrak{t}_*|_{Z(\tau)})\chi(\overline{\mathfrak{t}_*})$ are reciprocal, that is, palindromic or anti-palindromic.*

This paper is primarily based on work done as part of a Master's thesis [Miz26] by the second author.

The organization of the paper is as follows. In Section 2 we recall basic definitions and prove some results of train tracks. In Section 3 we introduce the train track polynomials and prove Theorem 1.1. In Section 4 we introduce the BBK polynomials and review some results of [BBK12]. In Section 5 we prove Theorem 1.2 and the conjugacy invariance of the BBK polynomials in the mapping class groups. In Section 6 we give examples.

2. Preliminaries

2.1. **Braid groups.** Let B_s be the braid group with s strands. The group B_s has a presentation with generators $\sigma_1, \ldots, \sigma_{s-1}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$ and $\sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i$ if $|i-j| = 1$.

Let D_s denote a disk with s punctures $a_1, \dots, a_s$. The mapping class group $\mathrm{MCG}(D_s)$ is the group of isotopy classes of orientation preserving self-homeomorphisms on D_s preserving the boundary ∂D of the disk setwise. There is a surjective homomorphism (strictly speaking, anti-homomorphism since we read braid words from left to right):

$$\Gamma : B_s \to \mathrm{MCG}(D_s)$$

sending the generator σ_i to the right-handed half twist h_i that interchanges the punctures a_i and a_{i+1} counterclockwise for $i = 1, \dots, s-1$. The kernel of Γ is an infinite cyclic group generated by a full twist. Collapsing the boundary ∂D of D_s into a point yields an $s+1$-punctured sphere. We may regard $\Gamma(b) \in \mathrm{MCG}(D_s)$ as an element of $\mathrm{MCG}(\Sigma_{0,s+1})$.

2.2. **Train tracks.** A *train track* τ is a finite C^1 graph embedded in a surface Σ and equipped with a well-defined tangent line at each vertex. It also requires that each component of the complement $\Sigma \setminus \tau$ is either a non-punctured k-gon for $k \geq 3$ or a once-punctured k-gon for $k \geq 1$. Edges of a train track are called *branches* and vertices are called *switches*. Two branches of τ that are tangent at a switch make a *cusp*. See Figure 2(1).

Definition 2.1. Let v be a switch of τ. We label branches $b'_1, \dots, b'_m$, $b_1, \dots, b_n$ emanating from v that are ordered counterclockwise so that the angle between each branch b'_i $(i = 1, \dots, m)$ and each branch b_j $(j = 1, \dots, n)$ is π (Figure 2(1)(1')). We form a *tie neighborhood* $\mathcal{N}(\tau) \subset \Sigma$ by taking the union of polygons associated with switches of τ and rectangles associated with branches of τ. Each rectangle connects the polygons along the branches of τ, and both polygons and rectangles are foliated by intervals transverse to the branches of τ. See Figure 2(2). We call the intervals *ties*.

Definition 2.2 (Section 3.1 in [HT25])**.** Each connected component of $\mathcal{N}(\tau) \setminus \tau$ is homeomorphic to an annulus. One boundary component of such an annulus is along the boundary of $\mathcal{N}(\tau)$ and the other boundary component is along τ. The boundary component along τ is called a *boundary component of* τ. The boundary components of τ are in one to one correspondence with the boundary components of $\mathcal{N}(\tau)$. We say that a boundary component c of τ is *k-pronged* if it contains k cusps. We say that c is *even-pronged* (resp. *odd-pronged*) if it is k-pronged for some even (resp. odd) k.

Definition 2.3. A train track τ is *orientable* if one can orient all branches in such a way that at each switch v of τ, the angle between each incoming branch toward v and each outgoing branch emanating from v is π. If τ has an odd-pronged boundary component c, then τ is non-orientable, since there is no consistent orientation on c.

2.3. **The weight spaces of train tracks.** Let $\mathcal{E}(\tau)$ be the set of all branches of τ. Let $V(\tau) \simeq \mathbb{R}^{|\mathcal{E}(\tau)|}$ denote the vector space of real weights on the branches of τ. We denote the weight (coefficient) that $\eta \in V(\tau)$ assigns to $b \in \mathcal{E}(\tau)$ by $\eta(b)$. The *weight space* $W(\tau)$ is a subspace of $V(\tau)$ consisting of vectors $\eta \in V(\tau)$ that satisfy the switch condition

$$\textstyle\sum_{i=1}^{m} \eta(b'_i) = \sum_{j=1}^{n} \eta(b_j)$$

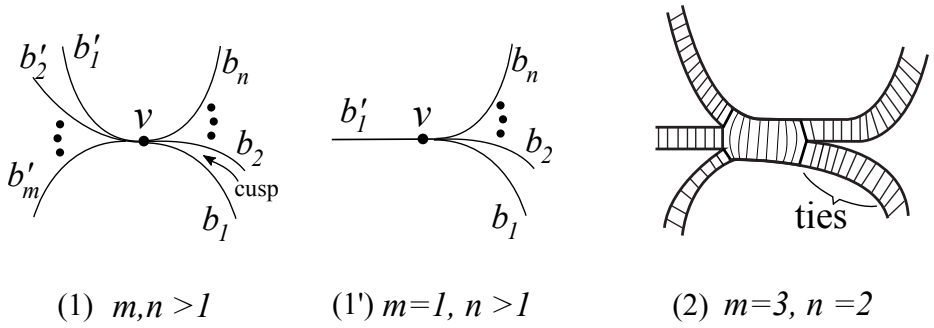


FIGURE 2. (1)(1') Train track τ near a switch v. (2) Tie neighborhood $\mathcal{N}(\tau)$.

at each switch v of τ (see Figure 2(1)).

Let σ be a train track in Σ. We say that σ is *carried by* τ, and we write $\sigma < \tau$, if σ is isotopic to a train track σ' contained in $\mathcal{N}(\tau)$ and σ' is transverse to the ties in $\mathcal{N}(\tau)$. Let $f : \Sigma \to \Sigma$ be a homeomorphism. A train track τ is *f-invariant* if $f(\tau) < \tau$. In this case, collapsing each tie in $\mathcal{N}(\tau)$ into a point, we obtain a graph map $\mathfrak{t} : \tau \to \tau$ such that $\mathfrak{t}$ maps vertices to vertices, edges to edge paths. Moreover, $\mathfrak{t}$ is locally injective at any points that do not map into vertices. This map $\mathfrak{t} : \tau \to \tau$ is called a *train track representative* of $[f]$.

Definition 2.4 (Bestvina-Handel [BH95])**.** A branch e of τ is said to be *infinitesimal* if e is eventually periodic under the train track representative $\mathfrak{t} : \tau \to \tau$, that is, $\mathfrak{t}^{m+n}(e) = \mathfrak{t}^n(e)$ for some integers $m > 0$ and $n \geq 0$. Other branches of τ are said to be *real*.

Definition 2.5. From a train track representative $\mathfrak{t} : \tau \to \tau$, we define a non-negative integer matrix $M = (m_{ij})$ of degree $|\mathcal{E}(\tau)|$, where (i, j)-entry m_{ij} is the number of times that the image $\mathfrak{t}(e_j)$ of the jth edge crosses the ith edge e_i in either direction. We call M the *transition matrix* of $\mathfrak{t} : \tau \to \tau$.

Suppose that $f : \Sigma \to \Sigma$ is a pseudo-Anosov map. Thurston showed that there exists an f-invariant train track τ that induces a train track representative $\mathfrak{t} : \tau \to \tau$. Moreover, if M is the transition matrix for $\mathfrak{t} : \tau \to \tau$, its spectral radius is the stretch factor of $[f]$. (See [FLP12] for example.)

The train track representative $\mathfrak{t} : \tau \to \tau$ induces a linear map $\mathfrak{t}_* : V(\tau) \to V(\tau)$. Notice that if $\eta \in V(\tau)$ that satisfies the switch condition at a switch v, then $\mathfrak{t}_*(\eta) \in V(\tau)$ also satisfies the switch condition at the switch $\mathfrak{t}(v)$, since τ is f-invariant. This implies that $\mathfrak{t}_*(W(\tau)) \subset W(\tau)$ and we have the linear map $\mathfrak{t}_*|_{W(\tau)} : W(\tau) \to W(\tau)$. This restriction map is denoted by

$$\mathfrak{t}_* : W(\tau) \to W(\tau).$$

Since the f-invariant train track τ is not unique, the characteristic polynomial of $\mathfrak{t}_* : V(\tau) \to V(\tau)$ (that is, the characteristic polynomial $\chi(M)$ of the above transition matrix M) is not a well-defined invariant of $[f]$. We will verify in Theorem 3.2 that the characteristic polynomial of the restriction map $\mathfrak{t}_* : W(\tau) \to W(\tau)$ is an invariant of $[f]$.

2.4. **The Thurston symplectic form.** Following [BBK12, HT25], we introduce the Thurston symplectic form ω on $W(\tau)$. This is a generalization of the symplectic form on $W(\tau)$ for trivalent train tracks [PH92]. Recall that we label branches $b'_1, \dots, b'_m$, $b_1, \dots, b_n$ emanating from a switch v of τ that are ordered counterclockwise as in Figure 2(1)(1'). Then the Thurston symplectic form ω is defined as follows. For $\eta, \zeta \in W(\tau)$,

$$\omega(\eta,\zeta) := \sum_{\substack{\text{switches}\\ \text{in } \tau}} \sum_{\substack{i<j\\ k<\ell}} \left\{ \begin{vmatrix} \eta(b_i) & \eta(b_j) \\ \zeta(b_i) & \zeta(b_j) \end{vmatrix} + \begin{vmatrix} \eta(b'_k) & \eta(b'_\ell) \\ \zeta(b'_k) & \zeta(b'_\ell) \end{vmatrix} \right\},$$

where the second summation is taken over all pairs of branches b_i, b_j (resp. b'_k, b'_ℓ) with $1 \le i < j \le n$ (resp. $1 \le k < \ell \le m$). In the case $m = 1$ (Figure 2(1')), we ignore $\left|\begin{smallmatrix} \eta(b'_k) & \eta(b'_\ell) \\ \zeta(b'_k) & \zeta(b'_\ell) \end{smallmatrix}\right|$ and we only take into account of $\left|\begin{smallmatrix} \eta(b_i) & \eta(b_j) \\ \zeta(b_i) & \zeta(b_j) \end{smallmatrix}\right|$. The case $n = 1$ is similar.

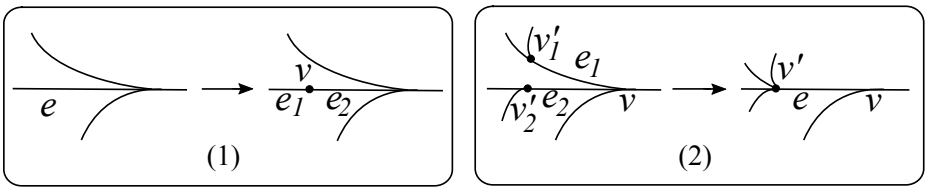


FIGURE 3. (1) Subdivision move. (2) Elementary folding move.

We borrow the following definition from [HT25].

Definition 2.6 (Elementary moves)**.** Let $\tau \subset \Sigma$ be a train track.

(1) Suppose that e is an edge of τ. We pick an interior point v of e. We define a new train track τ' obtained from τ by replacing e with two new edges e_1, e_2 whose common endpoint is v. (Although τ and τ' are not isomorphic as a graph, they are homeomorphic as a topological space.) We have a natural graph map $\tau \to \tau'$ between the two graphs. This map is called the *subdivision move*. See Figure 3(1).

(2) Suppose that e_1 and e_2 are two edges of τ that form a cusp at a switch v. Let $v'_1 \neq v$ and $v'_2 \neq v$ be the other endpoints of e_1 and e_2, respectively. We define a new train track τ' obtained from τ by combining v'_1 and v'_2 into one switch v' and replacing e_1 and e_2 by a single edge e which connects v and v'. There is a graph map $\tau \to \tau'$ sending v_1 and v_2 to v, e_1 and e_2 to e and sending the rest of vertices and edges to themselves. This map is called the *elementary folding move*. See Figure 3(2).

We refer to subdivision moves and elementary folding moves as *elementary moves*.

Lemma 2.7 (Lemmas 4.4 and 4.5 in [HT25])**.** *If $m : \tau \to \tau'$ is an elementary move, then $m_* : W(\tau) \to W(\tau')$ is an isomorphism that preserves the Thurston symplectic form ω.*

The following result follows from a theorem by Stallings [Sta83].

Proposition 2.8. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map. A train track representative $\mathfrak{t} : \tau \to \tau$ of $[f]$ admits a folding decomposition as follows.*

$$\mathfrak{t} = \rho \circ m_k \circ \cdots \circ m_1$$

where $m_i : \tau_i \to \tau_{i+1}$ are elementary folding moves with $\tau_1 = \tau_{k+1} = \tau$ and $\rho : \tau \to \tau$ is a homeomorphism.

By Proposition 2.8 together with Lemma 2.7, we obtain the following.

Corollary 2.9. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map. Let $\mathfrak{t} : \tau \to \tau$ be a train track representative of $[f]$. Then $\mathfrak{t}_* : W(\tau) \to W(\tau)$ is an isomorphism that preserves the Thurston symplectic form ω.*

2.5. **Unstable/stable stars.** The notion of unstable/stable stars introduced in [HT25] plays an important role in the later section.

Definition 2.10. Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map.

(1) Suppose that f is fully punctured. Let x be a puncture of Σ. *An unstable prong at x* is a connected subset of an unstable leaf of the unstable foliation $\mathcal{F}^u$ that limits to x. *An unstable star σ_x at x* is a maximal disjoint union of unstable prongs at x. In other words, if $\mathcal{F}^u$ is k-pronged at x, an unstable star σ_x at x is a disjoint union of k unstable prongs at x. (For example, if x is a puncture of Σ as shown in Figure 4(1), the three unstable prongs at x as in the same figure is an unstable star at x.) The union $\sigma = \bigcup_x \sigma_x$, where x ranges over all punctures of Σ is called *unstable stars of f*. Stable stars are defined similarly.

(2) Suppose that f is not fully punctured. In this case the *unstable stars of f* is the unstable stars of the fully punctured pseudo-Anosov map f°.

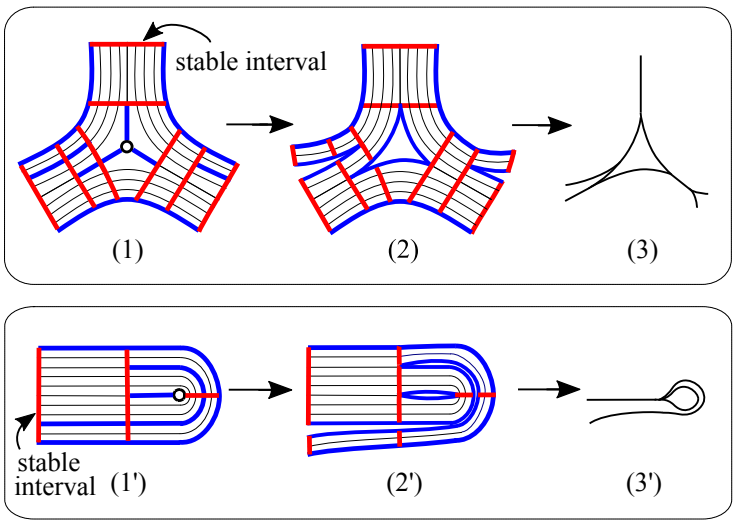


FIGURE 4. (1)(1') Parts of unstable stars. (2)(2') Cutting the surface along unstable stars. (3)(3') Train tracks.

Let $\tau \subset \Sigma$ be an f-invariant train track. A tie neighborhood $\mathcal{N}(\tau)$ can be thought of as the surface cut along some unstable stars σ of f. Conversely, given unstable stars, one can find a train track (not necessarily f-invariant) with this property:

Lemma 2.11. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map. Let σ be unstable stars of f. Then there is a train track τ whose tie neighborhood $\mathcal{N}(\tau)$ can be thought of as the surface cut along $\sigma \subset \Sigma$.*

Proof. Every stable half-leaf is dense in Σ. (See Lemma 9.6 in [FLP12].) Now we cut Σ along the unstable stars σ. The stable leaves become compact intervals, since each stable leaf eventually runs into an unstable star of f. Then a desired train track can be obtained from the resulting surface collapsing each stable interval (corresponding to a tie) to a point. □

We call τ in Lemma 2.11 the *train track corresponding to* σ. The authors learn the following result from Chi Cheuk Tsang.

Proposition 2.12. *Let σ and σ' be unstable stars of a pseudo-Anosov map $f : \Sigma \to \Sigma$. Let τ and τ' be train tracks corresponding to σ and σ', respectively. Then, there exists an elementary folding move $\tau \to \tau'$ if and only if $\sigma \supset \sigma'$.*

Proof. (If part.) The idea comes from the proof [HT25, Proposition 3.18]. Suppose that $\sigma \supset \sigma'$. By Lemma 2.11, tie neighborhoods $\mathcal{N}(\tau)$ and $\mathcal{N}(\tau')$ can be thought of as the surfaces cut along unstable stars σ and σ', respectively. Since σ contains σ', one can retract σ to σ'. Then the stable intervals (ties) in $\mathcal{N}(\tau)$ merge in $\mathcal{N}(\tau')$, and this effects an elementary folding move $\tau \to \tau'$.

(Only if part.) Recall the construction of train tracks corresponding to unstable stars (Figure 4). Then it is clear from Figure 5 that if there is an elementary folding move $\tau \to \tau'$, then σ contains σ'. □

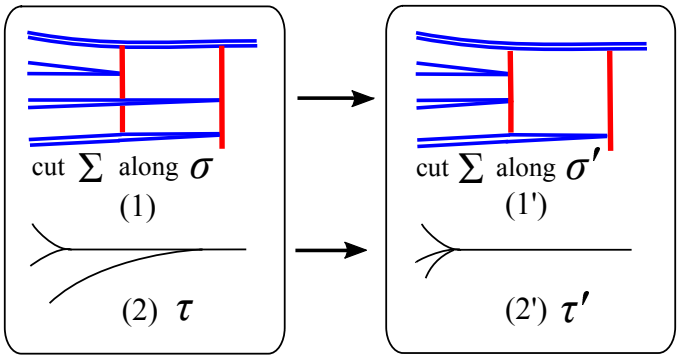


FIGURE 5. Proof of Proposition 2.12.

Corollary 2.13. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map. Let σ be unstable stars of f. Suppose that τ is the train track corresponding to σ. Then $f(\tau) < \tau$ (i.e., f-invariant) if and only if $f(\sigma) \supset \sigma$.*

Proof. The statement follows from Propositions 2.8 and 2.12. □

2.6. **The radical of the Thurston symplectic form.** The Thurston symplectic form ω is not a symplectic form in general, that is, the *radical* $Z(\tau)$ of ω is not trivial in general.

$$Z(\tau) = \{\eta \in W(\tau) \mid \omega(\eta, \zeta) = 0 \text{ for all } \zeta \in W(\tau)\}.$$

Definition 2.14 (The radical element [HT25])**.** Let c be an even-pronged boundary component of τ. We label the intervals on c between consecutive cusps by $I_1, \ldots, I_{2n}$ in cyclic order. (Hence, the endpoints of each I_i are cusps of τ.) The *radical element* r_c is an element of $W(\tau)$ defined as follows. First, assign weight $(-1)^i$ to each edge of τ that lies on I_i (Figure 6(1)(2)), and assign weight 0 to each edge disjoint from all the intervals I_i. Then, for each edge e of τ, define the coefficient $r_c(e)$ to be the sum of weights assigned to e over all its occurrences in the intervals (Figure 6(2)(3)). In other words, if e appears multiple times, possibly in multiple intervals I_i, then $r_c(e)$ is the sum of weights assigned to e at each occurrence. In particular, if an edge e appears exactly once in exactly one of the intervals I_i, then $r_c(e) = (-1)^i$. It is proved in [HT25, Proposition 5.2] that r_c is an element of the radical $Z(\tau) \subset W(\tau)$.

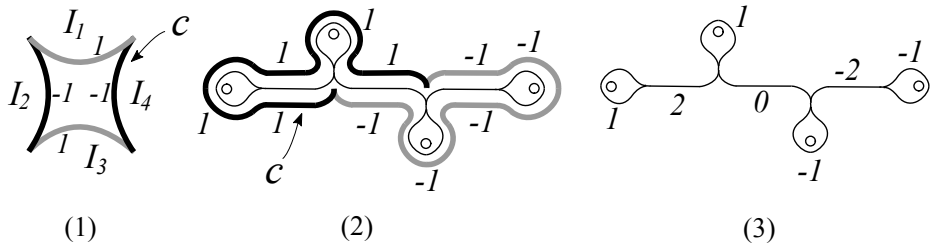


FIGURE 6. (1)4-pronged boundary component. (2) 2-pronged boundary component of τ. (3) Radical element r_c.

Suppose that τ' is obtained from τ by an elementary folding move. Then there is a natural one to one correspondence between the boundary components of τ and those of τ'. The following result can be found in [HT25, Proposition 5.3].

Proposition 2.15. *Let $m : \tau \to \tau'$ be an elementary move. Suppose that c' is a boundary component of τ' that corresponds to an even-pronged boundary component c of τ. Then the isomorphism $m_* : W(\tau) \to W(\tau')$ sends r_c to $r_{c'}$.*

Definition 2.16. Let l be a lamination in Σ. Let $\tau \subset \Sigma$ be a train track. We say that τ *carries* l if leaves of l are contained in a tie neighborhood $\mathcal{N}(\tau)$ and transverse to the ties in $\mathcal{N}(\tau)$. We say that τ *fully carries* l if τ carries l and every tie in $\mathcal{N}(\tau)$ intersects some leaf of l.

If τ is a train track corresponding to some unstable stars of a pseudo-Anosov map, then τ fully carries the unstable lamination of f.

The following definition is used in the proof of Proposition 2.18.

Definition 2.17 (Definition 4.7 in [HT25])**.** We consider the equivalence class on the set of all train tracks that is generated by there being an

elementary move between two train tracks. We say that train tracks τ and τ' are *equivalent* if τ and τ' are in the same equivalence class.

The following result is a refinement of Proposition 8.1 in [HT25] and the proof below is due to Chi Cheuk Tsang.

Proposition 2.18. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map, and let σ be unstable stars of f. Then the train track τ corresponding to σ satisfies the following.*

(1) *The radical $Z(\tau)$ is given by*

$$Z(\tau) = \mathrm{span}\{r_c\},$$

where c ranges over all even-pronged boundary components of τ.

(2) *The radical elements $\{r_c\}$ are linearly independent if and only if τ is non-orientable.*

(3) *The dimension of $Z(\tau)$ is*

$$\begin{cases} \#\{\text{even-pronged boundary components}\} - 1 & \text{if } \tau \text{ is orientable,} \\ \#\{\text{even-pronged boundary components}\} & \text{if } \tau \text{ is non-orientable.} \end{cases}$$

We will need *standardly embedded train tracks* introduced by Hironaka-Tsang. See Definition 3.1 of [HT25]. The property we need on standardly embedded train tracks τ is that one side of a switch of τ consists of a number of real edges (possibly one real edge) and the other side consists of two infinitesimal edges on infinitesimal polygons. See Figure 7(1)(2). For example, the train track in Figure 6(3) is a standardly embedded train track.

Proof of Proposition 2.18. The claim (1) holds from Proposition 8.1 in [HT25].

We turn to (2). For the proof of 'the only if part' ($\Longrightarrow$), we prove the contrapositive. Suppose that τ is orientable. Then we can consistently orient all edges of τ, and all boundary components of τ are even-pronged. (See Definition 2.3.) Let c be any boundary component of τ and let $I_1, \dots, I_{2n}$ be the intervals on c as in Definition 2.14. We fix a sign of the radical element r_c (i.e. signs of coefficients of edges in r_c) by specifying that I_{2i+1} are oriented clockwise and I_{2i} are oriented counterclockwise. Then $\sum_c r_c = 0$, since each edge of τ appears twice in the sum, once on a clockwise interval and the other time on a counterclockwise interval. (If τ has a single boundary component c, then $r_c = 0$.) Hence $\{r_c\}$ are linearly dependent.

We turn to 'the if part' ($\Longleftarrow$) of (2). Suppose that τ is a non-orientable train track. We prove that $\{r_c\}$ are linearly independent. To do this, we first consider the case where τ has at least two boundary components. The validity of the claim is invariant under equivalence of train tracks (see Definition 2.17). We note that any two train tracks τ and τ' corresponding to unstable stars σ and σ' are equivalent. Hence, we can take particular unstable stars σ and the corresponding train rack τ. From the proof of Proposition 5.4 in [HT25], we can assume that all boundary components of τ, except a specified boundary component O, are infinitesimal polygons, that is, the polygons whose sides are infinitesimal edges. (In fact, choose unstable stars σ so that the length of the unstable star at a puncture corresponding to O is long enough. Then the corresponding train track τ has the above property.) We can further assume that τ is a standardly embedded train track (c.f. Construction 3.6 of [HT25]).

If τ has an odd-pronged boundary component, then we can choose O to be odd-pronged. In this case, all even-pronged boundary components are disjoint. This means that $\{r_c\}$ are linearly independent.

We are left with the case where τ is non-orientable and all boundary components are even-pronged. We assume that there is a relation $\sum_c a_c r_c = 0$. Since the boundary components other than O are disjoint, $\{r_c\}_{c \neq O}$, that is, the radical elements other than r_O, are linearly independent, and hence the coefficient a_O is nonzero. We claim that one can construct orientations of edges on τ using the relation $\sum_c a_c r_c = 0$ and this would contradicts the assumption that τ is non-orientable.

To prove this claim, we choose the labeling $I_1, \ldots, I_{2n}$ on O as in Definition 2.14. Then orientations on the edges on O are defined by orientating all edges on I_{2i+1} clockwise and all edges on I_{2i} counterclockwise. We first verify that this is well defined: The only way this can fail is that an edge appears twice on O, on I_{2i} and I_{2j}, or I_{2i+1} and I_{2j+1}. If this is the case, the coefficient of the edge in the vector $\sum_c a_c r_c$ must be $\pm 2a_O$ which is nonzero. This contradicts the relation $\sum_c a_c r_c = 0$.

We continue with the proof of the claim. We check that the orientations induced by O matches up at each switch of τ: Since τ is a standardly embedded train track, one side of a switch of τ consists of a number of real edges and the other side consists of two infinitesimal edges, say e_1 and e_2. In the former case, each pair of adjacent real edges is consistently oriented as we proved above. In the latter case, we suppose that the two infinitesimal edges e_1 and e_2 belong to an infinitesimal polygon C. We look at the relation $\sum_c a_c r_c = 0$. Then e_1 and e_2 meet the two boundary components of τ, namely O and C. Hence, we have the two equations coming from the coefficients of e_1 and e_2 of the form $\pm a_O \pm a_C = 0$, since the coefficients of e_1 and e_2 in the two radical elements r_O and r_C are 1 or -1. Note that the coefficients of e_1 and e_2 in r_C have the opposite signs, since they are adjacent on C. Hence, the coefficients of e_1 and e_2 in r_O have the opposite signs, for otherwise $a_O = a_C = 0$. This implies that one infinitesimal edge, say e_1 lies on I_{2i} (with even index) of the side on O, while the other infinitesimal edge e_2 lies on I_{2j+1} (with odd index) of the side on O (see Figure 7(3)). Therefore, the orientations induced by O on e_1 and e_2 match up. This tells us that τ is orientable, which is a contradiction.

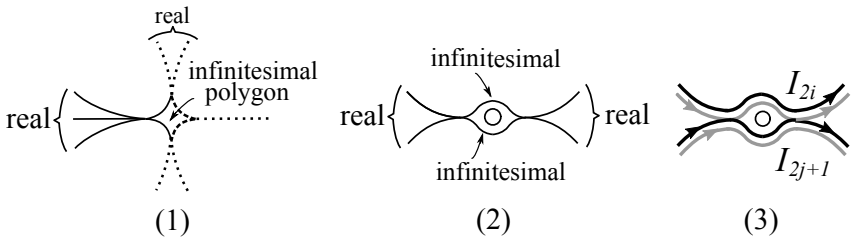


FIGURE 7. (1)(2) Standardly embedded train tracks near switches. (3) Proof of Proposition 2.18(2). (The symbol $\circ$ denotes a puncture of the surface.)

We next consider the remaining case where τ is a non-orientable train track with a single even-pronged boundary component O. Choose the labeling $I_1, \dots, I_{2n}$ on O as in Definition 2.14. The orientations on the edges of O (hence on the edges of τ) are defined by orientating all edges on I_{2i+1} clockwise and all edges on I_{2i} counterclockwise. By a similar argument as above, the relation $r_O = 0$ implies that one can consistently orient all edges of τ, which means τ is orientable. This contradicts the assumption that τ is non-orientable. This completes the proof of (2).

The claim (3) follows from (1) and (2) together with the proof of (2). □

3. Train track polynomials

In this section, we introduce the first kind of polynomials, which we call the train track polynomials, and prove that they are conjugacy invariants of pseudo-Anosov mapping classes. We will need the following lemma.

Lemma 3.1. *Let A, B and C be vector spaces. Suppose that we have the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

where the rows are exact and α, β and γ are representation matrices corresponding to the linear transformations. Then there exists a choice of basis such that $\beta = \begin{pmatrix} \alpha & * \\ 0 & \gamma \end{pmatrix}$. In particular, the characteristic polynomial of β is factorized as $\chi(\beta) = \chi(\alpha)\chi(\gamma)$.

The proof of the lemma is straightforward.

Theorem 3.2. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map. Let $\mathfrak{t} : \tau \to \tau$ be a train track representative of $[f]$. Then the characteristic polynomial $\chi(\mathfrak{t}_*)$ of $\mathfrak{t}_* : W(\tau) \to W(\tau)$ is a conjugacy invariant of $[f] \in \mathrm{MCG}(\Sigma)$.*

Proof. Suppose that $\mathfrak{t} : \tau \to \tau$ and $\mathfrak{t}' : \tau' \to \tau'$ are train track representatives of $[f]$. Let $\mathfrak{t}_* : W(\tau) \to W(\tau)$ and $\mathfrak{t}'_* : W(\tau') \to W(\tau')$ be the linear maps. Tie neighborhoods of τ and τ' can be thought of as the surfaces cut along unstable stars, say σ and σ', respectively of the common unstable foliation of f. Since both τ and τ' are f-invariant, we have $f(\sigma) \supset \sigma$ and $f(\sigma') \supset \sigma'$ by Corollary 2.13. We set $\sigma'' := \sigma \cap \sigma'$. Then the unstable stars σ'' also satisfy $f(\sigma'') \supset \sigma''$. This tells us that the train track τ'' corresponding to σ'' is also f-invariant by Corollary 2.13. Hence, we have a train track representative $\mathfrak{t}'' : \tau'' \to \tau''$ of $[f]$.

By definition of the unstable stars $\sigma'' = \sigma \cap \sigma'$, there exist homotopy equivalence graph maps $m : \tau \to \tau''$ and $m' : \tau' \to \tau''$. By a result of Stallings [Sta83], each of the graph maps $m : \tau \to \tau''$ and $m' : \tau' \to \tau''$ can be factored as a composition of elementary folding moves. Then the following

commutative diagram holds.

$$\begin{array}{ccc} W(\tau) & \xrightarrow{m_*} & W(\tau'') \\ \downarrow{\scriptstyle \mathfrak{t}_*} & & \downarrow{\scriptstyle \mathfrak{t}''_*} \\ W(\tau) & \xrightarrow{m_*} & W(\tau'') \end{array} \tag{3.1}$$

where $m_* : W(\tau) \to W(\tau'')$ is the isomorphism induced by m (see Lemma 2.7). To see the commutative square $m_* \circ \mathfrak{t}_* = \mathfrak{t}'' \circ m_*$, we note that the train track representatives $\mathfrak{t}$ and $\mathfrak{t}''$ can be factorized into elementary folding moves (see Proposition 2.8). For each element $\eta \in W(\tau)$, we keep track of the weights on the branches, and add them up when we perform elementary folding moves. Then we obtain the desired claim $m_* \circ \mathfrak{t}_*(\eta) = \mathfrak{t}'' \circ m_*(\eta)$ for $\eta \in W(\tau)$.

By (3.1), we have $\chi(\mathfrak{t}_*) = \chi(\mathfrak{t}''_*)$. Applying the other homotopy equivalence graph map $m' : \tau' \to \tau''$ to the same argument, we have $\chi(\mathfrak{t}'_*) = \chi(\mathfrak{t}''_*)$. Hence $\chi(\mathfrak{t}_*) = \chi(\mathfrak{t}'_*)$ is an invariant of $[f]$.

We turn to the conjugacy invariance of $\chi(\mathfrak{t}_*)$. Suppose that pseudo-Anosov maps $f, f' : \Sigma \to \Sigma$ are conjugate, that is, there exists an orientation preserving homeomorphism $u : \Sigma \to \Sigma$ such that $f' = u \circ f \circ u^{-1}$. Let $\mathfrak{t} : \tau \to \tau$ be a train track representative of $[f]$. We set $\tau' := u(\tau)$. Then the train track $f'(\tau') = u \circ f \circ u^{-1}(\tau')$ is carried by τ', and we obtain a train track representative $\mathfrak{t}' : \tau' \to \tau'$ of $[f'] \in \mathrm{MCG}(\Sigma)$. Then the following commutative diagram holds.

$$\begin{array}{ccc} W(\tau) & \xrightarrow{u_*} & W(\tau') \\ \downarrow{\scriptstyle \mathfrak{t}_*} & & \downarrow{\scriptstyle \mathfrak{t}'_*} \\ W(\tau) & \xrightarrow{u_*} & W(\tau') \end{array}$$

where $u_* : W(\tau) \to W(\tau')$ is the isomorphism induced by $u|_\tau : \tau \to \tau'$. Hence, $\chi(\mathfrak{t}_*) = \chi(\mathfrak{t}'_*)$ is a conjugacy invariant of $[f]$. □

By Corollary 2.9 we have the restriction map

$$\mathfrak{t}_*|_{Z(\tau)} : Z(\tau) \to Z(\tau).$$

The quotient space $W(\tau)/Z(\tau)$ inherits the Thurston symplectic form ω and is a non-degenerate symplectic vector space. The induced map $\overline{\mathfrak{t}_*}$ on $W(\tau)/Z(\tau)$ preserves the form ω. Then, it is a well-known fact that $\chi(\overline{\mathfrak{t}_*})$ is palindromic or anti-palindromic. The following commutative diagram holds.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(\tau) & \longrightarrow & W(\tau) & \longrightarrow & W(\tau)/Z(\tau) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathfrak{t}_*|_{Z(\tau)}} & & \downarrow{\scriptstyle \mathfrak{t}_*} & & \downarrow{\scriptstyle \overline{\mathfrak{t}_*}} & & \\ 0 & \longrightarrow & Z(\tau) & \longrightarrow & W(\tau) & \longrightarrow & W(\tau)/Z(\tau) & \longrightarrow & 0 \end{array}$$

By the above diagram together with Lemma 3.1, one has the following.

Proposition 3.3. *The characteristic polynomial of* $\mathfrak{t}_* : W(\tau) \to W(\tau)$ *has a factorization* $\chi(\mathfrak{t}_*) = \chi(\mathfrak{t}_*|_{Z(\tau)})\chi(\overline{\mathfrak{t}_*})$.

We call $\chi(\mathfrak{t}_*|_{Z(\tau)})$ and $\chi(\overline{\mathfrak{t}_*})$ the *radical polynomial* and the *(train track) symplectic polynomial* of $[f]$, respectively. We collectively refer to $\chi(\mathfrak{t}_*|_{Z(\tau)})$, $\chi(\overline{\mathfrak{t}_*})$ and $\chi(\mathfrak{t}_*) = \chi(\mathfrak{t}_*|_{Z(\tau)})\chi(\overline{\mathfrak{t}_*})$ as the *train track polynomials*.

Theorem 3.4. *The radial polynomial $\chi(\mathfrak{t}_*|_{Z(\tau)})$ and the train track symplectic polynomial $\chi(\overline{\mathfrak{t}_*})$ are conjugacy invariants of $[f] \in \mathrm{MCG}(\Sigma)$.*

Proof. By Proposition 3.3, $\chi(\overline{\mathfrak{t}_*})$ is given by $\chi(\overline{\mathfrak{t}_*}) = \frac{\chi(\mathfrak{t}_*)}{\chi(\mathfrak{t}_*|_{Z(\tau)})}$. By Theorem 3.2, it is enough to prove that $\chi(\mathfrak{t}_*|_{Z(\tau)})$ is a conjugacy invariant of $[f]$.

Suppose that $\mathfrak{t} : \tau \to \tau$ and $\mathfrak{t}' : \tau' \to \tau'$ are train track representatives of $[f]$. We take a third train track τ'' and a train track representative $\mathfrak{t}'' : \tau'' \to \tau''$ as in the proof of Theorem 3.2. Recall that the homotopy equivalence graph map $m : \tau \to \tau''$ is factored as a composition of elementary folding moves. Lemma 2.7 tells us that the isomorphism $m_* : W(\tau) \to W(\tau'')$ preserves the Thurston symplectic form ω, and we have the restriction map $m_*|_{Z(\tau)} : Z(\tau) \to Z(\tau'')$. This together with (3.1) yields the following commutative diagram.

$$\begin{array}{ccc} Z(\tau) & \xrightarrow{m_*|_{Z(\tau)}} & Z(\tau'') \\ \downarrow{\scriptstyle \mathfrak{t}_*|_{Z(\tau)}} & & \downarrow{\scriptstyle \mathfrak{t}''_*|_{Z(\tau'')}} \\ Z(\tau) & \xrightarrow{m_*|_{Z(\tau)}} & Z(\tau'') \end{array} \tag{3.2}$$

Hence, we have $\chi(\mathfrak{t}_*|_{Z(\tau)}) = \chi(\mathfrak{t}''_*|_{Z(\tau'')})$. Applying the same argument to the other homotopy equivalence graph map $m' : \tau' \to \tau''$, we obtain $\chi(\mathfrak{t}'_*|_{Z(\tau')}) = \chi(\mathfrak{t}''_*|_{Z(\tau'')})$. Thus, $\chi(\mathfrak{t}_*|_{Z(\tau)}) = \chi(\mathfrak{t}'_*|_{Z(\tau')})$ is an invariant of $[f]$. One can prove the conjugacy invariance of $\chi(\mathfrak{t}_*|_{Z(\tau)})$ in the same manner as in the proof of Theorem 3.2. □

Proof of Theorem 1.1. See Theorems 3.2 and 3.4. □

The train track polynomials are defined directly from train track representatives. Pseudo-Anosov maps on different surfaces may have the same train track representative: Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map. A singularity q of the unstable foliation $\mathcal{F}^u$ is either unpunctured or punctured. Suppose that q is an unpunctured singularity. Removing the periodic orbit $\mathcal{O}(q)$ under the action of f, we obtain a surface $\Sigma' := \Sigma \setminus \mathcal{O}(q)$ and a new pseudo-Anosov map $f' := f|_{\Sigma'} : \Sigma' \to \Sigma'$. Then a train track representative $\mathfrak{t} : \tau \to \tau$ of $[f] \in \mathrm{MCG}(\Sigma)$ is also a train track representative of $[f'] \in \mathrm{MCG}(\Sigma')$. This implies that the train track polynomials of $[f]$ and $[f']$ are the same:

Proposition 3.5. *The radical polynomial and the (train track) symplectic polynomial of $[f]$ are the same as those of $[f']$.*

The train track polynomials cannot distinguish between the two mapping classes $[f]$ and $[f']$. In Section 4, we introduce the second kind of polynomials, the BBK polynomials. We will prove in Section 5 that they have a finer structure than the train track polynomials.

4. BBK polynomials

4.1. **Bestvina-Handel algorithm.** Let G be a finite graph, and $\mathcal{V}(G)$ the set of vertices of G. We denote by $\mathcal{E}_{\mathrm{dir}}(G)$ and $\mathcal{E}(G)$, the set of oriented edges

of G and the set of unoriented edges of G, respectively. If $e \in \mathcal{E}_{\mathrm{dir}}(G)$, we use the notation $\overline{e}$ to denote the edge e with the reversed orientation. The *link* $\mathrm{Lk}(v) \subset \mathcal{E}_{\mathrm{dir}}(G)$ is the set of oriented edges emanating from a vertex v of G.

A *fibered surface* F is a compact surface with boundary which is decomposed into arcs and polygons modeled on k-junctions where $k = 1, 2, \ldots$ (Figure 8). An arc or a junction is called a *decomposition element*. Components fibered by arcs are called *strips*. Collapsing each arc to a point and each junction to a point gives a graph G whose vertices come from junctions and edges come from strips. We can regard G as a graph embedded in F and G represents the spine of F.

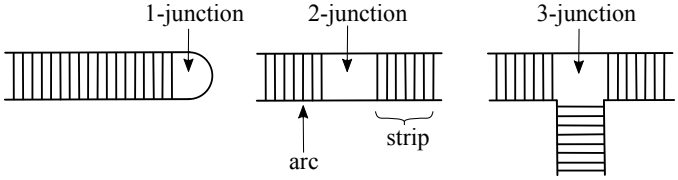


FIGURE 8. Fibered surface.

Let $\Sigma = \Sigma_{g,s}$ be a punctured surface for some $s \geq 1$. Consider a homeomorphism $f : \Sigma \to \Sigma$. We say that a fibered surface $F \subset \Sigma$ *carries* f if the inclusion $F \to \Sigma$ is a homotopy equivalence, f maps each decomposition element of F into a decomposition element of F, and f maps each junction into a junction. (In particular, f maps F into itself.) If F carries f, collapsing each decomposition element of F to a point, f determines a graph map $\mathfrak{g} : G \to G$ on the spine G of F sending vertices to vertices, and each edge to an edge-path. We call $\mathfrak{g} : G \to G$ an *induced graph map* of $[f] \in \mathrm{MCG}(\Sigma)$. From such a graph map $\mathfrak{g} : G \to G$, we define the transition matrix T of degree $|\mathcal{E}(G)|$ in the same manner as in Definition 2.5. The spectral radius of T is called the *growth rate* of $\mathfrak{g}$. The graph map $\mathfrak{g} : G \to G$ is *efficient* if for each $k \geq 1$, the graph map $\mathfrak{g}^k : G \to G$ has no back track, i.e., $\mathfrak{g}^k$ maps each oriented edge of G to an edge path that does not contain an edge $e \in \mathcal{E}_{\mathrm{dir}}(G)$ followed by $\overline{e}$.

Since Σ is a punctured surface, we can always find a fibered surface that carries f and we have an induced graph map $\mathfrak{g} : G \to G$ (which is not necessarily efficient). The Bestvina-Handel algorithm proceeds as follows. If the initial induced graph map $\mathfrak{g} : G \to G$ is not efficient, then either there is a $\mathfrak{g}$-invariant proper subgraph of G (corresponding to reducing curves for the reducible mapping class $[f]$), or it is possible to replace G with a graph G' such that the inclusion $G' \to \Sigma$ is a homotopy equivalence and the induced graph map $\mathfrak{g}' : G' \to G'$ has a smaller growth rate than $\mathfrak{g} : G \to G$. There is a number N depending on the initial graph map $\mathfrak{g} : G \to G$ and the surface Σ such that, after at most N steps, either a reduction or an efficient graph map $\mathfrak{g} : G \to G$ can be found, where we use the same notation $\mathfrak{g} : G \to G$ for the resulting graph map as the initial one. Such an efficient graph map can be thought of as a simplest possible induced graph map of $[f]$ in the sense of the growth rate. Once an efficient graph map $\mathfrak{g} : G \to G$ is found, one can determine the Nielsen-Thurston type of f. In the pseudo-Anosov case, we call the resulting graph G a *Bestvina-Handel graph* and $\mathfrak{g} : G \to G$ a *train*

track map of $[f]$. In this case, the growth rate of $\mathfrak{g}$ is equal to the stretch factor of $[f]$.

In the case where Σ is an s-punctured disk D_s, the program "Trains" developed by Toby Hall is available [Hal]. Using this program. one can easily find train track maps of pseudo-Anosov braids.

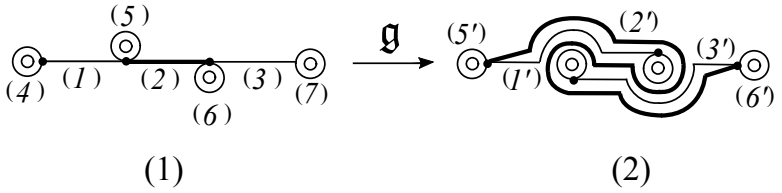


FIGURE 9. Example 4.1. The label (i') means the image of the edge labeled (i).

Example 4.1. Let $\Gamma : B_s \to \mathrm{MCG}(D_s)$ be the surjective homomorphism as in Section 2.1. Consider $[f] = \Gamma(\sigma_1\sigma_3\sigma_2^{-1}) \in \mathrm{MCG}(D_4)$. This is a pseudo-Anosov mapping class. The graph map $\mathfrak{g} : G \to G$ in Figure 9 is a train track map of $[f]$, where the edges are labeled $(1), \ldots, (7)$. The transition matrix of $\mathfrak{g}$ is given by $T = \begin{pmatrix} M & 0 \\ * & Q \end{pmatrix}$ where $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then $\chi(T) = \chi(M)\chi(Q) = (t-1)(t^2-4t+1)(t^4-1)$ and the largest real root $2+\sqrt{3}$ is the stretch factor of $[f]$.

4.2. **Construction of train track representatives.** Suppose that $[f]$ is pseudo-Anosov and $\mathfrak{g} : G \to G$ is a train track map of $[f]$. We define the *derivative* $D_\mathfrak{g} : \mathcal{E}_{\mathrm{dir}}(G) \to \mathcal{E}_{\mathrm{dir}}(G)$ as follows. Given an edge $e \in \mathcal{E}_{\mathrm{dir}}(G)$, we write the edge path $\mathfrak{g}(e) = e_1 e_2 \ldots e_\ell$, where $e_i \in \mathcal{E}_{\mathrm{dir}}(G)$. We set $D_\mathfrak{g}(e) := e_1$, that is, the initial edge of $\mathfrak{g}(e)$.

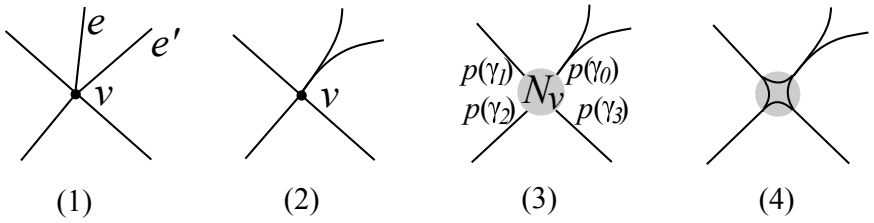


FIGURE 10. Edges e and e' are equivalent. Gray disks illustrate N_v. (1) Link $\mathrm{Lk}(v)$. (2) Step 1. (3) Step 2. (4) Step 3.

Definition 4.2. Consider the link $\mathrm{Lk}(v) \subset \mathcal{E}_{\mathrm{dir}}(G)$. Given $e_1, e_2 \in \mathrm{Lk}(v)$, we say that e_1 and e_2 are *equivalent* if $D_\mathfrak{g}^k(e_1) = D_\mathfrak{g}^k(e_2)$ for some $k \geq 1$. The equivalence classes are called *gates* incident to the vertex v.

Definition 4.3. From a train track map $\mathfrak{g} : G \to G$ of $[f]$, we construct a train track $\tau = \tau(G)$ taking the following 3 steps at each vertex v of G. (See Figure 10.)

Step 1. Deform the equivalent edges of $\mathrm{Lk}(v)$ in a neighborhood of v so that they are tangent at v.

Step 2. Replace the vertex v with a small disk N_v. For each gate γ incident to v, we assign a point $p(\gamma)$ on the boundary of N_v.

Step 3. If there exist an edge $e \in \mathcal{E}_{\mathrm{dir}}(G)$ and an integer $k \geq 1$ such that $\mathfrak{g}^k(e) = \cdots \overline{e_j} e_\ell \cdots$ for some $e_j, e_\ell \in \mathrm{Lk}(v)$ with $\gamma_j = [e_j] \neq \gamma_\ell = [e_\ell]$, then we join $p(\gamma_j)$ and $p(\gamma_\ell)$ by a smooth arc in N_v. The arc intersects the boundary of N_v transversely at $p(\gamma_j)$ and $p(\gamma_\ell)$, and no two such arcs intersect in the interior of N_v. The resulting graph is $\tau = \tau(G)$.

Remark 4.4. When we illustrate a Bestvina-Handel graph G, we may highlight the vertices with gray disks to show the additional structures of infinitesimal edges. With this convention, G can also represent the track $\tau(G)$. For example, if $\mathfrak{g} : G \to G$ is the train track map of $[f] = \Gamma(\sigma_1\sigma_3\sigma_2^{-1})$ in Example 4.1, Figure 11(2) depicts its train track $\tau(G)$. It indicates that each vertex of G is a partial vertex with three gates, and the graph in Figure 11(1) also represents the train track $\tau(G)$.

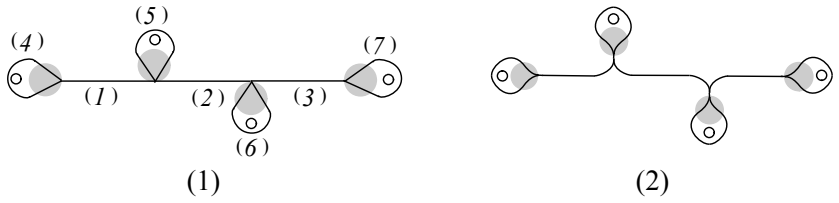


FIGURE 11. (1) Bestvina-Handel graph G of Example 4.1. (2) Train track $\tau(G)$.

We denote by $\mathcal{I}(\tau)$, the set of edges contained in N_v for all vertices v of G. Let $\mathcal{R}(\tau)$ be the set of other edges of τ.

Remark 4.5. An edge in $\mathcal{R}(\tau)$ is called "real" in [BBK12]. This terminology is not the same as that in Definition 2.4. See also Remark 4.7.

The set of all edges $\mathcal{E}(\tau)$ of τ is a union of $\mathcal{R}(\tau)$ and $\mathcal{I}(\tau)$:

$$\mathcal{E}(\tau) = \mathcal{R}(\tau) \cup \mathcal{I}(\tau). \tag{4.1}$$

There is a natural surjection

$$\pi : \tau \to G$$

that collapses all infinitesimal edges contained in N_v to the vertices v of G and that takes non-infinitesimal edges in $\mathcal{R}(\tau)$ to the corresponding edges of G. Since τ is f-invariant, the train track map $\mathfrak{g} : G \to G$ extends to a train track representative $\mathfrak{t} : \tau \to \tau$ of $[f]$ with the following commutative diagram.

$$\begin{array}{ccc} \tau & \xrightarrow{\pi} & G \\ \downarrow{\scriptstyle \mathfrak{t}} & & \downarrow{\scriptstyle \mathfrak{g}} \\ \tau & \xrightarrow{\pi} & G \end{array}$$

It is proved in [BH95, Proposition 3.3.4] that if v is a vertex of G with d gates, one of the following cases occurs.

Case V1. $d = 2$, and there exists a unique infinitesimal edge in N_v that joins the two gates.

Case V2. $d > 2$, and there exists an infinitesimal edge that joins a pair of adjacent gates except for one pair. Infinitesimal edges in N_v form a d-gon missing one side which is called a *partial polygon* (Figure 12(2)(2')).

Case V3. $d > 2$, and there exists an infinitesimal edge that joins a pair of adjacent gates. Infinitesimal edges in N_v form a d-gon which is called an *infinitesimal polygon* (Figure 12(3)(4)).

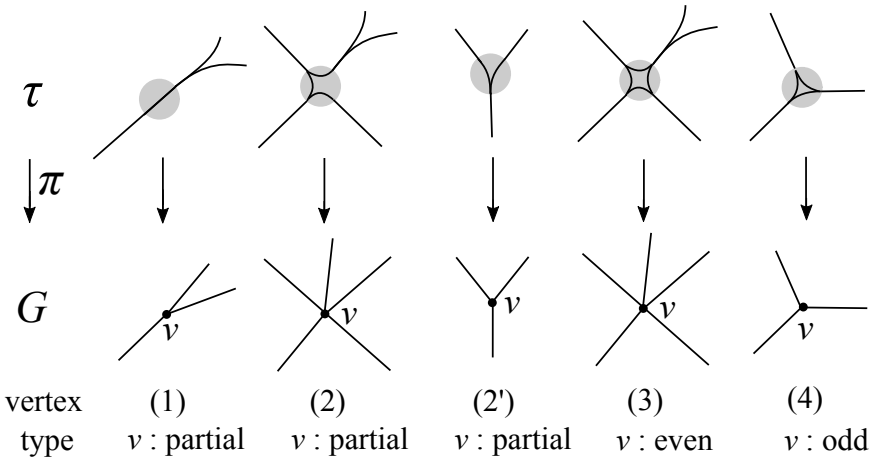


FIGURE 12. The surjection $\pi : \tau \to G$. (1) Case V1. (2)(2') Case V2. (3)(4) Case V3.

Definition 4.6. Let G be a Bestvina-Handel graph.

(1) Let v be a vertex of G with d gates. We say that v is *even* (resp. *odd*) if case V3 occurs and d is even (resp. odd). The vertex v is *partial* if case V1 or V2 occurs. See Figure 12.

(2) We denote by $\mathcal{V}_{\mathrm{e}}(G)$, the set of all even vertices of G, possibly $\mathcal{V}_{\mathrm{e}}(G) = \emptyset$.

Remark 4.7. Each edge in $\mathcal{I}(\tau)$ is infinitesimal (see Definition 2.4). On the other hand, an edge in $\mathcal{R}(\tau)$ is not necessarily a real edge. In fact, the train track τ in Figure 11(2) contains a $\mathfrak{t}$-invariant (peripheral) subgraph P whose components are circles, representing the peripheral curves in D_4. Each edge in P is an element of $\mathcal{R}(\tau)$ and an infinitesimal edge.

Recall that the inclusion $G \to \Sigma$ is a homotopy equivalence. If $\mathfrak{g} : G \to G$ is a train track map, one can construct a pseudo-Anosov representative of the corresponding mapping class and the invariant foliations. See [BH95, Section 3.4]. The following lemma is a consequence of the Bestvina-Handel algorithm.

Lemma 4.8. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map with unstable and stable foliations $\mathcal{F}^u$ and $\mathcal{F}^s$. Let $\mathfrak{g} : G \to G$ be a train track map of $[f]$. Then there is a one to one correspondence between the set of even (resp. odd) vertices of G and the set of unpunctured even- (resp. odd-)pronged singularities of $\mathcal{F}^u$ and $\mathcal{F}^s$.*

4.3. **The weight spaces of Bestvina-Handel graphs.** Let G be a Bestvina-Handel graph of $[f]$. Let $V(G) \simeq \mathbb{R}^{|\mathcal{E}(G)|}$ denote the vector space of real weights on the edges of G. The surjection $\pi_*\colon V(\tau) \to V(G)$ is induced by $\pi\colon \tau \to G$. Consider the weight space $W(\tau)$ of the train track τ. The *weight space* $W(G) \subset V(G)$ of G is defined by

$$W(G) := \pi_*(W(\tau)).$$

The restriction $\pi_*|_{W(\tau)} : W(\tau) \to W(G)$ is denoted by $\pi_* : W(\tau) \to W(G)$.

Let v be a vertex of G with d gates. We label $0, \dots, d-1$ for the gates counterclockwise. If v is a partial vertex, the labeling starts at an end of the partial polygon as shown in Figure 13(2)(2'). Given an element $\eta \in V(G)$, let $w_i^v(\eta)$ be the weight (coefficient) of the ith gate incident to the vertex v, that is, $w_i^v(\eta)$ is the sum of weights (coefficients) of all edges in the ith gate.

Lemma 4.9 (Lemma 2.3 in [BBK12])**.** *Let $\eta \in V(G)$. Then $\eta \in W(G)$ if and only if for each non-odd vertex, that is, even or partial vertex v with d gates, we have $\sum_{i=0}^{d-1}(-1)^i w_i^v(\eta) = 0$.*

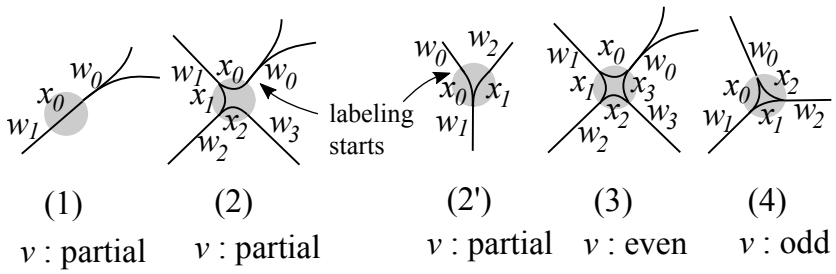


FIGURE 13. Gray disks illustrate N_v for the vertex v. For $\eta \in V(G)$, the weight of the ith gate incident to v is denoted by $w_i := w_i^v(\eta)$, and the weights on infinitesimal edges are denoted by x_j's.

Remark 4.10. The odd vertices of G are free in the statement of Lemma 4.9. In fact, if G contains an odd vertex v, a simple calculation shows that the switch condition is satisfied at every switch in $\tau(G)$ associated with the vertex v. (See also the proof of Lemma 2.3 in [BBK12].) For example, if v is an odd vertex with three gates, then there is a 3-pronged boundary component $\pi^{-1}(v)$ of $\tau(G)$. (See Definition 2.2.) Given any real weights w_0, w_1 and w_2 at the three gates, there exist unique real weights x_0, x_1 and x_2 on the infinitesimal edges such that the switch conditions $x_{i-1} + x_i = w_i$ hold, where the index i is considered mod 3. See Figure 13(4).

4.4. **Homology polynomials.** Since the train track $\tau = \tau(G)$ is f-invariant, the linear map $\mathfrak{t}_* : V(\tau) \to V(\tau)$ preserves $W(\tau)$, that is, $\mathfrak{t}_*(W(\tau)) \subset W(\tau)$. We can define an induced linear map $\mathfrak{g}_*\colon W(G) \to W(G)$ by the following commutative diagram

$$\begin{array}{ccc} W(\tau) & \xrightarrow{\pi_*} & W(G) \\ \downarrow{\scriptstyle \mathfrak{t}_*} & & \downarrow{\scriptstyle \mathfrak{g}_*} \\ W(\tau) & \xrightarrow{\pi_*} & W(G) \end{array}$$

Birman-Brinkmann-Kawamuro stated in Corollary 2.2 of [BBK12, p.31] that the characteristic polynomial $h(t) := \chi(\mathfrak{g}_*)$ of $\mathfrak{g}_* : W(G) \to W(G)$ is an invariant of the pseudo-Anosov mapping class $[f]$. Dodge-Kawamuro [DK26] gave more details on the proof of this result.

In [BBK12], $h(t) = \chi(\mathfrak{g}_*)$ is called the *homology polynomial* of $[f]$. We prove in Section 5.2 that the polynomial $h(t)$ is a conjugacy invariant of the pseudo-Anosov mapping class $[f]$. See also Remark 1.4.

Remark 4.11. As stated in Theorem 1.1 of [BBK12], the homology polynomial $h(t)$ is a factor of $\chi(T)$, where T is the transition matrix of the train track map $\mathfrak{g} : G \to G$. To see this, we note that $\chi(T)$ is the characteristic polynomial of $\mathfrak{g}_* : V(G) \to V(G)$ with respect to the standard basis of $V(G)$. Since $W(G)$ is the subspace of $V(G)$ preserved by $\mathfrak{g}_* : V(G) \to V(G)$, the polynomial $h(t)(= \chi(\mathfrak{g}_*|_{W(G)}))$ should be a factor of $\chi(T)$.

4.5. **Puncture polynomials and symplectic polynomials.** Following [BBK12], we introduce a skew-symmetric form on $W(G)$:

Definition 4.12. For $\eta, \zeta \in W(G)$, we choose vectors $\eta', \zeta' \in W(\tau)$ so that $\pi_*(\eta') = \eta$ and $\pi_*(\zeta') = \zeta$. A skew-symmetric form $\overline{\omega}$ on $W(G)$ is defined by

$$\overline{\omega}(\eta, \zeta) := \omega(\eta', \zeta'),$$

where ω is the Thurston symplectic form on $W(\tau)$. See Section 2.4.

The value of $\overline{\omega}(\eta, \zeta)$ does not depend on the choices of $\eta', \zeta' \in W(\tau)$. To see this, it suffices to show that $\omega(\hat{\eta}, \hat{\zeta}) = 0$ for any $\hat{\eta} \in \ker \pi_* \subset W(\tau)$ and $\hat{\zeta} \in W(\tau)$. This computation is included in the proof of Proposition 3.1(1) in [BBK12]. See also Remark 5.2.

The *radical* $Z(G)$ of $\overline{\omega}$ is defined by

$$Z(G) := \{\eta \in W(G) \mid \overline{\omega}(\eta, \zeta) = 0 \text{ for all } \zeta \in W(G)\}.$$

By definition, we have $\pi_*(Z(\tau)) = Z(G)$. Consider a direct sum decomposition $W(G) \simeq Z(G) \oplus W(G)/Z(G)$. Since $\mathfrak{t}_* \colon W(\tau) \to W(\tau)$ preserves the Thurston symplectic form ω and $\mathfrak{g}_*$ is of the form $\mathfrak{g}_* = \pi_* \circ \mathfrak{t}_* \circ \pi_*^{-1}$, the map $\mathfrak{g}_* : W(G) \to W(G)$ also preserves the form $\overline{\omega}$, and hence $\mathfrak{g}_*(Z(G)) \subset Z(G)$. We can define the restriction map $\mathfrak{g}_*|_{Z(G)}$ and have the following commutative diagram.

$$\begin{array}{ccc} Z(\tau) & \xrightarrow{\pi_*|_{Z(\tau)}} & Z(G) \\ \downarrow{\scriptstyle \mathfrak{t}_*|_{Z(\tau)}} & & \downarrow{\scriptstyle \mathfrak{g}_*|_{Z(G)}} \\ Z(\tau) & \xrightarrow{\pi_*|_{Z(\tau)}} & Z(G) \end{array}$$

The quotient space $W(G)/Z(G)$ inherits the form $\overline{\omega}$ and the induced map $\overline{\mathfrak{g}_*}$ preserves the form $\overline{\omega}$. The following commutative diagram holds.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(G) & \longrightarrow & W(G) & \longrightarrow & W(G)/Z(G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathfrak{g}_*|_{Z(G)}} & & \downarrow{\scriptstyle \mathfrak{g}_*} & & \downarrow{\scriptstyle \overline{\mathfrak{g}_*}} & & \\ 0 & \longrightarrow & Z(G) & \longrightarrow & W(G) & \longrightarrow & W(G)/Z(G) & \longrightarrow & 0 \end{array}$$

The above diagram together with Lemma 3.1 yields the following.

Proposition 4.13 (Theorem 3.1 in [BBK12])**.** *The homology polynomial* $h(t) = \chi(\mathfrak{g}_*)$ *has a factorization* $\chi(\mathfrak{g}_*) = \chi(\mathfrak{g}_*|_{Z(G)})\chi(\overline{\mathfrak{g}_*})$.

The polynomial $p(t) := \chi(\mathfrak{g}_*|_{Z(G)})$ is called the *puncture polynomial* and $s(t) := \chi(\overline{\mathfrak{g}_*})$ is called the *symplectic polynomial*. We collectively refer to $p(t)$, $s(t)$ and $h(t) = p(t)s(t)$ as the *BBK polynomials* of $[f]$.

Birman-Brinkmann-Kawamuro stated in Theorem 3.1 of [BBK12, p.36] that the polynomials $p(t)$ and $s(t)$ are invariants of $[f]$, and the symplectic polynomial $s(t)$ contains the stretch factor of $[f]$ as its largest real root. In [DK26], more details on the proof of the invariance of the polynomials $p(t)$ and $s(t)$ are provided. We prove in Theorem 5.10 that the polynomials $p(t)$ and $s(t)$ are conjugacy invariants of $[f]$. See also Remark 1.4.

5. Proof of conjugacy invariance of the BBK polynomials

In this section, we introduce a polynomial conjugacy invariant of pseudo-Anosov maps, which we call the unpunctured singularity polynomial $u(t)$. The polynomial $u(t)$ encodes the transitions among unpunctured even-pronged singularities of pseudo-Anosov maps. We prove that the BBK polynomials can be described by the train track polynomials together with $u(t)$. This gives us an alternative proof of the conjugacy invariance of the BBK polynomials.

Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map, where $\Sigma = \Sigma_{g,s}$ with $s \geq 1$. Let $\mathfrak{g} : G \to G$ be a train track map of $[f]$ and $\mathfrak{t} : \tau \to \tau$ the train track representative on $\tau = \tau(G)$. Several of the commutative diagrams that appear in this section are based on the following commutative diagram (5.1). The reader may find it helpful to keep this diagram (5.1) in mind while reading this section.

(5.1)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \pi_*|_{Z(\tau)} & \longrightarrow & Z(\tau) & \xrightarrow{\pi_*|_{Z(\tau)}} & Z(G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \ker \pi_* & \longrightarrow & W(\tau) & \xrightarrow{\pi_*} & W(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \overline{\pi_*} & \longrightarrow & W(\tau)/Z(\tau) & \xrightarrow{\overline{\pi_*}} & W(G)/Z(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where both rows and columns are exact and ι denotes the inclusion. (We shall see $\ker \pi_*|_{Z(\tau)} = \ker \pi_*$, and hence $\ker \overline{\pi_*} = \ker \pi_* / \ker \pi_*|_{Z(\tau)} = \{0\}$.)

5.1. Unpunctured singularity polynomials.

We begin with the lemma.

Lemma 5.1. *The surjections* $\pi_* : W(\tau) \to W(G)$ *and* $\pi_*|_{Z(\tau)} : Z(\tau) \to Z(G)$ *satisfy the following.*

(1) $\ker \pi_* = \ker \pi_*|_{Z(\tau)}$.

(2) $\ker \pi_*|_{Z(\tau)}$ *has a basis* $\{r_{\pi^{-1}(v)} \mid v \text{ is an even vertex of } G\}$*, where* $r_{\pi^{-1}(v)}$ *is the radical element constructed in Definition 2.14.*

Proof. Take an element $\zeta \in \ker \pi_* \subset W(\tau)$. By the definition of $\pi : \tau \to G$, the coefficient of $\zeta \in \ker \pi_*$ on any edge in $\mathcal{R}(\tau)$ is 0. (For the definition of the set $\mathcal{R}(\tau)$, see the paragraph just after Remark 4.4.) Suppose that e is an (infinitesimal) edge on the infinitesimal polygon $\pi^{-1}(v)$ for some vertex v of G. In case v is an odd vertex, then $\zeta(e) = 0$ (see Remark 4.10). In case v is a partial vertex, it is easy to see that $\zeta(e) = 0$. Consider the remaining case where v is an even vertex. Then $\pi^{-1}(v)$ is an even-pronged boundary component of τ. Consider the radical element $r_{\pi^{-1}(v)} \in W(\tau)$. By the definition of $\pi : \tau \to G$, we have $r_{\pi^{-1}(v)} \in \ker \pi_*$. On the other hand, if there is a vector $\zeta \in \ker \pi_*$ where $\zeta(e) = a$ for an edge $e \subset \pi^{-1}(v)$, then the coefficients of other edges $e' \subset \pi^{-1}(v)$ in $\zeta \in \ker \pi_*$ should be a or $-a$ by the switch conditions, and it follows that $\zeta = \sum a_v r_{\pi^{-1}(v)}$ where v rages over all even-pronged vertex of G. Hence, we have

$$\ker \pi_* = \operatorname{span}\{r_{\pi^{-1}(v)} \mid v \text{ is an even vertex of } G\}. \tag{5.2}$$

Since $r_{\pi^{-1}(v)} \in Z(\tau)$ (Proposition 2.18), it holds $\ker \pi_* = \ker \pi_*|_{Z(\tau)}$. This completes the proof of (1).

We turn to the proof of (2). Consider all even vertices v of G. Then $\{r_{\pi^{-1}(v)}\}$ are linearly independent, since the infinitesimal polygons of the form $\pi^{-1}(v)$ are disjoint. This together with (5.2) implies claim (2). □

Remark 5.2. The fact that the skew-symmetric form $\overline{\omega}$ on $W(G)$ is well defined follows from Lemma 5.1. In fact, Lemma 5.1 tells us that if $\eta' \in \ker \pi_*$, then $\eta' \in Z(\tau)$. Hence, $\omega(\eta', \zeta') = 0$ for any $\zeta' \in W(\tau)$. This implies that $\overline{\omega}$ is well defined.

By Lemma 5.1, the dimension of $\ker \pi_* (= \ker \pi_*|_{Z(\tau)})$ equals $|\mathcal{V}_{\mathrm{e}}(G)|$, that is, the number of even vertices of G. Then the surjections $\pi_* : W(\tau) \to W(G)$ and $\pi_*|_{Z(\tau)} : Z(\tau) \to Z(G)$ yield the following result.

Corollary 5.3. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map. Let G be a Bestvina-Handel graph of $[f]$ and $\tau = \tau(G)$ the train track. We have the following about the dimensions of vector spaces.*

(1) $\dim W(\tau) = \dim W(G) + |\mathcal{V}_{\mathrm{e}}(G)|$.
(2) $\dim Z(\tau) = \dim Z(G) + |\mathcal{V}_{\mathrm{e}}(G)|$.

In particular, if f is a fully punctured pseudo-Anosov map, then $\dim W(\tau) = \dim W(G)$ *and* $\dim Z(\tau) = \dim Z(G)$.

Since $\mathfrak{g}$ preserves the set of even vertices, we have $\mathfrak{t}_*(\ker \pi_*) \subset \ker \pi_*$. Hence, the restriction map $\mathfrak{t}_*|_{\ker \pi_*} : \ker \pi_* \to \ker \pi_*$ is defined.

Theorem 5.4. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map, where $\Sigma = \Sigma_{g,s}$ with $s \geq 1$. The characteristic polynomial $\chi(\mathfrak{t}_*|_{\ker \pi_*})$ of $\mathfrak{t}_*|_{\ker \pi_*} : \ker \pi_* \to \ker \pi_*$ is a conjugacy invariant of $[f] \in \mathrm{MCG}(\Sigma)$.*

Proof. By Lemma 5.1(1) we have

$$\chi(\mathfrak{t}_*|_{\ker \pi_*}) = \chi(\mathfrak{t}_*|_{\ker \pi_*|_{Z(\tau)}}),$$

where $\mathfrak{t}_*|_{\ker \pi_*|_{Z(\tau)}} : \ker \pi_*|_{Z(\tau)} \to \ker \pi_*|_{Z(\tau)}$. We prove that $\chi(\mathfrak{t}_*|_{\ker \pi_*|_{Z(\tau)}})$ is a conjugacy invariant of $[f]$.

Suppose that $\mathfrak{g} : G \to G$ and $\mathfrak{g}' : G' \to G'$ are train track maps on Bestvina-Handel graphs G and G' for $[f]$, respectively. We have f-invariant train tracks $\tau = \tau(G)$ and $\tau' = \tau(G')$. Consider the train track representatives $\mathfrak{t} : \tau \to \tau$ and $\mathfrak{t}' : \tau' \to \tau'$. We take the third train track τ'' together with the train track representative $\mathfrak{t}'' : \tau'' \to \tau''$ as in the proof of Theorem 3.2.

Recall that the homotopy equivalence graph map $m : \tau \to \tau''$ yields the isomorphism $m_*|_{Z(\tau)} : Z(\tau) \to Z(\tau'')$ with the commutative diagram (3.2). By Lemma 5.1 we have

$$\ker \pi_*|_{Z(\tau)} = \operatorname{span}\{r_{\pi^{-1}(v)} \mid v \text{ is an even vertex of } G\},$$

where $\pi : \tau \to G$ is the surjection. We take any even vertex v of G and consider the even-pronged boundary component $\pi^{-1}(v)$ of τ. Propositions 2.15 and 2.18(1) tell us that there is a boundary component $c''(v)$ of τ'' corresponding to $\pi^{-1}(v)$ such that the radical element $r_{\pi^{-1}(v)} \in Z(\tau)$ maps to the radical element $r_{c''(v)} \in Z(\tau'')$ under the isomorphism $m_*|_{Z(\tau)} : Z(\tau) \to Z(\tau'')$.

We turn to the other homotopy equivalence graph map $m' : \tau' \to \tau''$ which gives the isomorphism $m'_*|_{Z(\tau')} : Z(\tau') \to Z(\tau'')$. By Lemma 5.1 again, we have

$$\ker \pi'_*|_{Z(\tau')} = \operatorname{span}\{r_{\pi'^{-1}(v)} \mid v' \text{ is an even vertex of } G'\},$$

where $\pi' : \tau' \to G'$ is the surjection. By Lemma 4.8, there is a natural one to one correspondence between the set of even vertices of G and the set of even vertices of G' via the set of unpunctured even-pronged singularities of $\mathcal{F}^u$ and $\mathcal{F}^s$. For each even vertex v of G, we pick an even vertex v' of G' corresponding to v, and consider the even-pronged boundary component $\pi'^{-1}(v')$ of τ'. By Propositions 2.15, the isomorphism $m'_*|_{Z(\tau')} : Z(\tau') \to Z(\tau'')$ takes $r_{\pi'^{-1}(v')}$ to $r_{c''(v)} \in Z(\tau'')$ for the above boundary component $c''(v)$ of τ''.

Putting them together, one sees that for the above even vertices v of G and v' of G', the isomorphism $(m'_*|_{Z(\tau')})^{-1} \circ m_*|_{Z(\tau)} : Z(\tau) \to Z(\tau')$ takes the radical element $r_{\pi^{-1}(v)}$ to the radical element $r_{\pi'^{-1}(v')}$, and gives the isomorphism between $\ker \pi_*|_{Z(\tau)}$ to $\ker \pi'_*|_{Z(\tau')}$. Hence, the following commutative diagram holds.

$$\begin{array}{ccc} \ker \pi_*|_{Z(\tau)} & \longrightarrow & \ker \pi'_*|_{Z(\tau')} \\ {\scriptstyle \mathfrak{t}_*|_{\ker \pi_*|_{Z(\tau)}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{t}'_*|_{\ker \pi'_*|_{Z(\tau')}}} \\ \ker \pi_*|_{Z(\tau)} & \longrightarrow & \ker \pi'_*|_{Z(\tau')} \end{array}$$

Thus, $\chi(\mathfrak{t}_*|_{\ker \pi_*|_{Z(\tau)}}) = \chi(\mathfrak{t}'_*|_{\ker \pi_*|_{Z(\tau')}})$ is an invariant of $[f]$.

The conjugacy invariance can be proved in the same manner as in the proof of Theorem 3.2. This completes the proof. □

We call $u(t) := \chi(\mathfrak{t}_*|_{\ker \pi_*})$ the *unpunctured singularity polynomial* of $[f]$.

Corollary 5.5. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map, where $\Sigma = \Sigma_{g,s}$ with $s \geq 1$. Suppose that the unstable foliation $\mathcal{F}^u$ has no unpunctured even-pronged singularities. Then $u(t) = 1$. In particular, if f is fully punctured, then $u(t) = 1$.*

Proof. By Lemmas 4.8 and 5.1, it follows that $\dim(\ker \pi_*) = 0$. Hence, $u(t) = \chi(\mathfrak{t}_*|_{\ker \pi_*}) = 1$. □

5.2. **Proof of conjugacy invariance of homology polynomials.** Let $\mathfrak{g}\colon G \to G$ be a train track map of $[f]$. Let $\tau = \tau(G)$, and $\mathfrak{t}\colon \tau \to \tau$ a train track representative of $[f]$. We recall the linear maps $\mathfrak{g}_*\colon W(G) \to W(G)$ and $\mathfrak{t}_*\colon W(\tau) \to W(\tau)$. The following commutative diagram holds.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \pi_* & \longrightarrow & W(\tau) & \xrightarrow{\pi_*} & W(G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathfrak{t}_*|_{\ker \pi_*}} & & \downarrow{\scriptstyle \mathfrak{t}_*} & & \downarrow{\scriptstyle \mathfrak{g}_*} & & \\ 0 & \longrightarrow & \ker \pi_* & \longrightarrow & W(\tau) & \xrightarrow{\pi_*} & W(G) & \longrightarrow & 0 \end{array}$$

Applying Lemma 3.1 to the above diagram, we obtain the following.

Proposition 5.6. *The characteristic polynomial of* $\mathfrak{t}_* : W(\tau) \to W(\tau)$ *has a factorization* $\chi(\mathfrak{t}_*) = \chi(\mathfrak{g}_*)\chi(\mathfrak{t}_*|_{\ker \pi_*})$.

Theorem 5.7. *Let* $f : \Sigma \to \Sigma$ *be a pseudo-Anosov map, where* $\Sigma = \Sigma_{g,s}$ *with* $s \geq 1$. *The homology polynomial* $h(t) = \chi(\mathfrak{g}_*)$ *is a conjugacy invariant of* $[f] \in \mathrm{MCG}(\Sigma)$.

Proof. By Proposition 5.6, we can write $\chi(\mathfrak{g}_*)$ as $\chi(\mathfrak{g}_*) = \chi(\mathfrak{t}_*)/\chi(\mathfrak{t}_*|_{\ker \pi_*})$. By Theorems 3.2 and 5.4, the polynomials $\chi(\mathfrak{t}_*)$ and $\chi(\mathfrak{t}_*|_{\ker \pi_*})$ are conjugacy invariants of $[f]$. Thus, $\chi(\mathfrak{g}_*)$ is a conjugacy invariant of $[f]$. □

5.3. **Proof of conjugacy invariance of puncture/symplectic polynomials.** Recall that $\overline{\omega}$ is the skew-symmetric form on $W(G)$, and $Z(G)$ is the radical of $\overline{\omega}$. Since $\mathfrak{t}_*\colon W(\tau) \to W(\tau)$ preserves the Thurston symplectic form ω on $W(\tau)$, we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \pi_*|_{Z(\tau)} & \longrightarrow & Z(\tau) & \xrightarrow{\pi_*|_{Z(\tau)}} & Z(G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathfrak{t}_*|_{\ker \pi_*|_{Z(\tau)}}} & & \downarrow{\scriptstyle \mathfrak{t}_*|_{Z(\tau)}} & & \downarrow{\scriptstyle \mathfrak{g}_*|_{Z(G)}} & & \\ 0 & \longrightarrow & \ker \pi_*|_{Z(\tau)} & \longrightarrow & Z(\tau) & \xrightarrow{\pi_*|_{Z(\tau)}} & Z(G) & \longrightarrow & 0 \end{array}$$

Recall that $\chi(\mathfrak{t}_*|_{\ker \pi_*}) = \chi(\mathfrak{t}_*|_{\ker \pi_*|_{Z(\tau)}})$. Applying Lemma 3.1 to the above diagram, we obtain the following.

Proposition 5.8. *The characteristic polynomial* $\chi(\mathfrak{t}_*|_{Z(\tau)})$ *of* $\mathfrak{t}_*|_{Z(\tau)} : Z(\tau) \to Z(\tau)$ *has a factorization* $\chi(\mathfrak{t}_*|_{Z(\tau)}) = \chi(\mathfrak{g}_*|_{Z(G)})\chi(\mathfrak{t}_*|_{\ker \pi_*})$.

Let $\overline{\mathfrak{t}_*}\colon W(\tau)/Z(\tau) \to W(\tau)/Z(\tau)$ be the map induced by $\mathfrak{t}_* : W(\tau) \to W(G)$, and let $\overline{\mathfrak{g}_*}\colon W(G)/Z(G) \to W(G)/Z(G)$ be the map induced by $\mathfrak{g}_* : W(G) \to W(G)$. We have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \overline{\pi_*} & \longrightarrow & W(\tau)/Z(\tau) & \xrightarrow{\overline{\pi_*}} & W(G)/Z(G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\mathfrak{t}_*}|_{\ker \overline{\pi_*}}} & & \downarrow{\scriptstyle \overline{\mathfrak{t}_*}} & & \downarrow{\scriptstyle \overline{\mathfrak{g}_*}} & & \\ 0 & \longrightarrow & \ker \overline{\pi_*} & \longrightarrow & W(\tau)/Z(\tau) & \xrightarrow{\overline{\pi_*}} & W(G)/Z(G) & \longrightarrow & 0 \end{array} \tag{5.3}$$

By Lemma 5.1(1), $\ker \overline{\pi_*} = \ker \pi_* / \ker \pi_*|_{Z(\tau)} = \{0\}$. See also diagram (5.1). Hence, the map $\overline{\pi_*} : W(\tau)/Z(\tau) \to W(G)/Z(G)$ in (5.3) (induced by $\pi_* : W(\tau) \to W(G)$) is an isomorphism, and we obtain the following.

Proposition 5.9. $\chi(\overline{\mathfrak{t}_*}) = \chi(\overline{\mathfrak{g}_*})(= s(t))$.

Theorem 5.10. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map, where $\Sigma = \Sigma_{g,s}$ with $s \geq 1$. The puncture polynomial $p(t) = \chi(\mathfrak{g}_*|_{Z(G)})$ and the symplectic polynomial $s(t) = \chi(\overline{\mathfrak{g}_*})$ are conjugacy invariants of $[f] \in \mathrm{MCG}(\Sigma)$.*

Proof. By Proposition 5.8, we can write $p(t) = \chi(\mathfrak{g}_*|_{Z(G)})$ as

$$\chi(\mathfrak{g}_*|_{Z(G)}) = \chi(\mathfrak{t}_*|_{Z(\tau)})/\chi(\mathfrak{t}_*|_{\ker \pi_*}).$$

Then $p(t)$ is a conjugacy invariant, since the radical polynomial $\chi(\mathfrak{t}_*|_{Z(\tau)})$ and the unpunctured singularity polynomial $\chi(\mathfrak{t}_*|_{\ker \pi_*})$ are conjugacy invariants of $[f]$ (see Theorems 3.4 and 5.4).

The conjugacy invariance of $s(t) = \chi(\overline{\mathfrak{g}_*})$ follows from Theorem 3.4 and Proposition 5.9. □

Theorem 1.2 is a straightforward modification of the following result, which is a characterization of the BBK polynomials.

Theorem 5.11. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map, where $\Sigma = \Sigma_{g,s}$ with $s \geq 1$. Let $\mathfrak{g}\colon G \to G$ be a train track map and $\mathfrak{t}\colon \tau \to \tau$ the train track representative of $[f]$. Then we have the following.*

(1) $h(t)(= \chi(\mathfrak{g}_*)) = \chi(\mathfrak{t}_*)/\chi(\mathfrak{t}_*|_{\ker \pi_*})$
(2) $p(t)(= \chi(\mathfrak{g}_*|_{Z(G)})) = \chi(\mathfrak{t}_*|_{Z(\tau)})/\chi(\mathfrak{t}_*|_{\ker \pi_*})$
(3) $s(t)(= \chi(\overline{\mathfrak{g}_*})) = \chi(\overline{\mathfrak{t}_*})$.

In particular, if f is fully punctured, then $h(t) = \chi(\mathfrak{t}_*)$, $p(t) = \chi(\mathfrak{t}_*|_{Z(\tau)})$ and $s(t) = \chi(\overline{\mathfrak{t}_*})$.

Proof. For claims (1), (2) and (3), see Propositions 5.6, 5.8 and 5.9. If f is fully punctured, then $u(t)(= \chi(\mathfrak{t}_*|_{\ker \pi_*})) = 1$ (Corollary 5.5). This completes the proof. □

Proof of Corollary 1.3. The conjugacy invariance of the BBK polynomials follows from Theorems 5.7 and 5.10. □

Definition 5.12. Let $Q(t) = \sum_{i=1}^{n} a_i t^{\alpha_i}$ be a polynomial. Its *degree* and its *reciprocal polynomial* are defined by $\deg(Q) := \max \alpha_i$ and $Q_*(t) := t^{\deg(Q)} Q(t^{-1})$. We say that $Q(t)$ is *reciprocal* if $Q(t) = \pm Q_*(t)$; it is *palindromic* if $Q(t) = Q_*(t)$, and *anti-palindromic* if $Q(t) = -Q_*(t)$.

We now prove the proposition stated in Section 1.

Proof of Proposition 1.5. Suppose that f has at least two singularity orbits. In the proof of Theorem 6.1 of [HT25], it is proved that the polynomials $\chi(\overline{\mathfrak{t}_*})$, $\chi(\mathfrak{t}_*|_{Z(\tau)})$ and $\chi(\mathfrak{t}_*)$ are reciprocal for a particular train track representative $\mathfrak{t} : \tau \to \tau$. Since the train track polynomials are conjugacy invariants of pseudo-Anosov mapping classes $[f]$ (Theorems 3.2 and 3.4), the same properties hold for all train track representatives of $[f]$. In the case where f has a single singularity orbit, one can prove the statement in the same manner.

As an alternative proof, it is proved in [BBK12] that the BBK polynomials $s(t) = \chi(\overline{\mathfrak{g}_*})$, $p(t) = \chi(\mathfrak{g}_*|_{Z(G)})$ and $h(t) = \chi(\mathfrak{g}_*)$ are reciprocal. If f is a fully punctured pseudo-Anosov map, the BBK polynomials coincide with the train

track polynomials (Theorem 5.11). Therefore, the train track polynomials $\chi(\overline{\mathfrak{t}_*})$, $\chi(\mathfrak{t}_*|_{Z(\tau)})$ and $\chi(\mathfrak{t}_*)$ are reciprocal. □

The following lemma can be found in [BBK12, Lemma 2.5].

Lemma 5.13. *The dimension of $W(G)$ is*

$$\begin{array}{ll} \#\{\text{edges of } G\} - \#\{\text{vertices of } G\} + 1 & \text{if } \tau \text{ is orientable,} \\ \#\{\text{edges of } G\} - \#\{\text{non-odd vertices of } G\} & \text{if } \tau \text{ is non-orientable.} \end{array}$$

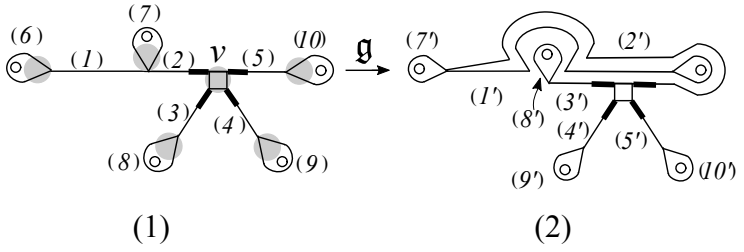


FIGURE 14. Example 5.14. Bold edges are half edges emanating from the even vertex v.

Example 5.14. Consider $[f] = \Gamma(\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}) \in \mathrm{MCG}(D_5)$ which is pseudo-Anosov. We compute the BBK polynomials. Figure 14 illustrates a train track map $\mathfrak{g} : G \to G$ for $[f]$. The edges of the graph G are labeled $(1), \dots, (10)$, and the set of vertices consists of an even vertex v and five partial vertices. See the gray disks in Figure 14(1) and Remark 4.4.

Since $\tau(G)$ is non-orientable, Lemma 5.13 tells us $\dim W(G) = 10 - 6 = 4$. Take vectors $\eta_{(i)} \in V(G)$ $(i = 7, 8, 9, 10)$ as shown in Figure 15. For example, $\eta_{(7)} = (\,2\,0\,0\,0\,0\,1\,1\,0\,0\,0\,)$. By Lemma 4.9, one sees that $\eta_{(i)} \in W(G)$. Among them, the vector $\eta_{(i)}$ is the only one such that the edge labeled (i) has non-zero weight (coefficient). Hence, they are linearly independent and yield a basis of $W(G)$. The representation matrix of $\mathfrak{g}_* : W(G) \to W(G)$ with respect to this basis is given by $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$. Hence, $h(t) = \chi(\mathfrak{g}_*) = t^4 - 3t^3 + 3t^2 - 3t + 1$. Let $\mathfrak{t} : \tau \to \tau$ be the train track representative of $[f]$ obtained from $\mathfrak{g} : G \to G$. All the boundary components of τ have 1-pronged, except the infinitesimal polygon $\pi^{-1}(v)$ which is 4-pronged. From the transition of half edges emanating from v (Figure 14), we see that $\mathfrak{t}_*(\gamma_{\pi^{-1}(v)}) = -\gamma_{\pi^{-1}(v)}$. (See Definition 2.14 for the radical element.) Hence, the radical polynomial $\chi(\mathfrak{t}|_{Z(\tau)})$ and the unpunctured singularity polynomial $u(t) = \chi(\mathfrak{t}_*|_{\ker \pi_*})$ satisfy $\chi(\mathfrak{t}|_{Z(\tau)}) = u(t) = t + 1$. This together with Proposition 5.8 implies that $p(t) = 1$. By factorization $h(t) = p(t)s(t)$ (Proposition 4.13), we obtain $s(t)(= h(t)) = t^4 - 3t^3 + 3t^2 - 3t + 1$. The largest real root equals the stretch factor $\lambda \approx 2.15372$ of $[f]$.

6. Examples

Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map. We define a pseudo-Anosov map f_{even} obtained from f. To do this, let $\mathcal{S}_{\mathrm{e}}$ be the set of all unpunctured

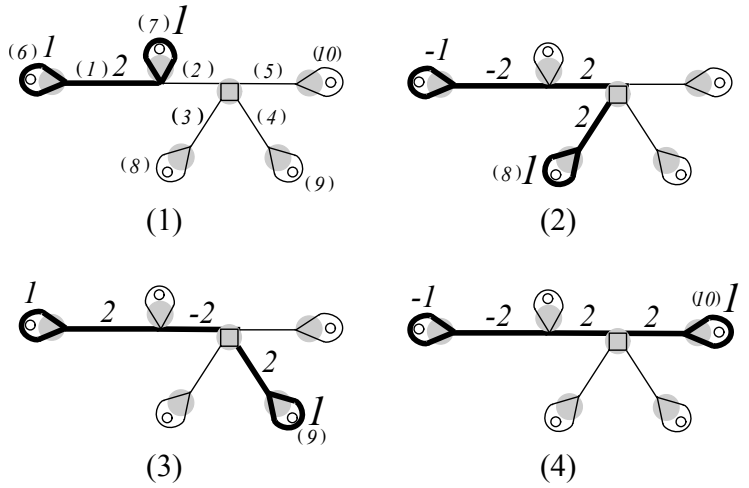


FIGURE 15. Vectors (1) $\eta_{(7)}$, (2) $\eta_{(8)}$, (3) $\eta_{(9)}$ and (4) $\eta_{(10)}$ in $W(G)$. An edge without number means its weight is 0.

even-pronged singularities of $\mathcal{F}^u$, possibly $\mathcal{S}_{\mathrm{e}} = \emptyset$. We set $\Sigma_{\mathrm{e}} := \Sigma \setminus \mathcal{S}_{\mathrm{e}}$. Then we have a pseudo-Anosov map

$$f_{\mathrm{even}} := f|_{\Sigma_{\mathrm{e}}} : \Sigma_{\mathrm{e}} \to \Sigma_{\mathrm{e}}$$

that is obtained from f by removing all unpunctured even-pronged singularities. If there are no unpunctured odd-pronged singularities, then $f_{\mathrm{even}} = f^\circ$ is a fully punctured pseudo-Anosov map.

Theorem 6.1. *Let $f : \Sigma \to \Sigma$ be a pseudo-Anosov map, where $\Sigma = \Sigma_{g,s}$ with $s \geq 1$. Then the pseudo-Anosov maps $f_{\mathrm{even}} : \Sigma_{\mathrm{e}} \to \Sigma_{\mathrm{e}}$ and $f^\circ : \Sigma^\circ \to \Sigma^\circ$ have the same BBK polynomials. More precisely, the following properties hold.*

1. *The symplectic polynomials of $[f_{\mathrm{even}}]$, $[f^\circ]$ and $[f]$ are the same.*
2. *The puncture polynomials of $[f_{\mathrm{even}}]$ and $[f^\circ]$ are both equal to $p(t)u(t)$, where $p(t)$ and $u(t)$ are the puncture polynomial and the unpunctured singularity polynomial of $[f]$, respectively.*

Proof. The claim (1) holds from Propositions 3.5 and 5.9.

We turn to the proof of (2) for $[f_{\mathrm{even}}]$. The proof of (2) for $[f^\circ]$ is similar. By Proposition 5.8, the radical polynomial $\chi(\mathfrak{t}_*|_{Z(\tau)})$ of $[f]$ is given by $p(t)u(t)(= \chi(\mathfrak{g}_*|_{Z(G)})\chi(\mathfrak{t}_*|_{\ker \pi_*}))$. Let $p'(t)$ be the puncture polynomial of $[f_{\mathrm{even}}]$. By Corollary 5.5, the unpunctured singularity polynomial of $[f_{\mathrm{even}}]$ is 1. This together with Proposition 5.8 implies that the radical polynomial of $[f_{\mathrm{even}}]$ is equal to the puncture polynomial $p'(t)$ of $[f_{\mathrm{even}}]$. Since the radical polynomials of $[f_{\mathrm{even}}]$ and $[f]$ are the same (Proposition 3.5), we obtain $p'(t) = p(t)u(t)$. This completes the proof. □

Example 6.2. Let $f : D_5 \to D_5$ be a pseudo-Anosov map that represents $\Gamma(\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}) \in \mathrm{MCG}(D_5)$. See Example 5.14. Consider the fully punctured pseudo-Anosov map $f^\circ : D_6 \to D_6$. Let $p^\circ(t)$ and $s^\circ(t)$ be the puncture polynomial and the symplectic polynomial of $[f^\circ]$, respectively. By

Theorem 6.1 together with Example 5.14, we have $p^\circ(t) = p(t)u(t) = t + 1$ and $s^\circ(t) = t^4 - 3t^3 + 3t^2 - 3t + 1$.

The next example tells us that puncture polynomials are useful, since one can detect that mapping classes are not conjugate.

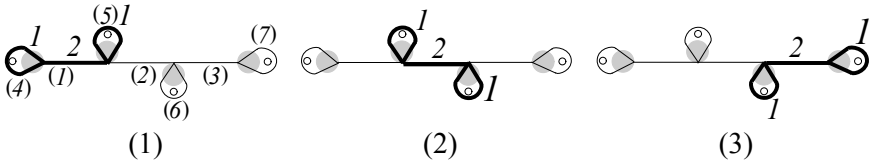


FIGURE 16. Vectors (1) $\eta_{(1)}$, (2) $\eta_{(2)}$ and (3) $\eta_{(3)}$ in $W(G)$.

Example 6.3. Let $\Delta_4 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 \in B_4$ be the positive half twist. Consider the two pseudo-Anosov mapping classes $\phi := \Gamma(\sigma_1\sigma_3\sigma_2^{-1})$ and $\phi' = \Gamma(\sigma_1\sigma_3\sigma_2^{-1}\Delta_4)$ in $\mathrm{MCG}(D_4)$. The former mapping class ϕ is the one in Example 4.1. We claim that the puncture polynomials can detect ϕ and ϕ' are not conjugate, although their symplectic polynomials are the same: The graph G in Figure 9(1) is a Bestvina-Handel graph of ϕ, and it is also a Bestvina-Handel graph of ϕ'. Note that G is preserved by the π rotation $r : G \to G$ about the middle point on the edge labeled (2).

Recall the train track map $\mathfrak{g} : G \to G$ of $[\phi]$ as in Figure 9. One can check that the graph map $\mathfrak{g}' := r \circ \mathfrak{g} : G \to G$ is a train track map of $[\phi']$. Let $\mathfrak{t} : \tau \to \tau$ and $\mathfrak{t}' : \tau \to \tau$ be the train track representatives of ϕ and ϕ', respectively. All the boundary components of the common train track $\tau = \tau(G)$ are 1-pronged, except for the 2-boundary component c (Figure 6(2)). Consider the radical element r_c (Figure 6(3)). By computations, one sees that $\mathfrak{t}_*(r_c) = r_c$ and $\mathfrak{t}'_*(r_c) = -r_c$. This tells us that the radical polynomials of ϕ and ϕ' are $t - 1$ and $t + 1$, respectively. Since ϕ (resp. ϕ') is a fully punctured pseudo-Anosov mapping class, by Theorem 5.11 the radical polynomial is equal to the puncture polynomial of ϕ (resp. ϕ'). Hence, the puncture polynomials of ϕ and ϕ' differ, and this means that $\phi, \phi \in \mathrm{MCG}(D_4)$ are not conjugate.

Let us compute the symplectic polynomials. Since τ is non-orientable, by Lemma 5.13 we have $\dim W(G) = 7 - 4 = 3$. The vectors $\eta_{(i)}$ $(i = 1, 2, 3)$ as shown in Figure 16 is a basis of $W(G)$. If M and M' are the representation matrices of $\mathfrak{g}_* : W(G) \to W(G)$ and $\mathfrak{g}'_* : W(G) \to W(G)$, respectively, then

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } M' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

where M is the block matrix of the transition matrix T of $\mathfrak{g} : G \to G$ as in Example 4.1. Hence, the homology polynomials ϕ and ϕ' are $\chi(M) = (t - 1)(t^2 - 4t + 1)$ and $\chi(M') = (t + 1)(t^2 - 4t + 1)$, respectively. By the factorization of the homology polynomial (Proposition 4.13), ϕ and ϕ' have the same symplectic polynomial $t^2 - 4t + 1$.

Example 6.4. On the twice-punctured torus $\Sigma_{1,2}$, let δ_i be the right-handed Dehn twist about the simple closed curve c_i $(i = 1, 2, 3)$ as in Figure 17(1).

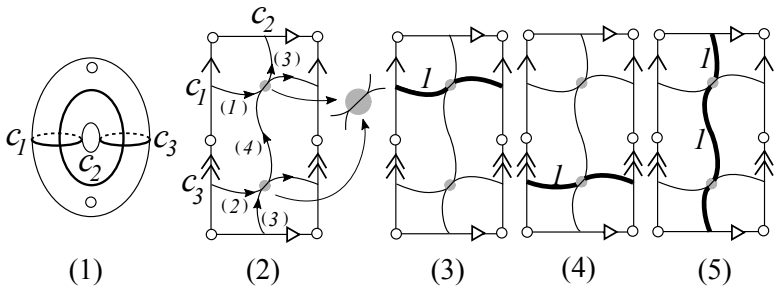


FIGURE 17. (1) Curves c_i. (2) Orientable train track $\tau = \tau(G)$. Vectors (3) $\eta_{(1)}$, (4) $\eta_{(2)}$ and (5) $\eta_{(3)}$ in $W(\tau)$. An edge without number means its weight is 0.

Consider a rectangle with its four corners and the two middle points of its sides removed, and identify the sides of the rectangle according to the arrows of the same kinds as shown in Figure 17(2). We then obtain a twice-punctured torus $\Sigma_{1,2}$. Consider $[f] = \delta_2^{-1} \circ \delta_3 \circ \delta_1 \in \mathrm{MCG}(\Sigma_{1,2})$. This is a pseudo-Anosov mapping class obtained from Penner's construction [Pen88]. Figure 17(2) illustrates an invariant train track τ which is orientable. Collapsing the two gray disks into points, we obtain a Bestvina-Handel graph G of $[f]$. By Lemma 5.13, $\dim W(G) = 4 - 2 + 1 = 3$. Since the pseudo-Anosov map $f : \Sigma_{1,2} \to \Sigma_{1,2}$ is fully punctured, by Corollary 5.3 we have $\dim W(\tau) = \dim W(G) = 3$. Take vectors $\eta_{(i)}$ $(i = 1, 2, 3)$ as shown in Figure 17. The set of vectors $\{\eta_{(i)}\}$ is a basis of $W(\tau)$. Let $\mathfrak{t} : \tau \to \tau$ be the train track representative of $[f]$. The representation matrix of $\mathfrak{t}_* : W(\tau) \to W(\tau)$ is $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence, by Theorem 5.11, $\chi(\mathfrak{t}_*) = \chi(\mathfrak{g}_*)(= h(t)) = (t-1)(t^2 - 4t + 1)$. By Proposition 2.18(3), $\dim Z(\tau) = 2 - 1 = 1$, and by the factorization of the homology polynomial (Proposition 4.13), we have $\chi(\mathfrak{t}_*|_{Z(\tau)}) = \chi(\mathfrak{g}_*|_{Z(G)})(= p(t)) = t - 1$ and $s(t) = t^2 - 4t + 1$.

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