A STUDY OF BRAIDS ARISING FROM SIMPLE CHOREOGRAPHIES OF THE PLANAR NEWTONIAN N-BODY PROBLEM

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Dedicated to the memory of Prof. Masaya Yamaguti on the 100th anniversary of his birth

Abstract. We study periodic solutions of the planar Newtonian N-body problem with equal masses. Each periodic solution traces out a braid with N strands in 3-dimensional space. When the braid is of pseudo-Anosov type, it has an associated stretch factor greater than 1, which reflects the complexity of the corresponding periodic solution. For each $N \geq 3$, Guowei Yu established the existence of a family of simple choreographies to the planar Newtonian N-body problem. We prove that braids arising from Yu's periodic solutions are of pseudo-Anosov types, except in the special case where all particles move along a circle. We also identify the simple choreographies whose braid types have the largest and smallest stretch factors, respectively.

1. Introduction

We consider a periodic solution

$$\boldsymbol{z}(t) = (z_0(t), \dots, z_{N-1}(t)),\ z_i(t) \in \mathbb{R}^2\ (i = 0, \dots, N-1)$$

of the planar Newtonian N-body problem with equal masses. Let $T > 0$ be the period of the solution $\boldsymbol{z}(t)$. We take time to be a third axis orthogonal to the plane. For each fixed $t_0 \in \mathbb{R}$, the trajectory of $\boldsymbol{z}(t)$ from t_0 to $t_0 + T$ traces a pure braid $b(\boldsymbol{z}([t_0, t_0+T]))$ (see Section 2.3). The following question is a starting point for our study.

Question 1.1 (Montgomery [Mon25] (cf. Moore [Moo93]))**.** *Is every pure braid type with N strands realized by a periodic solution of the planar Newtonian N-body problem?*

See Definition 2.1 for the notion of braid types. Question 1.1 has been resolved for $N = 3$, as shown by Moeckel-Montgomery [MM15], yet it remains wide open for $N \geq 4$.

A *simple choreography* of the planar N-body problem is a periodic solution in which all N particles chase each other along a single closed curve.

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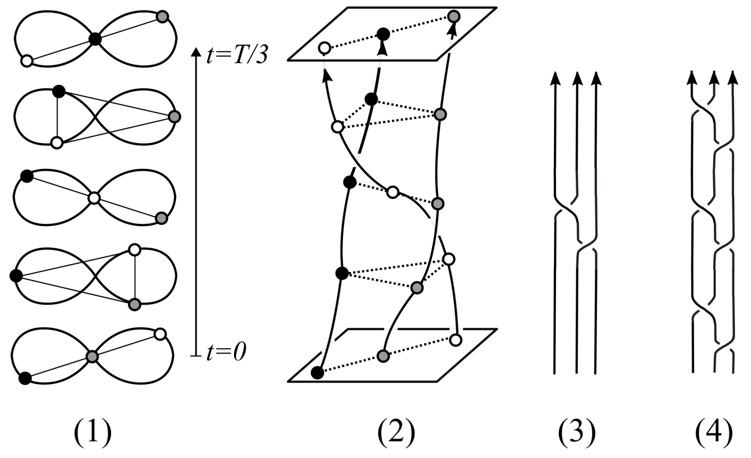


FIGURE 1. (1) The figure-eight solution $\boldsymbol{z}(t)$ with period T. (2)(3) The primitive braid $b := b(\boldsymbol{z}([0, \frac{T}{3}])) = \sigma_1^{-1}\sigma_2$. (4) The braid $b^3 = (\sigma_1^{-1}\sigma_2)^3$ represents the braid type of the figure-eight.

We require that the phase shift between consecutive particles be constant. If $\boldsymbol{z}(t)$ is a simple choreography with period T, then there exists a cyclic permutation σ of N elements $\{0, \dots, N-1\}$ such that

$$z_i(t + \tfrac{T}{N}) = z_{\sigma(i)}(t) \text{ for } i \in \{0, \dots, N-1\} \text{ and } t \in \mathbb{R}.$$

We call $\frac{T}{N}$ the *primitive period* of the simple choreography $\boldsymbol{z}(t)$. For any fixed $t_0 \in \mathbb{R}$, the trajectory of $\boldsymbol{z}(t)$ from t_0 to $t_0 + \frac{T}{N}$ determines a braid $b := b(\boldsymbol{z}([t_0, t_0 + \frac{T}{N}]))$ that is called a *primitive braid* of $\boldsymbol{z}(t)$. We call its braid type the *primitive braid type* of $\boldsymbol{z}(t)$ (see Figure 1).

Applying the Nielsen-Thurston classification of surface automorphisms [Thu88], we classify braids into three types: periodic, reducible, and pseudo-Anosov. For a braid of pseudo-Anosov type, there is an associated stretch factor greater than 1, which is a conjugacy invariant of the braid (see Section 2.2). We use stretch factors as a measure of the complexity of periodic solutions to the planar N-body problem.

For each integer $N \geq 3$, let us set

$$\Omega_N = \{\boldsymbol{\omega} = (\omega_1, \dots, \omega_{N-1}) \mid \omega_i \in \{1, -1\} \text{ for } i \in \{1, \dots, N-1\}\}.$$

The main theorem of this paper is as follows.

Theorem 1.2. *For each $N \geq 3$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{N-1}) \in \Omega_N$, there exists a simple choreography of the planar Newtonian N-body problem whose primitive braid type is given by the braid $\sigma_1^{\omega_1}\sigma_2^{\omega_2}\cdots\sigma_{N-1}^{\omega_{N-1}}$. In particular, the braid type of the simple choreography is given by $(\sigma_1^{\omega_1}\sigma_2^{\omega_2}\cdots\sigma_{N-1}^{\omega_{N-1}})^N$, and it is a pseudo-Anosov type if $\boldsymbol{\omega}$ is neither $(1, 1, \dots, 1)$ nor $(-1, -1, \dots, -1)$. Otherwise, the braid type of the simple choreography is periodic.*

See Figure 5(1) in Section 2.1 for the generator σ_i of the braid group. In 2017, Guowei Yu established the existence of a simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t) = (z_i(t))_{i=0}^{N-1}$ for each $\boldsymbol{\omega} \in \Omega_N$ to the planar Newtonian N-body problem with

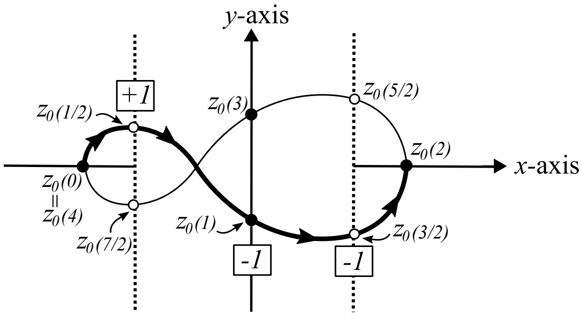


FIGURE 2. The closed curve obtained from the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ in the case $N = 4$ and $\boldsymbol{\omega} = (1, -1, -1)$. Thick arrows illustrate the trajectory of the 0th particle $z_0(t)$ from $t = 0$ to $\frac{N}{2} = 2$.

equal masses [Yu17]. The period of the periodic solution $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ is N, and $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ fulfills

$$z_i(t) = z_0(t + i) \quad \text{for } t \in \mathbb{R} \text{ and } i \in \{0, 1, \dots, N - 1\}.$$

Moreover, the closed curve on which N particles travel is symmetric with respect to the x-axis. The element $\boldsymbol{\omega} = (\omega_i)_{i=1}^{N-1} \in \Omega_N$ determines the shape of the closed curve (Figure 2). See Section 3 for more details. We will prove that the periodic solution $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ by Yu satisfies the statement of Theorem 1.2.

The figure-eight solution of the planar 3-body problem [CM00] has the same braid type as the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ in the case $\boldsymbol{\omega} = (1, -1)$ (Example 4.2). The super-eight solution of the planar 4-body problem [KZ03, Shi14] and the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ for $\boldsymbol{\omega} = (1, -1, 1)$ have the same braid type (Corollary 4.3). See Figure 3.

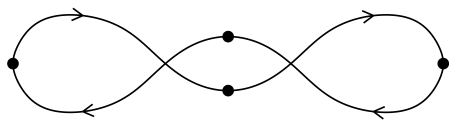


FIGURE 3. The super-eight of the planar 4-body problem.

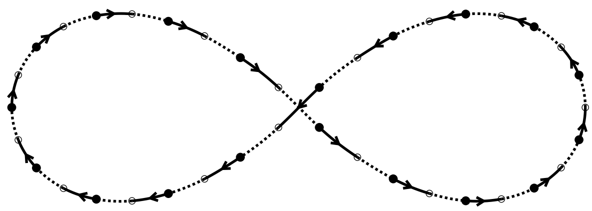
To state the next result, we define $\boldsymbol{\omega}_{\max}, \boldsymbol{\omega}_{\min} \in \Omega_N$ as follows.

$$\begin{aligned}
\boldsymbol{\omega}_{\max} &:= ((-1)^{i-1})_{i=1}^{N-1} = (1, -1, \dots, (-1)^{N-2}). \\
\boldsymbol{\omega}_{\min} &:= (\omega_i)_{i=1}^{N-1}, \\
\text{where} \quad \omega_i &= \begin{cases} 1 & \text{for } i = 1, \dots, \lfloor \frac{N}{2} \rfloor \\ -1 & \text{for } i = \lfloor \frac{N}{2} \rfloor + 1, \dots, N - 1. \end{cases}
\end{aligned}$$

The simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}_{\max}}(t)$ travels a chain made of $N - 1$ loops. On the other hand, $\boldsymbol{z}_{\boldsymbol{\omega}_{\min}}(t)$ moves on a figure-eight curve and approximately half of the N particles stay on each loop at every time. See Figure 4.

Theorem 1.3. *Among all $\boldsymbol{\omega} \in \Omega_N$ except for the two elements $(1, 1, \dots, 1)$ and $(-1, -1, \dots, -1)$, the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ whose braid type having*

(1) $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ for $\boldsymbol{\omega} = \boldsymbol{\omega}_{\max}$.



(2) $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ for $\boldsymbol{\omega} = \boldsymbol{\omega}_{\min}$.

FIGURE 4. Simple choreographies $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ in the case $N = 19$. The dots denote the initial condition. The arrows indicate the trajectory of $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ from $t = 0$ to $\frac{1}{2}$.

the largest stretch factor is realized by $\boldsymbol{\omega}_{\max}$*, while the one whose braid type having the smallest stretch factor is realized by* $\boldsymbol{\omega}_{\min}$.

A *multiple choreography* of the planar N-body problem is a periodic solution such that particles travel on k different closed curves for some $k > 1$. In [Shi06], the third author established the existence of a family of multiple choreographies to the planar Newtonian $2N$-body problem. These periodic solutions have pseudo-Anosov braid types whose stretch factors are quadratic irrational [KKS23]. In this sense, the multiple choreographies given in [Shi06] have an algebraic restriction from view points of pseudo-Anosov stretch factors. On the other hand, simple choreographies in Theorem 1.2 do not have such a restriction. In particular, the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ in the case $\boldsymbol{\omega} = (-1, 1, 1)$ gives us the following result.

Corollary 1.4. *There exists a simple choreography of the planar Newtonian* 4*-body problem whose primitive braid type is given by the pseudo-Anosov braid* $\sigma_1\sigma_2^{-1}\sigma_3^{-1}$*. The primitive braid type of the simple choreography has the stretch factor* ≈ 2.2966 *which is the largest real root of the polynomial* $t^4 - 2t^3 - 2t + 1$ *with degree* 4.

For other periodic solutions that have been proven to exist, see, for example, [Che03a, Che03b, Che08, FT04]. See also [Sim01, DPK$^+$03, ŠD13] for periodic solutions that have been obtained numerically.

The organization of the paper is as follows. In Section 2, we recall basic results on the braid groups and mapping class groups. In Section 3, we review the simple choreographies by Yu. Section 4 and Section 5 contain the proofs of results and the conclusion in this paper.

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2. Preliminaries

2.1. **Braid groups and mapping class groups.** We review the basics of the braid groups. See also [Bir74, Chapters 1, 4]. Let B_n be the braid group of n strands generated by $\sigma_1, \ldots, \sigma_{n-1}$. The group B_n has the following presentation.

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1 \end{array} \right\rangle$$

The generator σ_i corresponds to a geometric braid as in Figure 5(1). There is a surjective homomorphism

$$\hat{s} : B_n \to S_n \tag{2.1}$$

from B_n to the symmetry group S_n of n elements sending each σ_j to the transposition $(j, j+1)$. The kernel of $\hat{s}$ is called the *pure braid group* $P_n < B_n$. An element of P_n is called a *pure braid*.

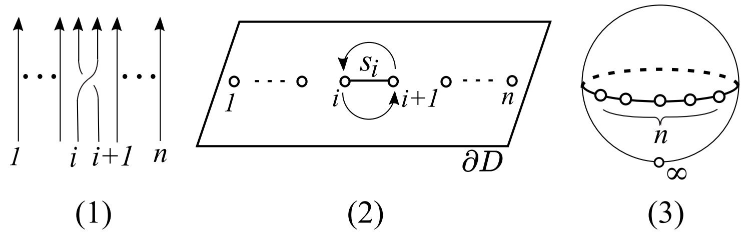


Figure 5. (1) $\sigma_i \in B_n$. (2) $h_i = \Gamma(\sigma_i) \in \mathrm{MCG}(D_n)$. (3) $\Sigma_{0,n+1}$.

Let $Z(B_n)$ be the center of the n-braid group B_n. The subgroup $Z(B_n)$ is an infinite cyclic group generated by the full twist $\Delta^2 \in B_n$ ([Bir74, Corollary 1.8.4]), where $\Delta \in B_n$ is the half twist that is defined by

$$\Delta = (\sigma_1\sigma_2 \ldots \sigma_{n-1})(\sigma_1\sigma_2 \ldots \sigma_{n-2}) \ldots (\sigma_1\sigma_2)\sigma_1.$$

Note that Δ^2 is obtained by rotating the set of n points one full revolution.

Let Σ be an orientable, connected surface, possibly with punctures and boundary. The *mapping class group* $\mathrm{MCG}(\Sigma)$ of Σ is the group of isotopy classes of orientation preserving homeomorphisms of Σ which preserve the punctures and boundary setwise. Let $\Sigma_{g,n}$ be an orientable, connected surface of genus g and n punctures. In this paper, we consider the mapping class groups of an n-punctured disk D_n and an n-punctured sphere $\Sigma_{0,n}$. The group $\mathrm{MCG}(D_n)$ is generated by $h_1, \ldots, h_{n-1}$, where h_i is the right-handed half twist about a segment s_i connecting the ith puncture and $i+1$th puncture, i.e., h_i interchanges the ith puncture and $i+1$th puncture as in Figure 5(2). A relation between B_n and $\mathrm{MCG}(D_n)$ is given by the following surjective homomorphism

$$\Gamma : B_n \to \mathrm{MCG}(D_n) \tag{2.2}$$

which sends σ_i to h_i for each $i \in \{1, \ldots, n-1\}$. The kernel of Γ is the center $Z(B_n)$ of B_n.

Definition 2.1. The *braid type* $\langle b \rangle$ of a braid $b \in B_n$ is the conjugacy class of $\Gamma(b)$ in $\mathrm{MCG}(D_n)$. Since $\mathrm{MCG}(D_n)$ is isomorphic to $B_n/Z(B_n)$, the braid type $\langle b \rangle$ can be identified with a conjugacy class in $B_n/Z(B_n)$. We may call the braid type of a pure braid the *pure braid type*.

2.2. **Nielsen-Thurston classification.** We assume that $3g-3+n \geq 1$. According to the Nielsen-Thurston classification of surface automorphisms [Thu88], elements of $\mathrm{MCG}(\Sigma_{g,n})$ are classified into three types: periodic, reducible and pseudo-Anosov as we recall now. A mapping class $\phi \in \mathrm{MCG}(\Sigma_{g,n})$ is *periodic* if ϕ is of finite order. A simple closed curve C in $\Sigma_{g,n}$ is *essential* if it is not homotopic to a point (possibly a point corresponding to a puncture). A mapping class $\phi \in \mathrm{MCG}(\Sigma_{g,n})$ is *reducible* if there is a collection of mutually disjoint and non-homotopic essential simple closed curves $C_1, \dots, C_j$ in $\Sigma_{g,n}$ (possibly $j=1$) such that $C_1 \cup \cdots \cup C_j$ is preserved by ϕ. Notice that there is a mapping class that is periodic and reducible. A mapping class $\phi \in \mathrm{MCG}(\Sigma_{g,n})$ is *pseudo-Anosov* if ϕ is neither periodic nor reducible. The Nielsen-Thurston type is a conjugacy invariant, i.e., two mapping classes are conjugate to each other in $\mathrm{MCG}(\Sigma_{g,n})$, then their Nielsen-Thurston types are the same.

We review properties of pseudo-Anosov mapping classes. For more details, see [FLP79, FM12, Boy94]. A homeomorphism $\Phi : \Sigma_{g,n} \to \Sigma_{g,n}$ is a *pseudo-Anosov map* if there exist a constant $\lambda = \lambda(\Phi) > 1$ and a pair of transverse measured foliations $(\mathcal{F}^+, \mu^+)$ and $(\mathcal{F}^-, \mu^-)$ so that Φ preserves both foliations $\mathcal{F}^+$ and $\mathcal{F}^-$, and it contracts the leaves of $\mathcal{F}^-$ by $\frac{1}{\lambda}$ and it expands the leaves of $\mathcal{F}^+$ by λ. More precisely, Φ fulfills

$$\Phi((\mathcal{F}^+, \mu^+)) = (\mathcal{F}^+, \lambda\mu^+) \quad \text{and} \quad \Phi((\mathcal{F}^-, \mu^-)) = (\mathcal{F}^-, \tfrac{1}{\lambda}\mu^-).$$

The constant $\lambda > 1$ is called the *stretch factor* of Φ. For each pseudo-Anosov mapping class $\phi \in \mathrm{MCG}(\Sigma_{g,n})$, there exists a pseudo-Anosov homeomorphism $\Phi : \Sigma_{g,n} \to \Sigma_{g,n}$ that is a representative of ϕ. The *stretch factor* $\lambda(\phi)$ of ϕ is defined by $\lambda(\phi) = \lambda(\Phi)$, and it is a conjugacy invariant of pseudo-Anosov mapping classes. We understand that stretch factors measure the complexity of pseudo-Anosov mapping classes.

A square matrix M with nonnegative integer entries is *Perron-Frobenius* if some power of M is a positive matrix. In this case, the Perron-Frobenius theorem [Sen06, Theorem 1.1] tells us that M has a real eigenvalue $\lambda(M) > 1$ which exceeds the moduli of all other eigenvalues. We call $\lambda(M)$ the *Perron-Frobenius eigenvalue*. It is known that the stretch factor $\lambda(\phi)$ of a pseudo-Anosov mapping class ϕ is the largest eigenvalue of a Perron-Frobenius matrix, that is $\lambda(\phi)$ is a Perron number. Observe that if ϕ is a pseudo-Anosov mapping class, then ϕ^k is pseudo-Anosov for all $k \geq 1$, and it holds

$$\lambda(\phi^k) = (\lambda(\phi))^k. \tag{2.3}$$

We recall the homomorphism $\Gamma : B_n \to \mathrm{MCG}(D_n)$ as in (2.2). Collapsing the boundary of the disk to a point ∞ in the sphere, we obtain the $n+1$-punctured sphere $\Sigma_{0,n+1}$ (see Figure 5(3)) and a homomorphism

$$\mathfrak{c} : \mathrm{MCG}(D_n) \to \mathrm{MCG}(\Sigma_{0,n+1}).$$

We may identify a mapping class $\Gamma(b) \in \mathrm{MCD}(D_n)$ for $b \in B_n$ with $\mathfrak{c}(\Gamma(b)) \in \mathrm{MCG}(\Sigma_{0,n+1})$. We say that a braid $b \in B_n$ is *periodic* (resp. *reducible*, *pseudo-Anosov*) if the mapping class $\mathfrak{c}(\Gamma(b))$ is of the corresponding type. When b is pseudo-Anosov, its *stretch factor* $\lambda(b)$ is defined by the stretch factor of the pseudo-Anosov mapping class $\mathfrak{c}(\Gamma(b))$. In this case, it makes sense to say that the braid type $\langle b \rangle$ is pseudo-Anosov, since the Nielsen-Thurston type is a conjugacy invariant. The *stretch factor* $\lambda(\langle b \rangle)$ of the braid type $\langle b \rangle$ can be defined by $\lambda(\langle b \rangle) = \lambda(b)$.

2.3. **Braids as particle dances.** We consider the motion of N points in the plane $\mathbb{R}^2$

$$\boldsymbol{z}(t) = (z_0(t), \dots, z_{N-1}(t)),\ z_i(t) \in \mathbb{R}^2\ (i = 0, \dots, N-1)$$

where $z_i(t) \in \mathbb{R}^2$ is the position of the ith point at $t \in \mathbb{R}$. We assume the following conditions.

- (collision-free) $z_i(t) \neq z_j(t)$ for $i \neq j$ and $t \in \mathbb{R}$.
- (periodicity) There exists $T > 0$ such that

$$\{z_0(t), \dots, z_{N-1}(t)\} = \{z_0(t+T), \dots, z_{N-1}(t+T)\} \quad \text{for } t \in \mathbb{R}.$$

We take time to be a third axis orthogonal to the plane. Fixing $t_0 \in \mathbb{R}$, we have mutually disjoint N curves

$$\begin{array}{rcl} [t_0, t_0+T] & \to & \mathbb{R}^2 \times \mathbb{R} \\ t & \mapsto & (z_i(t), t). \end{array}$$

The union of the curves forms a braid, denoted by $b(\boldsymbol{z}([t_0, t_0+T]))$. Such a braid is sometimes referred to as a *particle dance*.

We turn to a periodic solution $\boldsymbol{z}(t) = (z_0(t), \dots, z_{N-1}(t))$ of the planar Newtonian N-body problem. Suppose that the periodic solution $\boldsymbol{z}(t)$ has a period T, that is T is the smallest positive number such that $z_i(t) = z_i(t+T)$ for all $i = 0, \dots, N-1$ and $t \in \mathbb{R}$. Choosing any $t_0 \in \mathbb{R}$, we obtain a pure braid $b(\boldsymbol{z}([t_0, t_0+T]))$ with N strands that obviously depends on the choice of t_0. Although the set of base points $\{z_0(t_0), \dots, z_{N-1}(t_0)\}$ of $b(\boldsymbol{z}([t_0, t_0+T]))$ does not necessarily lie along a straight line in the plane, it still makes sense to consider its braid type $\big\langle b(\boldsymbol{z}([t_0, t_0+T]))\big\rangle$. See [KKS23, Section 3.1]. Such a braid type $\big\langle b(\boldsymbol{z}([t_0, t_0+T]))\big\rangle$ does not depend on the choice of t_0 and we call it the *braid type of the periodic solution* $\boldsymbol{z}(t)$.

Suppose that $\boldsymbol{z}(t)$ is a simple choreography with period T of the planar Newtonian N-body problem. Choosing any $t_0 \in \mathbb{R}$, we have

$$\{z_0(t_0), \dots, z_{N-1}(t_0)\} = \{z_0(t_0 + \tfrac{T}{N}), \dots, z_{N-1}(t_0 + \tfrac{T}{N})\},$$

and we obtain a braid $b := b(\boldsymbol{z}\big([t_0, t_0 + \frac{T}{N}]\big))$ as a particle dance. We call $\frac{T}{N}$ the *primitive period* of the simple choreography $\boldsymbol{z}(t)$ and call the braid $b(\boldsymbol{z}\big([t_0, t_0+\frac{T}{N}]\big))$ a *primitive braid* of $\boldsymbol{z}(t)$. Its braid type $\big\langle b(\boldsymbol{z}\big([t_0, t_0+\frac{T}{N}]\big))\big\rangle$ is called the *primitive braid type* of the simple choreography $\boldsymbol{z}(t)$. Clearly, the Nth powers b^N of the primitive braid b equals $b(\boldsymbol{z}([t_0, t_0+T]))$, and b^N represents the braid type $\big\langle b(\boldsymbol{z}([t_0, t_0+T]))\big\rangle$ of the periodic solution $\boldsymbol{z}(t)$.

2.4. **Compositions of integers.** A *composition* of a positive integer n is a representation of n as a sum of positive integers. For example, there are four compositions of 3:

$$3 = 3,\ 3 = 1 + 2,\ 3 = 2 + 1, 3 = 1 + 1 + 1.$$

(There is another notion 'partition' of a positive integer. A *partition* of n is a representation of n as a sum of positive integers, where the order of the summands is *not* taken into account.)

A composition of n is written by a $(k+1)$-tuple $\boldsymbol{m} = (m_1, \cdots, m_{k+1})$ of positive integers m_i with $n = \sum_{i=1}^{k+1} m_i$ and $k \geq 0$. Let Ψ_n denote the set of all compositions of n. For example, Ψ_3 consists of the four compositions

$$(3),\ (1,2),\ (2,1),\ (1,1,1).$$

The cardinality of the set Ψ_n is 2^{n-1}. To see this, consider n circles.

$$\circ_1 \quad \circ_2 \quad \circ_3 \quad \cdots \quad \circ_n$$

There are $n-1$ spaces between them. In each space, one can choose to place a bar or leave it empty. There exist 2^{n-1} possible ways to place the bars, and each configuration corresponds to a composition of n. This means that the number of compositions of n is 2^{n-1}. For example, the four compositions of 3 are written by

$$\circ\ \circ\ \circ \leftrightarrow (3),\ \ \circ|\circ\ \circ \leftrightarrow (1,2),\ \ \circ\ \circ|\circ \leftrightarrow (2,1),\ \ \circ|\circ|\circ \leftrightarrow (1,1,1) \tag{2.4}$$

We turn to another set that is related to compositions of integers. Recall the set Ω_N as in Section 1. Given an integer $N \geq 3$, let Ω_N^+ denote the set of all elements $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{N-1}) \in \Omega_N$ whose first component ω_1 is 1:

$$\Omega_N^+ := \{\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{N-1}) \in \Omega_N \mid \omega_1 = 1\}.$$

Lemma 2.2. *There is a bijection $\Theta : \Psi_{N-1} \to \Omega_N^+$.*

Proof. We represent a composition $\boldsymbol{m} \in \Psi_{N-1}$ by using $N-1$ circles together with bars. We now define $\Theta(\boldsymbol{m}) = (\omega_1 = 1, \omega_2 \ldots, \omega_{N-1})$. For each $i \in \{1, \ldots, N-1\}$, we replace the ith circle with 1 or -1 which indicates $\omega_i = 1$ or $\omega_i = -1$ as we explain now. We set $\omega_1 = 1$. Suppose that we have replaced the ith circle with 1 or -1. (Then ω_i is determined.) If there is a bar between ith circle and $(i+1)$th circle, then we set $\omega_{i+1} = -\omega_i$. Otherwise, we set $\omega_{i+1} = \omega_i$. In other words, if $\boldsymbol{m} = (m_1, \ldots, m_{k+1})$ is an element of Ψ_{N-1}, then

$$\Theta(\boldsymbol{m}) = (\underbrace{1, \ldots, 1}_{m_1}, \underbrace{-1, \ldots, -1}_{m_2}, \ldots, \underbrace{(-1)^k, \ldots, (-1)^k}_{m_{k+1}}) \in \Omega_N^+.$$

Clearly, $\Theta : \Psi_{N-1} \to \Omega_N^+$ is injective. To see $\Theta : \Psi_{N-1} \to \Omega_N^+$ is surjective, we take any $\boldsymbol{\omega} = (\omega_1 = 1, \omega_2, \ldots, \omega_{N-1}) \in \Omega_N^+$. By applying the reverse operation, one can determine a composition $\boldsymbol{m}_{\boldsymbol{\omega}} \in \Psi_{N-1}$ so that $\Theta(\boldsymbol{m}_{\boldsymbol{\omega}}) = \boldsymbol{\omega}$. This completes the proof. $\square$

For example, the bijection $\Theta : \Psi_3 \to \Omega_4^+$ is described as follows.

$$\begin{array}{cccc} \Theta: & \Psi_3 & \to & \Omega_4^+ \\ & \circ\ \circ\ \circ & \mapsto & (1,1,1), \\ & \circ|\circ\ \circ & \mapsto & (1,-1,-1), \\ & \circ\ \circ|\circ & \mapsto & (1,1,-1), \\ & \circ|\circ|\circ & \mapsto & (1,-1,1). \end{array}$$

We write the n-tuple $(1,1,\cdots,1) \in \Psi_n$ correspoinding to the composition $n = 1+1+\cdots+1$ as $\mathbf{1}_n$.

Example 2.3. Consider the bijection $\Theta : \Psi_{N-1} \to \Omega_N^+$.

(1) The image of $\mathbf{1}_{N-1}$ under Θ is

$$\Theta(\mathbf{1}_{N-1}) = ((-1)^{i-1})_{i=1}^{N-1} = (1,-1,\dots,(-1)^{N-2}).$$

(2) The image of the composition $(N-1)$ under Θ is $(1,1,\dots,1)$.

2.5. **Braids associated with compositions of integers.** We introduce braids $\alpha_{\boldsymbol{\omega}}$, $e_{\boldsymbol{\omega}}$, $o_{\boldsymbol{\omega}}$ for each $\boldsymbol{\omega} \in \Omega_N$ and $\beta_{\boldsymbol{m}}$ for each composition $\boldsymbol{m} \in \Psi_{N-1}$. We first define an N-braid $\alpha_{\boldsymbol{\omega}}$ for $\boldsymbol{\omega} = (\omega_1,\dots,\omega_{N-1})$ as follows.

$$\alpha_{\boldsymbol{\omega}} := \sigma_1^{\omega_1}\sigma_2^{\omega_2}\cdots\sigma_{N-1}^{\omega_{N-1}}. \tag{2.5}$$

Notice that the Nth power $\alpha_{\boldsymbol{\omega}}^N$ of $\alpha_{\boldsymbol{\omega}}$ is a pure braid. We next define N-braids $e_{\boldsymbol{\omega}}$ and $o_{\boldsymbol{\omega}}$ as follows:

$$e_{\boldsymbol{\omega}} := \prod_{\substack{i\in\{1,\dots,N-1\}\\ i \text{ even}}} \sigma_i^{\omega_i}, \qquad o_{\boldsymbol{\omega}} := \prod_{\substack{i\in\{1,\dots,N-1\}\\ i \text{ odd}}} \sigma_i^{\omega_i}. \tag{2.6}$$

Example 2.4. For each $\boldsymbol{\omega} \in \Omega_7$, we demonstrate that $e_{\boldsymbol{\omega}}o_{\boldsymbol{\omega}}$ is conjugate to $\alpha_{\boldsymbol{\omega}}$ in B_7. Recall that the braid group has a relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$. We set $b_0 := e_{\boldsymbol{\omega}}^{-1}(e_{\boldsymbol{\omega}}o_{\boldsymbol{\omega}})e_{\boldsymbol{\omega}} = o_{\boldsymbol{\omega}}e_{\boldsymbol{\omega}}$. Then we have

$$b_0 = o_{\boldsymbol{\omega}} \cdot e_{\boldsymbol{\omega}} = \sigma_5^{\omega_5}\sigma_3^{\omega_3}\sigma_1^{\omega_1} \cdot \sigma_6^{\omega_6}\sigma_4^{\omega_4}\sigma_2^{\omega_2} = (\sigma_5^{\omega_5}\sigma_6^{\omega_6})(\sigma_3^{\omega_3}\sigma_4^{\omega_4}\sigma_1^{\omega_1}\sigma_2^{\omega_2}).$$

Set $b_1 := (\sigma_5^{\omega_5}\sigma_6^{\omega_6})^{-1}b_0(\sigma_5^{\omega_5}\sigma_6^{\omega_6})$. We have

$$b_1 = (\sigma_3^{\omega_3}\sigma_4^{\omega_4}\sigma_1^{\omega_1}\sigma_2^{\omega_2})(\sigma_5^{\omega_5}\sigma_6^{\omega_6}) = (\sigma_3^{\omega_3}\sigma_4^{\omega_4}\sigma_5^{\omega_5}\sigma_6^{\omega_6})(\sigma_1^{\omega_1}\sigma_2^{\omega_2}).$$

Set $b_2 := (\sigma_3^{\omega_3}\sigma_4^{\omega_4}\sigma_5^{\omega_5}\sigma_6^{\omega_6})^{-1}b_1(\sigma_3^{\omega_3}\sigma_4^{\omega_4}\sigma_5^{\omega_5}\sigma_6^{\omega_6})$ that is of the form

$$b_2 = \sigma_1^{\omega_1}\sigma_2^{\omega_2}\sigma_3^{\omega_3}\sigma_4^{\omega_4}\sigma_5^{\omega_5}\sigma_6^{\omega_6} = \alpha_{\boldsymbol{\omega}}.$$

Thus, $e_{\boldsymbol{\omega}}o_{\boldsymbol{\omega}}$, b_0, b_1, and $b_2 = \alpha_{\boldsymbol{\omega}}$ are conjugate to each other in B_7.

The following lemma is used in the proof of Theorem 4.1.

Lemma 2.5. *For each $\boldsymbol{\omega} \in \Omega_N$, the braids $e_{\boldsymbol{\omega}}o_{\boldsymbol{\omega}}$ and $\alpha_{\boldsymbol{\omega}}$ are conjugate to each other in the N-braid group B_N. In particular, $(e_{\boldsymbol{\omega}}o_{\boldsymbol{\omega}})^n$ is conjugate to $(\alpha_{\boldsymbol{\omega}})^n$ in B_N for each $n \geq 1$.*

Proof. In the same manner as in Example 2.4, one can show that $e_{\boldsymbol{\omega}}o_{\boldsymbol{\omega}}$ is conjugate to $\alpha_{\boldsymbol{\omega}}$ in B_N. Take an N-braid h so that $e_{\boldsymbol{\omega}}o_{\boldsymbol{\omega}} = h\alpha_{\boldsymbol{\omega}}h^{-1}$. Then we have $(e_{\boldsymbol{\omega}}o_{\boldsymbol{\omega}})^n = (h\alpha_{\boldsymbol{\omega}}h^{-1})^n = h(\alpha_{\boldsymbol{\omega}})^nh^{-1}$. This completes the proof. $\square$

We define a map (anti-homomorphism)

$$\begin{array}{rcl} \mathrm{rev}: B_n & \to & B_n \\ \sigma_{i_1}^{\mu_1}\sigma_{i_2}^{\mu_2}\cdots\sigma_{i_k}^{\mu_k} & \mapsto & \sigma_{i_k}^{\mu_k}\cdots\sigma_{i_2}^{\mu_2}\sigma_{i_1}^{\mu_1}, \quad \mu_j = \pm 1. \end{array}$$

The image of $\alpha_{\boldsymbol{\omega}} = \sigma_1^{\omega_1}\sigma_2^{\omega_2}\cdots\sigma_{N-1}^{\omega_{N-1}}$ under the map rev is given by

$$\mathrm{rev}(\alpha_{\boldsymbol{\omega}}) = \sigma_{N-1}^{\omega_{N-1}}\cdots\sigma_2^{\omega_2}\sigma_1^{\omega_1}.$$

Lemma 2.6. *For each $\boldsymbol{\omega} \in \Omega_N$, $\alpha_{\boldsymbol{\omega}}$ is conjugate to* $\mathrm{rev}(\alpha_{\boldsymbol{\omega}})$ *in B_N.*

Proof. Suppose that $N = 3$. For each $\boldsymbol{\omega} \in \Omega_3$, we have $\alpha_{\boldsymbol{\omega}} = \sigma_1^{\omega_1}\sigma_2^{\omega_2}$ and $\mathrm{rev}(\alpha_{\boldsymbol{\omega}}) = \sigma_2^{\omega_2}\sigma_1^{\omega_1}$. Clearly, the statement follows in this case.

Suppose that $N = 4$. For each $\boldsymbol{\omega} \in \Omega_4$, we have $\alpha_{\boldsymbol{\omega}} = \sigma_1^{\omega_1}\sigma_2^{\omega_2}\sigma_3^{\omega_3}$ and $\mathrm{rev}(\alpha_{\boldsymbol{\omega}}) = \sigma_3^{\omega_3}\sigma_2^{\omega_2}\sigma_1^{\omega_1}$. First, we consider the braid $b_0 := \sigma_3^{-\omega_3}(\mathrm{rev}(\alpha_{\boldsymbol{\omega}}))\sigma_3^{\omega_3}$. Then we have

$$b_0 = \sigma_3^{-\omega_3}(\sigma_3^{\omega_3}\sigma_2^{\omega_2}\sigma_1^{\omega_1})\sigma_3^{\omega_3} = \sigma_2^{\omega_2}\sigma_1^{\omega_1}\sigma_3^{\omega_3} = \sigma_2^{\omega_2}\sigma_3^{\omega_3}\sigma_1^{\omega_1}.$$

Next, we consider the braid $b_1 := (\sigma_2^{\omega_2}\sigma_3^{\omega_3})^{-1}b_0(\sigma_2^{\omega_2}\sigma_3^{\omega_3})$. It is written by $b_1 = \sigma_1^{\omega_1}\sigma_2^{\omega_2}\sigma_3^{\omega_3} = \alpha_{\boldsymbol{\omega}}$. Hence $\mathrm{rev}(\alpha_{\boldsymbol{\omega}})$, b_0 and $\alpha_{\boldsymbol{\omega}}$ are conjugate to each other in B_4.

In the same argument as above, one can verify the statement of general N. We leave the rest of the proof to the reader. □

We turn to the definition of the braid $\beta_{\boldsymbol{m}}$. Let $\boldsymbol{m} = (m_1, \dots, m_{k+1})$ be a $(k+1)$-tuple of positive integers with $k \geq 0$. Let $\beta_{\boldsymbol{m}} = \beta_{(m_1,\dots,m_{k+1})}$ denote the braid with $(1 + \sum_{i=1}^{k+1} m_i)$ strands as in Figure 6. If we set $N := 1 + \sum_{i=1}^{k+1} m_i$, then $\sum_{i=1}^{k+1} m_i = N - 1$ and $\boldsymbol{m}$ is a composition of the integer $N - 1$.

Observe that if $k = 0$ and $\boldsymbol{m} = (N-1) \in \Psi_{N-1}$, then

$$\beta_{\boldsymbol{m}} = \beta_{(N-1)} = \sigma_1\sigma_2\cdots\sigma_{N-1}.$$

Here are some examples written by Artin generators: $\beta_{(3,2)} = \sigma_1\sigma_2\sigma_3\sigma_4^{-1}\sigma_5^{-1} \in B_6$, $\beta_{(1,1,1,1,1)} = \sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_5 \in B_6$. See Figure 6.

Lemma 2.7. *If $\boldsymbol{m} = (N-1) \in \Psi_{N-1}$, then $\beta_{\boldsymbol{m}}$ is a periodic braid.*

Proof. If $\boldsymbol{m} = (N-1)$, then $(\beta_{\boldsymbol{m}})^N = (\sigma_1\sigma_2\cdots\sigma_{N-1})^N$ equals the full twist Δ^2 corresponding to the identity element in the group $B_N/Z(B_N)$. Hence $\beta_{\boldsymbol{m}}$ is a periodic braid. □

2.6. **Stretch factors of pseudo-Anosov braids.** In this section, we study the stretch factor of the braid $\beta_{\boldsymbol{m}}$ when it is of pseudo-Anosov type. To do this, let $f(t)$ be an integral polynomial of degree d. The *reciprocal* of $f(t)$, denoted by $f_*(t)$, is defined by

$$f_*(t) = t^d f(\tfrac{1}{t}). \tag{2.7}$$

The following result gives a recursive formula for the stretch factor of $\beta_{\boldsymbol{m}}$.

Theorem 2.8. *Let $\boldsymbol{m} = (m_1, \dots, m_{k+1})$ be a $(k+1)$-tuple of positive integers. If $k > 0$, then the braid $\beta_{\boldsymbol{m}}$ is pseudo-Anosov. The stretch factor $\lambda_{\boldsymbol{m}} = \lambda_{(m_1,\dots,m_{k+1})}$ of $\beta_{\boldsymbol{m}}$ is the largest real root of the polynomial*

$$F_{\boldsymbol{m}}(t) = F_{(m_1,\dots,m_{k+1})}(t) := t^{m_{k+1}}R_{(m_1,\dots,m_k)}(t) + (-1)^{k+1}R_{(m_1,\dots,m_k)_*}(t),$$

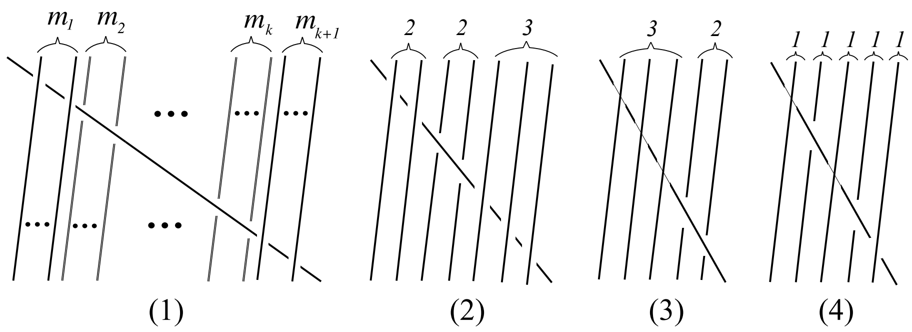


FIGURE 6. (1) $\beta_{\boldsymbol{m}} = \beta_{(m_1,\ldots,m_{k+1})}$. (2) $\beta_{(2,2,3)}$. (3) $\beta_{(3,2)}$. (4) $\beta_{(1,1,1,1,1)}$.

where $R_{(m_1,\ldots,m_i)}(t)$ is defined recursively as follows:

$$\begin{aligned} R_{(m_1)}(t) &:= t^{m_1+1}(t-1) - 2t, \\ R_{(m_1,\ldots,m_i)}(t) &:= t^{m_i}(t-1)R_{(m_1,\ldots,m_{i-1})}(t) + (-1)^i 2t R_{(m_1,\ldots,m_{i-1})_*}(t). \end{aligned}$$

See Proposition 4.1, Theorem 1.2 in [KT08] for the proof of Theorem 2.8. The next lemma explains a relation between braids $\alpha_{\boldsymbol{\omega}}$ (see (2.5)) and $\beta_{\boldsymbol{m}}$.

Lemma 2.9. *Let $\Theta : \Psi_{N-1} \to \Omega_N^+$ be the bijection as in Lemma 2.2. Then the identity $\beta_{\boldsymbol{m}} = \alpha_{\Theta(\boldsymbol{m})}$ holds for each $\boldsymbol{m} \in \Psi_{N-1}$. Moreover, $\beta_{\boldsymbol{m}}$ is pseudo-Anosov if and only if $\boldsymbol{m} \neq (N-1)$.*

Proof. The former statement is immediate by the definitions of $\alpha_{\boldsymbol{\omega}}$ and $\beta_{\boldsymbol{m}}$. For the latter statement, we represent a composition $\boldsymbol{m} \in \Psi_{N-1}$ by a $(k+1)$-tuple $\boldsymbol{m} = (m_1, \ldots, m_{k+1})$ of positive integers with $k \geq 0$ and $N - 1 = \sum_{i=1}^{k+1} m_i$. By Lemma 2.7 and Theorem 2.8, $\beta_{\boldsymbol{m}}$ is pseudo-Anosov if and only if $k > 0$, equivalently $\boldsymbol{m} \neq (N-1)$. $\square$

For convenience of the readers, we explain how to compute the stretch factor $\lambda_{\boldsymbol{m}}$ of the pseudo-Anosov braid $\beta_{\boldsymbol{m}}$. For more details, see [KT08].

How to compute $\lambda_{\boldsymbol{m}}$. We take a composition $\boldsymbol{m} = (m_1, \ldots, m_{k+1}) \in \Psi_{N-1}$ with $k > 0$. Theorem 2.8 tells us that $\mathfrak{c}(\Gamma(\beta_{\boldsymbol{m}})) \in \mathrm{MCG}(\Sigma_{0,N+1})$ is a pseudo-Anosov mapping class. We identify $\beta_{\boldsymbol{m}}$ with $\mathfrak{c}(\Gamma(\beta_{\boldsymbol{m}}))$. We view an $N+1$-punctured sphere $\Sigma_{0,N+1}$ as a sphere with $m_i + 1$ marked points X_i circling an unmarked point u_i for each $i \in \{1, \ldots, k+1\}$ and a single marked point ∞ (corresponding to ∂D). Note that $|X_i| = 1 + m_i$ for each $i \in \{1, \ldots, k+1\}$ and $|X_j \cap X_{j+1}| = 1$ for each $j \in \{1, \ldots, k\}$. See Figure 7(1).

We choose a finite graph $G_{\boldsymbol{m}} \subset \Sigma_{0,N+1}$ that is homotopy equivalent to the $N+1$-punctured sphere. The graph $G_{\boldsymbol{m}}$ has N loop edges, each of which encircles a puncture (a marked point), and $N + k$ non-loop edges. See Figure 7(2). Let P be the set of N loop edges of $G_{\boldsymbol{m}}$. The graph $G_{\boldsymbol{m}}$ has $N + k + 1$ vertices. For each loop edge, there is a vertex of degree 3 or 4. The unmarked point u_i corresponds to a vertex of degree $1 + m_i$ for $i \in \{1, \ldots, k+1\}$.

Given a mapping class $\psi \in \mathrm{MCG}(\Sigma_{0,N+1})$, one can pick an induced graph map $g : G_{\boldsymbol{m}} \to G_{\boldsymbol{m}}$ (see [BH95, Section 1]). We require that g sends vertices to vertices, edges to edge paths and satisfies $g(P) = P$. We may suppose

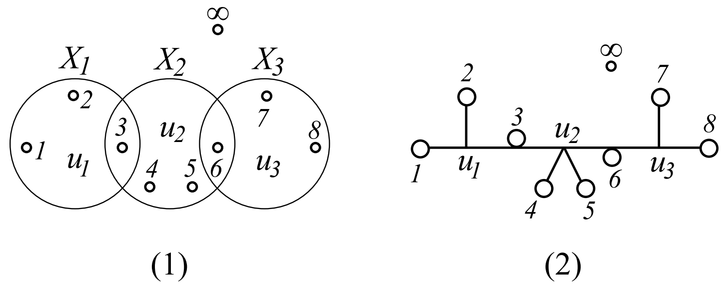


FIGURE 7. Case $N = 8$, $k = 2$ and $\boldsymbol{m} = (m_1, m_2, m_3) = (2, 3, 2) \in \Psi_7$. (1) $N+1$-marked points in the sphere. (Small circles indicate marked points (punctures).) (2) The graph $G_{\boldsymbol{m}}$. (Each loop edge encircles a marked point. In this case, $G_{\boldsymbol{m}}$ has 8 loop edges and 10 non-loop edges.)

that g has *no backtracks*, i.e., g maps each oriented edge of $G_{\boldsymbol{m}}$ to an edge path which does not contain an oriented edge e followed by the reverse edge $\bar{e}$ of the oriented edge e. Then the graph map g defines an $N + k$ by $N+k$ *transition matrix* M with respect to the $N+k$ non-loop edges. More precisely, for $r, s \in \{1, \ldots, N + k\}$ the rs-entry M_{rs} is the number of times that the g-image of the sth edge runs the rth edge in either direction. We say that $g : G_{\boldsymbol{m}} \to G_{\boldsymbol{m}}$ is *efficient* if $g^n : G_{\boldsymbol{m}} \to G_{\boldsymbol{m}}$ has no backtracks for all $n > 0$.

We now define $\phi_{\boldsymbol{m}} \in \mathrm{MCG}(\Sigma_{0,N+1})$. We will see in (2.8) that $\beta_{\boldsymbol{m}}$ and $\phi_{\boldsymbol{m}}$ are conjugate in $\mathrm{MCG}(\Sigma_{0,N+1})$. Let $f_i = f_{\boldsymbol{m},i} : \Sigma_{0,N+1} \to \Sigma_{0,N+1}$ be a homeomorphism such that f_i rotates the marked points of X_i counterclockwise around u_i if i is odd, and f_i rotates the marked points of X_i clockwise around u_i if i is even. See Figure 8(1)(2). Define $\phi_i = [f_i] \in \mathrm{MCG}(\Sigma_{0,N+1})$. Figure 8(3)(4) illustrates the N-braid $b_i = b_{\boldsymbol{m},i}$ such that $\phi_i = \mathfrak{c}(\Gamma(b_i))$. We set $\phi_{\boldsymbol{m}} = \phi_{k+1} \circ \cdots \circ \phi_2 \circ \phi_1$. (c.f. [KT08, Figure 3] for $\phi_{\boldsymbol{m}} = \phi_{(4,2,1)}$.) Recall that rev : $B_n \to B_n$ is the map as in Section 2.5. By definitions of f_i and $\beta_{\boldsymbol{m}}$ (see Figure 6), we can verify that $\phi_{\boldsymbol{m}}$ is of the form

$$\phi_{\boldsymbol{m}} = \mathrm{rev}(\beta_{\boldsymbol{m}}) \in \mathrm{MCG}(\Sigma_{0,N+1}).$$

For example, when $N = 5$, $k = 1$ and $\boldsymbol{m} = (2, 2) \in \Phi_4$, we have

$$\phi_{\boldsymbol{m}} = \phi_2 \circ \phi_1 = \sigma_4^{-1}\sigma_3^{-1} \cdot \sigma_2\sigma_1 = \mathrm{rev}(\sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}) = \mathrm{rev}(\beta_{(2,2)}).$$

Let $\Theta : \Psi_{N-1} \to \Omega_N^+$ be the bijection as in Lemma 2.2. Then $\beta_{\boldsymbol{m}} = \alpha_{\Theta(\boldsymbol{m})}$ by Lemma 2.9. Moreover $\alpha_{\Theta(\boldsymbol{m})}$ and $\mathrm{rev}(\alpha_{\Theta(\boldsymbol{m})})$ are conjugate in B_N by Lemma 2.6. As a consequence, one sees that

(2.8)
$$\phi_{\boldsymbol{m}} (= \mathrm{rev}(\beta_{\boldsymbol{m}}) = \mathrm{rev}(\alpha_{\Theta(\boldsymbol{m})})) \text{ and } \beta_{\boldsymbol{m}} \text{ are conjugate in } \mathrm{MCG}(\Sigma_{0,N+1}).$$

The mapping class $\phi_i = [f_i]$ for $i \in \{1, \ldots, k+1\}$ induces a graph map $g_i = g_{\boldsymbol{m},i} : G_{\boldsymbol{m}} \to G_{\boldsymbol{m}}$ which has no backtracks as shown in Figure 8(5)(6). We denote by $M_i = M_{\boldsymbol{m},i}$, the transition matrix of g_i (with respect to the $N + k$ non-loop edges). The composition

$$g_{\boldsymbol{m}} := g_{k+1} \circ \cdots \circ g_2 \circ g_1 : G_{\boldsymbol{m}} \to G_{\boldsymbol{m}} \tag{2.9}$$

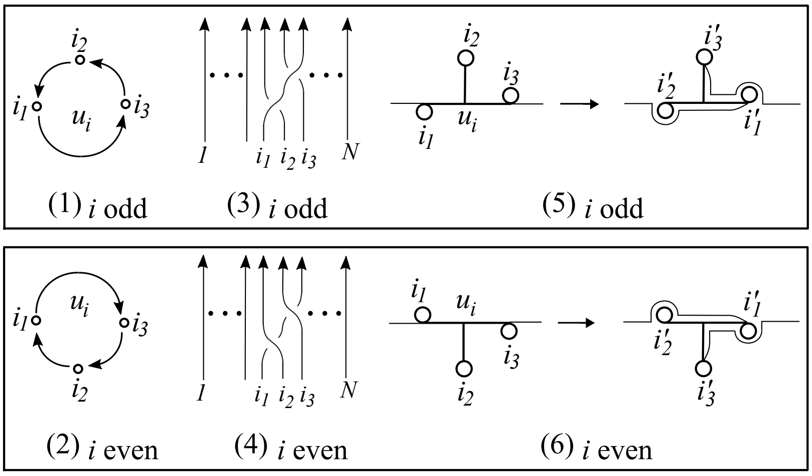


FIGURE 8. Case $m_i = 2$. (1)(2) $f_i : \Sigma_{0,N+1} \to \Sigma_{0,N+1}$ when i is odd/even. (3)(4) The braid b_i corresponding to ϕ_i when i is odd/even. (5)(6) $g_i : G_{\boldsymbol{m}} \to G_{\boldsymbol{m}}$ when i is odd/even.

is an induced graph map of $\phi_{\boldsymbol{m}}$. By the induction on k, one can show that $g_{\boldsymbol{m}} : G_{\boldsymbol{m}} \to G_{\boldsymbol{m}}$ has no backtracks. This implies that the transition matrix of $g_{\boldsymbol{m}}$ with respect to the non-loop edges is given by

$$M_{\boldsymbol{m}} := M_{k+1} \cdots M_2 \cdot M_1.$$

It is proved in [KT08] that $M_{\boldsymbol{m}}$ is a Perron-Frobenius matrix. Moreover one can show that $g_{\boldsymbol{m}} : G_{\boldsymbol{m}} \to G_{\boldsymbol{m}}$ is efficient. As a consequence of Bestvina-Handel algorithm [BH95], the Perron-Frobenius eigenvalue $\lambda(M_{\boldsymbol{m}})$ of $M_{\boldsymbol{m}}$ equals the stretch factor $\lambda(\phi_{\boldsymbol{m}})$ of $\phi_{\boldsymbol{m}}$. By (2.8), $\phi_{\boldsymbol{m}}$ is conjugate to $\beta_{\boldsymbol{m}}$ in $\mathrm{MCG}(\Sigma_{0,N+1})$. We obtain

$$\lambda_{\boldsymbol{m}} (= \lambda(\beta_{\boldsymbol{m}})) = \lambda(M_{\boldsymbol{m}}). \tag{2.10}$$

The Perron-Frobenius eigenvalue $\lambda(M_{\boldsymbol{m}})$ gives the stretch factor $\lambda_{\boldsymbol{m}}$. □

Fixing $N \geq 3$, let Y_N be the set of braids $\beta_{\boldsymbol{m}}$ over all $\boldsymbol{m} \in \Psi_{N-1}$.

$$Y_N := \{\beta_{\boldsymbol{m}} \mid \boldsymbol{m} = (m_1, \dots, m_{k+1}) \in \Psi_{N-1} \text{ with } k \geq 0\} \subset B_N.$$

By Lemma 2.9, all braids $\beta_{\boldsymbol{m}}$ in Y_N except the case $\boldsymbol{m} = (N-1)$ are pseudo-Anosov. By Lemma 2.2, Y_N can be written by

$$Y_N = \{\alpha_{\boldsymbol{\omega}} \mid \boldsymbol{\omega} \in \Omega_N^+\}. \tag{2.11}$$

The following result identifies the braids in Y_N with the largest and smallest stretch factor, respectively.

Theorem 2.10. *Among all braids in $Y_N \setminus \{\beta_{(N-1)}\}$, the braid $\beta_{\boldsymbol{1}_{N-1}} = \beta_{(1,1,\dots,1)}$ realizes the largest stretch factor, while the smallest stretch factor is realized by $\beta_{(n,n)}$ if $N = 2n+1$, and by $\beta_{(n-1,n)}$ (and $\beta_{(n,n-1)}$) if $N = 2n$.*

To prove Theorem 2.10, we need the following results.

Proposition 2.11 (Proposition 1.1 in [KT08]). *For $k > 0$, we consider $(k+1)$-tuples of positive integers $\boldsymbol{m} = (m_1, \dots, m_{k+1})$ and $\boldsymbol{m}' = (m'_1, \dots, m'_{k+1})$.*

Suppose that $m'_i = m_i + 1$ *for some* i *and* $m'_j = m_j$ *if* $j \neq i$*. Then we have* $\lambda_{\boldsymbol{m}'} < \lambda_{\boldsymbol{m}}$*.*

Using Proposition 2.11 repeatedly, we obtain the following.

Corollary 2.12. *For* $k > 0$*, we consider* $(k+1)$*-tuples of positive integers* $\boldsymbol{m} = (m_1, \dots, m_{k+1})$ *and* $\boldsymbol{m}' = (m'_1, \dots, m'_{k+1})$*. Suppose that* $m_j \leq m'_j$ *for all* j*. Then we have* $\lambda_{\boldsymbol{m}'} \leq \lambda_{\boldsymbol{m}}$*. The equality holds if and only if* $\boldsymbol{m} = \boldsymbol{m}'$*, i.e.,* $m_j = m'_j$ *for all* j*.*

Proposition 2.13. *For* $k > 0$*, we consider a* $(k+1)$*-tuple of positive integers* $\boldsymbol{m} = (m_1, \dots, m_{k+1})$ *and a* $(k+2)$*-tuple of positive integers* $\boldsymbol{m}' = (m'_1, \dots, m'_{k+1}, m'_{k+2})$*. We assume that* $\boldsymbol{m}'$ *is of the form*

$$\boldsymbol{m}' = (m_1, \dots, m_{k+1}, m'_{k+2}),$$

i.e., $m_j = m'_j$ *for all* $j = 1, \dots, k+1$*. Then we have* $\lambda_{\boldsymbol{m}} < \lambda_{\boldsymbol{m}'}$*.*

Proof. Consider the graph maps $g_{\boldsymbol{m}} : G_{\boldsymbol{m}} \to G_{\boldsymbol{m}}$ and $g_{\boldsymbol{m}'} : G_{\boldsymbol{m}'} \to G_{\boldsymbol{m}'}$ as in (2.9). Note that $G_{\boldsymbol{m}}$ is a subgraph of $G_{\boldsymbol{m}'}$ by the assumption on $\boldsymbol{m}'$. Let $M_{\boldsymbol{m}}$ and $M_{\boldsymbol{m}'}$ be the transtion matrices of $g_{\boldsymbol{m}}$ and $g_{\boldsymbol{m}'}$, respectively. Let $\lambda(M_{\boldsymbol{m}})$ and $\lambda(M_{\boldsymbol{m}'})$ be the corresponding Perron-Frobenius eigenvalues. By (2.10), it is enough to prove that $\lambda(M_{\boldsymbol{m}}) < \lambda(M_{\boldsymbol{m}'})$. Under the suitable labeling of non-loop edges of $G_{\boldsymbol{m}'}$, the matrix $M_{\boldsymbol{m}'}$ can be written by

$$M_{\boldsymbol{m}'} = \begin{bmatrix} M_{\boldsymbol{m}} & A \\ B & C \end{bmatrix},$$

where A, B and C are block matrices with nonnegative integer entries and C is a square matrix. Assume that $A = \mathbf{0}$ and $B = \mathbf{0}$. Then any power $M^k_{\boldsymbol{m}'}$ of $M_{\boldsymbol{m}'}$ is not a positive matrix, which contradicts the fact that $M_{\boldsymbol{m}'}$ is Perron-Frobenius. Hence, either A or B is a non-zero matrix. Consider the square matrix D with the same size as $M_{\boldsymbol{m}'}$ of the form $D = \begin{bmatrix} M_{\boldsymbol{m}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. Then $D \neq M_{\boldsymbol{m}'}$ since either A or B is a non-zero matrix. Since $A, B, C \geq \mathbf{0}$, we have $\mathbf{0} \leq D \leq M_{\boldsymbol{m}'}$, i.e., $0 \leq D_{st} \leq (M_{\boldsymbol{m}'})_{st}$ for each st-entry. Then the Perron-Frobenius theorem ([Sen06, Theorem 1.1(e)]) tells us that if λ is an eigenvalue of D, then $|\lambda| < \lambda(M_{\boldsymbol{m}'})$. Thus, we obtain $\lambda(M_{\boldsymbol{m}}) < \lambda(M_{\boldsymbol{m}'})$. This completes the proof. □

Proof of Theorem 2.10. We consider the braid $\beta_{\boldsymbol{m}}$ associated with a $(k+1)$-tuple $\boldsymbol{m} = (m_1, \dots, m_{k+1})$. Suppose that $\boldsymbol{m}$ is a composition of $N-1$. Clearly, $k+1 \leq N-1$ and $\beta_{\boldsymbol{m}} = \beta_{(m_1,\dots,m_{k+1})} \in Y_N$. By Corollary 2.12, we have the inequality

$$\lambda(\beta_{(m_1,\dots,m_{k+1})}) \leq \lambda(\beta_{\mathbf{1}_{k+1}}) \quad \text{for all integers } m_1, \dots, m_{k+1} \geq 1. \tag{2.12}$$

The equality holds if and only if $\boldsymbol{m} = \mathbf{1}_{k+1}$. The inequalities $k+1 \leq N-1$ and (2.12) together with Proposition 2.13 tell us that

$$\lambda(\beta_{(m_1,\dots,m_{k+1})}) \leq \lambda(\beta_{\mathbf{1}_{k+1}}) \leq \lambda(\beta_{\mathbf{1}_{N-1}}).$$

Thus, $\beta_{\mathbf{1}_{N-1}} \in Y_N$ realizes the largest stretch factor.

Next, we turn to the braid in Y_N with the smallest stretch factor. Let $\beta_{(m_1,\dots,m_{k+1},m)}$ be the braid associated with a $(k+2)$-tuple $(m_1, \dots, m_{k+1}, m)$. Suppose this $(k+2)$-tuple is a composition of $N-1$. Then $\beta_{(m_1,\dots,m_{k+1},m)} \in$

Y_N. Note that the braid $\beta_{(m_1,\ldots,m_k,m_{k+1}+m)}$ associated with the $(k+1)$-tuple $(m_1,\ldots,m_k,m_{k+1}+m) \in \Psi_{N-1}$ is also an element of Y_N. Proposition 2.13 tells us that

$$\lambda(\beta_{(m_1,\ldots,m_k,m_{k+1})}) < \lambda(\beta_{(m_1,\ldots,m_k,m_{k+1},m)}). \tag{2.13}$$

By Corollary 2.12, we have

$$\lambda(\beta_{(m_1,\ldots,m_k,m_{k+1}+m)}) < \lambda(\beta_{(m_1,\ldots,m_k,m_{k+1})}). \tag{2.14}$$

By (2.13) and (2.14), the stretch factor of $\beta_{(m_1,\ldots,m_k,m_{k+1}+m)} \in Y_N$ is smaller than that of $\beta_{(m_1,\ldots,m_k,m_{k+1},m)} \in Y_N$. This means that for the braid with the smallest stretch factor, it is enough to consider elements $\beta_{\boldsymbol{m}} \in Y_N$ associated with the compositions $\boldsymbol{m} \in \Psi_{N-1}$ of the form $\boldsymbol{m} = (m,n)$. The inverse $\beta^{-1}_{(m,n)}$ of $\beta_{(m,n)}$ satisfies $\Delta\beta^{-1}_{(m,n)}\Delta^{-1} = \beta_{(n,m)}$, where Δ is the half twist. Hence $\beta^{-1}_{(m,n)}$ and $\beta_{(n,m)}$ are conjugate in B_N. In particular, we have $\lambda(\beta^{-1}_{(m,n)}) = \lambda(\beta_{(n,m)})$. Since a pseudo-Anosov braid b and its inverse b^{-1} have the same stretch factor, we conclude that

$$\lambda(\beta_{(m,n)}) = \lambda(\beta^{-1}_{(m,n)}) = \lambda(\beta_{(n,m)}).$$

(The equality $\lambda(\beta_{(m,n)}) = \lambda(\beta_{(n,m)})$ also follows from Lemma 3.12.) Hence, we restrict our attention to the pairs (m,n) with $m \leq n$. The following inequalities are proved in [HK06, Proposition 3.33].

$$\begin{array}{rcll} \lambda(\beta_{(n,n)}) & < & \lambda(\beta_{(n-k,n+k)}) & \text{for } k = 1,2,\ldots,n-1, \\ \lambda(\beta_{(n-1,n)}) & < & \lambda(\beta_{(n-k-1,n+k)}) & \text{for } k = 1,2,\ldots,n-2. \end{array}$$

Thus, if $N = 2n+1$ (resp. $N = 2n$), then the smallest stretch factor is realized by $\beta_{(n,n)}$ (resp. $\beta_{(n-1,n)}$ and $\beta_{(n,n-1)}$). This completes the proof. $\square$

By Theorem 2.10, we are interested in the computation of the stretch factors of $\beta_{\mathbf{1}_{N-1}}$ and $\beta_{(m,n)}$ with $|m-n| = 0$ or 1. Examples 2.14 and 2.15 are useful.

Example 2.14. Let us compute the stretch factors of $\beta_{\mathbf{1}_3}$, $\beta_{\mathbf{1}_4}$ and $\beta_{\mathbf{1}_5}$.

(1) By recursive formulas of $R_{\boldsymbol{m}}(t)$ and $F_{\boldsymbol{m}}(t)$ (see Theorem 2.8) and the definition of $f_*(t)$ (see (2.7)), we obtain

$$\begin{array}{rcl} R_{(1)}(t) & = & t^3 - t^2 - 2t, \\ R_{(1,1)}(t) & = & t(t-1)R_{(1)}(t) + 2tR_{(1)_*}(t) = t^5 - 2t^4 - 5t^3 + 2t, \\ F_{(1,1,1)}(t) & = & tR_{(1,1)}(t) - R_{(1,1)_*}(t) = (t-1)(t+1)^3(t^2-4t+1). \end{array}$$

Hence, the largest real root of the third factor $t^2 - 4t + 1$ of $F_{(1,1,1)}(t)$ is equal to $\lambda(\beta_{\mathbf{1}_3}) = 2 + \sqrt{3}$.

(2) A computation shows that

$$\begin{array}{rcl} R_{(1,1,1)}(t) & = & t(t-1)R_{(1,1)}(t) - 2tR_{(1,1)_*}(t) \\ & = & t^7 - 3t^6 - 7t^5 + 5t^4 + 12t^3 + 2t^2 - 2t, \\ F_{(1,1,1,1)}(t) & = & tR_{(1,1,1)}(t) + R_{(1,1,1)_*}(t) \\ & = & (t+1)^4(t^4 - 7t^3 + 13t^2 - 7t + 1). \end{array}$$

The largest real root of the second factor of $F_{(1,1,1,1)}(t)$ gives the stretch factor $\lambda(\beta_{\mathbf{1}_4}) \approx 4.39026$.

(3) Lastly, we compute

$$\begin{aligned} R_{(1,1,1,1,1)}(t) &= t^9 - 4t^8 - 8t^7 + 16t^6 + 31t^5 - 18t^3 - 4t^2 + 2t, \\ F_{(1,1,1,1,1)}(t) &= tR_{(1,1,1,1,1)}(t) - R_{(1,1,1,1,1)_*}(t) \\ &= (t-1)(t+1)^5(t^2-3t+1)(t^2-5t+1). \end{aligned}$$

Hence, the largest real root of the last factor t^2-5t+1 of $F_{(1,1,1,1,1)}(t)$ gives us $\lambda(\beta_{\mathbf{1}_5}) \approx 4.79129$.

Example 2.15. A computation shows that

$$\begin{aligned} R_{(m)}(t) &= t^{m+1}(t-1) - 2t = t^{m+2} - t^{m+1} - 2t, \\ R_{(m)_*}(t) &= t^{m+2} R_{(m)}(\tfrac{1}{t}) = 1 - t - 2t^{m+1}. \end{aligned}$$

By Theorem 2.8, the stretch factor of $\beta_{(m,n)}$ is the largest real root of

$$F_{(m,n)}(t) = t^n R_{(m)}(t) + R_{(m)_*}(t) = t^n(t^{m+2} - t^{m+1} - 2t) - 2t^{m+1} - t + 1.$$

In Table 1, we list the smallest and largest stretch factors among all pseudo-Anosov braids in Y_N.

TABLE 1. Smallest and largest stretch factors in $Y_N \setminus \{\beta_{(N-1)}\}$.

N	$\min_{\beta \in Y_N \setminus \{\beta_{(N-1)}\}} \lambda(\beta)$	$\max_{\beta \in Y_N \setminus \{\beta_{(N-1)}\}} \lambda(\beta)$
3	$\lambda(\beta_{\mathbf{1}_2}) = \frac{3+\sqrt{5}}{2}$	$\lambda(\beta_{\mathbf{1}_2}) = \frac{3+\sqrt{5}}{2}$
4	$\lambda(\beta_{(1,2)}) \approx 2.29663$	$\lambda(\beta_{\mathbf{1}_3}) = 2+\sqrt{3}$
5	$\lambda(\beta_{(2,2)}) \approx 2.01536$	$\lambda(\beta_{\mathbf{1}_4}) \approx 4.39026$
6	$\lambda(\beta_{(2,3)}) \approx 1.8832$	$\lambda(\beta_{\mathbf{1}_5}) \approx 4.79129$
7	$\lambda(\beta_{(3,3)}) \approx 1.75488$	$\lambda(\beta_{\mathbf{1}_6}) \approx 5.04892$
8	$\lambda(\beta_{(3,4)}) \approx 1.6815$	$\lambda(\beta_{\mathbf{1}_7}) \approx 5.22274$
9	$\lambda(\beta_{(4,4)}) \approx 1.60751$	$\lambda(\beta_{\mathbf{1}_8}) \approx 5.345$
10	$\lambda(\beta_{(4,5)}) \approx 1.56028$	$\lambda(\beta_{\mathbf{1}_9}) \approx 5.43401$

3. Simple choreographies by Yu

In this section, we explain simple choreographies of the planar N-body problem obtained by Yu [Yu17]. His main theorem is:

Theorem 3.1 ([Yu17])**.** *For every $N \geq 3$, there exist at least $2^{N-3} + 2^{\lfloor (N-3)/2 \rfloor}$ different simple choreographies for the planar Newtonian N-body problem with equal masses, where $\lfloor \cdot \rfloor$ denotes the integer part of a real number.*

In the end of this section, we will explain the meaning of "different simple choreographies" in the statement of Theorem 3.1.

We identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. The planar N-body problem with equal masses is described by the following differential equation:

$$\ddot{z}_j = \sum_{k \in \{0,1,\ldots,N-1\} \setminus \{j\}} -\frac{z_j - z_k}{|z_j - z_k|^3} \quad (j \in \{0,1,\ldots,N-1\}) \tag{3.1}$$

where $\boldsymbol{z} = (z_j)_{j=0}^{N-1} \in \mathbb{C}^N$. The N-body problem has a variational structure. That is, the critical points of the functional

$$\mathcal{A}_{[a,b]}(\boldsymbol{z}) = \int_a^b L(\boldsymbol{z}, \dot{\boldsymbol{z}})dt$$

correspond to weak solutions of the N-body problem, where

$$\boldsymbol{z} \in H^1([a,b], \mathbb{C}^N) := \{\boldsymbol{x}\colon [a,b] \to \mathbb{C}^N \mid \boldsymbol{x}, \dot{\boldsymbol{x}} \in L^2([a,b], \mathbb{C}^N)\}$$

and

$$L(\boldsymbol{z}, \dot{\boldsymbol{z}}) = \frac{1}{2}\sum_{j=0}^{N-1} |\dot{z}_j|^2 + \sum_{\substack{j,k\in\{0,1,\dots,N-1\},\\ j<k}} \frac{1}{|z_j - z_k|}.$$

Theorem 3.1 was proved using this variational structure. More precisely, he showed the existence of a minimizer of $\mathcal{A}_{[0,N]}(\boldsymbol{z})$ in the N-periodic functional space

$$\Lambda_N := H^1(\mathbb{R}/N\mathbb{Z}, \mathbb{C}^N)$$

under symmetric and topological constraints.

Firstly, we explain the symmetric condition. Let G be a finite group and define three actions τ, ρ and σ as follows:

$$\begin{aligned}
&\tau\colon G \to O(2), \ \text{(the action of } G \text{ on the time circle } \mathbb{R}/2\pi\mathbb{Z}),\\
&\rho\colon G \to O(2), \ \text{(the action of } G \text{ on two-dimensional Euclidean space), and}\\
&\sigma\colon G \to S_N, \ \text{(the action of } G \text{ on the index set } \{0,1,\dots,N-1\}),
\end{aligned}$$

where $O(2)$ and S_N represent the two-dimensional orthogonal group and the symmetric group of N elements, respectively. For each $g \in G$, we define its action as follows:

$$g(\boldsymbol{z}(t)) = (\rho(g)z_{\sigma(g^{-1})(0)}(\tau(g^{-1})t), \dots, \rho(g)z_{\sigma(g^{-1})(N-1)}(\tau(g^{-1})t).$$

Set

$$\begin{aligned}
\Lambda_N^G &= \{\boldsymbol{z} \in \Lambda_N \mid g(\boldsymbol{z}(t)) = \boldsymbol{z}(t) \text{ for all } g \in G\},\\
\hat{\Lambda}_N &= \{\boldsymbol{z} \in \Lambda_N \mid z_i(t) \neq z_j(t) \text{ for } i \neq j\}, \text{ and}\\
\hat{\Lambda}_N^G &= \Lambda_N^G \cap \hat{\Lambda}_N.
\end{aligned}$$

The set $\hat{\Lambda}_N$ implies that each element has no collision. As a consequence, a critical point of $\mathcal{A}_{[0,N]}$ in $\hat{\Lambda}_N^G$ is also a critical point in $\hat{\Lambda}_N$ if the N masses are equal. This fact follows from the Palais principle:

Proposition 3.2 (Palais principle, [Pal79]). *Let M be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, and let G be a group such that each $g \in G$ is a linear operator on M satisfying*

$$\langle gx, gy\rangle = \langle x, y\rangle \quad \textit{for any } x, y \in M.$$

Define

$$\Sigma = \{x \in M \mid gx = x \ \textit{ for } g \in G\}.$$

Suppose that $f\colon M \to \mathbb{R}$ is of class C^1, G-invariant, and Σ is a closed subspace of M. If $p \in \Sigma$ is a critical point of $f|_\Sigma$, then $p \in \Sigma$ is also a critical point of f.

Example 3.3. Let G be the cyclic group, i.e., $G = \langle g \mid g^N = 1\rangle (=: \mathbb{Z}_N)$, and its actions are given by:

$$\tau(g)t = t - 1, \quad \rho(g) = \mathrm{id}, \quad \text{and} \quad \sigma(g) = (0, 1, \ldots, N-1). \tag{3.2}$$

Then, for any $\boldsymbol{z} \in \Lambda_N^{\mathbb{Z}_N}$,

$$z_j(t) = z_0(t + j) \quad \text{for } t \in \mathbb{R} \text{ and } j \in \{0, 1, \ldots, N-1\}, \tag{3.3}$$

and this implies that a critical point of $\mathcal{A}_{[0,N]}$ in $\Lambda_N^{\mathbb{Z}_N}$ describes a simple choreography if it has no collisions.

Example 3.3 is a standard setting in proofs of the existence of periodic solutions using the Palais principle. However, it is also known that the global minimizer in $\Lambda_N^{\mathbb{Z}_N}$ is only the rotating regular N-gon. Thus, additional constraints are needed to obtain nontrivial periodic orbits through minimizing methods.

Example 3.4 (Setting in [Yu17])**.** Set

$$D_N = \langle g, h \mid g^N = h^2 = 1, (gh)^2 = 1\rangle,$$

where the actions of g are the same as in (3.2) in Example 3.3, and

$$\begin{aligned} &\tau(h)t = -t + 1, \quad \rho(h)\boldsymbol{z} = \bar{\boldsymbol{z}}, \quad \text{and} \\ &\sigma(h) = (0, N-1)(1, N-2)\cdots(\mathbf{n}, N-1-\mathbf{n}), \end{aligned}$$

where $\mathbf{n} = \lfloor (N-1)/2 \rfloor$. Thus, any $\boldsymbol{z} \in \Lambda_N^{D_N}$ satisfies the following three properties. Firstly, the actions of h imply:

$$z_j(t) = \bar{z}_{N-1-j}(1-t) \text{ for } t \in \mathbb{R} \text{ and } j \in \{0, 1, \ldots, N-1\} \tag{3.4}$$

and combining (3.3) and (3.4) yields:

$$z_j(t) = \bar{z}_{N-j}(-t),$$

especially,

$$\mathrm{Re}(\dot{z}_j(0)) = -\mathrm{Re}(\dot{z}_{N-j}(0)). \tag{3.5}$$

Secondly, the actions of gh indicate:

$$z_0(t) = \bar{z}_0(-t) \text{ for } t \in \mathbb{R}. \tag{3.6}$$

By (3.6), we get $\mathrm{Im}(z_0(0)) = 0$.

Clrarly, $\Lambda_N^{D_N} \subset \Lambda_N^{\mathbb{Z}_N}$. Since the set $\Lambda_N^{D_N}$ contains an element representing the rotating N-gon, which is the global minimizer in $\Lambda_N^{\mathbb{Z}_N}$, we need additional assumptions.

Definition 3.5 (The $\boldsymbol{\omega}$-topological constraints, [Yu17])**.** For any $\boldsymbol{\omega} \in \Omega_N$, $\boldsymbol{z} \in \Lambda_N^{D_N}$ is said to satisfy the *$\boldsymbol{\omega}$-topological constraints* if

$$\mathrm{Im}(z_0(j/2)) = \omega_j |\mathrm{Im}(z_0(j/2))| \text{ for } j \in \{1, \ldots, N-1\}.$$

The periodic orbits obtained in Theorem 3.1 satisfy the $\boldsymbol{\omega}$-topological constraints and following monotonicity.

Theorem 3.6 ([Yu17])**.** *For each $\boldsymbol{\omega} \in \Omega_N$, there exists at least one simple choreography $\boldsymbol{z} = (z_j)_{j=0}^{N-1} \in \hat{\Lambda}_N^{D_N}$ satisfying (3.1), the $\boldsymbol{\omega}$-topological constraints and the following properties:*

(1) $\mathrm{Re}(\dot{z}_0(t)) > 0$ *for* $t \in (0, N/2)$*, and*

(2) $\mathrm{Re}(\dot{z}_0(0)) = \mathrm{Re}(\dot{z}_0(N/2)) = 0$.

Remark 3.7. Since the periodic orbits in Theorem 3.1 are collision-free, (3.3) and (3.4) imply $\mathrm{Im}(z_0(j/2)) \neq 0$ for any $j \in \{1, \ldots, N-1\}$.

Let $\boldsymbol{z}_{\boldsymbol{\omega}} \in \Lambda_N^{D_N}$ satisfy the properties of Theorem 3.6 for $\boldsymbol{\omega}$. If $\boldsymbol{z}_{\boldsymbol{\omega}} \in \hat{\Lambda}_N^{D_N}$, then it draws a trajectory that transits between line segments parallel to the x- or y-axis. The shape of the trajectories can be easily understood by showing examples as below.

For $z_j \colon \mathbb{R}/N\mathbb{Z} \to \mathbb{C}$, $\boldsymbol{z} \colon \mathbb{R}/N\mathbb{Z} \to \mathbb{C}^N$, and t_1, t_2 with $0 \leq t_1 < t_2 \leq N$, define

$$z_j([t_1, t_2]) = \{z_j(t) \mid t_1 \leq t \leq t_2\}, \quad j \in \{0, 1, \ldots, N-1\},$$

and

$$\boldsymbol{z}([t_1, t_2]) = \{\boldsymbol{z}(t) \mid t_1 \leq t \leq t_2\}.$$

We think of $\boldsymbol{z}([t_1, t_2])$ as the N oriented curves in the complex plane $\mathbb{C}$.

Note that if $\boldsymbol{z}(t)$ is a solution of (3.1), so is $\bar{\boldsymbol{z}}(-t)$. Thus we get

$$z_0([0, N]) = z_0([0, N/2]) \cup \bar{z}_0([0, N/2]). \tag{3.7}$$

Moreover, by the property (2) of Theorem 3.6, $z_0([0, N])$ forms a smooth closed curve.

Remark 3.8. Set $\omega_0 = \omega_N = 0$ and

$$z_0((t_1, t_2)) = \{z_0(t) \mid t_1 < t < t_2\}.$$

The proof of Proposition 3.1 in [Yu17] implies that for $j \in \{0, \ldots, N-1\}$, the trajectory $z_0((j/2, (j+1)/2))$ crosses the x-axis exactly once if $\omega_j \omega_{j+1} = -1$, and does not cross the x-axis otherwise.

Example 3.9. Set $N = 3$ and $\boldsymbol{\omega} = (1, -1)$. Figure 9(1) represents $z_0([0, N/2])$. Each arrow precisely indicates $z_0([i/2, (i+1)/2])$ for $i = 0, 1, 2$ and $\boldsymbol{\omega} = (1, -1)$ shows whether the trajectory passes through the positive or negative side of the y-axis. By (3.7), the case $\boldsymbol{\omega} = (1, -1)$ gives a periodic orbit like the figure-eight [CM00]. On the other hand, Figure 9(2) describes the trajectory $\boldsymbol{z}_{\boldsymbol{\omega}}([0, 1/2])$ and each arrow represents $z_0([0, 1/2])$, $z_1([0, 1/2])$ and $z_2([0, 1/2])$, respectively. The indices of the particles are determined from (3.3). See also Figure 9(3) for $\boldsymbol{z}_{\boldsymbol{\omega}}([1/2, 1])$.

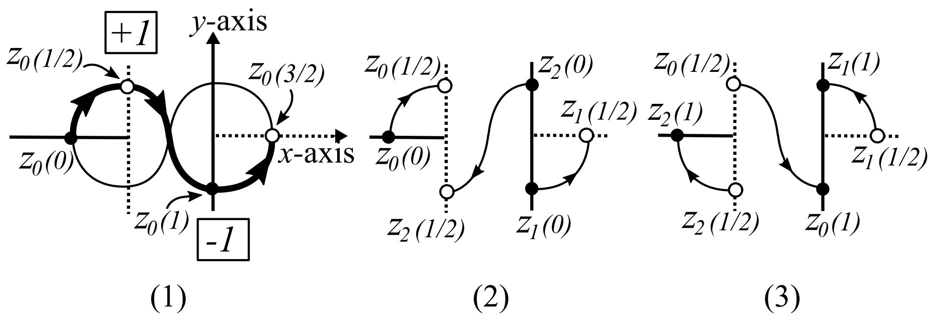


FIGURE 9. Case $N = 3$, $\boldsymbol{\omega} = (1, -1)$. (1) Thick arrows indiate $z_0([0, N/2])$. (2) $\boldsymbol{z}_{\boldsymbol{\omega}}([0, 1/2])$. (3) $\boldsymbol{z}_{\boldsymbol{\omega}}([1/2, 1])$.

Example 3.10. Set $N = 4$ and $\boldsymbol{\omega} = (1, -1, 1)$. As in the previous example, Figure 10(1) represents $z_0([0, N/2])$. Figures 10(2) and 10(3) depict $\boldsymbol{z}_{\boldsymbol{\omega}}([0, 1/2])$ and $\boldsymbol{z}_{\boldsymbol{\omega}}([1/2, 1])$, respectively. While the figure-eight consists of two connected loops, a trajectory forming three such loops is called the super-eight, and the existence of a periodic solution with this shape in the four-body problem has been established in [KZ03, Shi14].

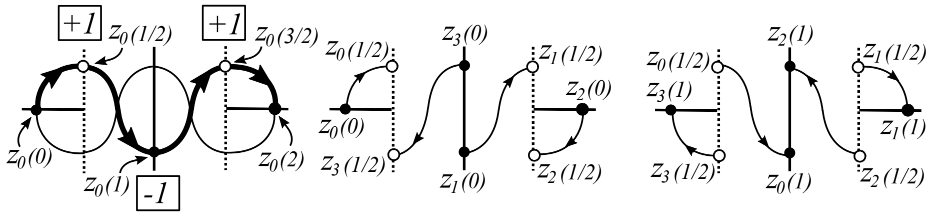


FIGURE 10. Case $N = 4$, $\boldsymbol{\omega} = (1, -1, 1)$. (1) Thick arrows indiate $z_0([0, N/2])$. (2) $\boldsymbol{z}_{\boldsymbol{\omega}}([0, 1/2])$. (3) $\boldsymbol{z}_{\boldsymbol{\omega}}([1/2, 1])$.

As seen above, each simple choreography in Theorem 3.1 travels a chain made of several loops. Moreover, Remark 3.8 implies that the number of loops is uniquely determined for each $\boldsymbol{\omega}$. For example, when $\boldsymbol{\omega} = (1, -1)$, the trajectory traces a chain made of two loops, whereas $\boldsymbol{\omega} = (1, -1, 1)$ results in a chain of three loops. More precisely, each trajectory of $\boldsymbol{z}_{\boldsymbol{\omega}}$ traces a chain consisting of $1 + |\boldsymbol{\omega}|$ loops, where $|\boldsymbol{\omega}|$ is defined by

$$|\boldsymbol{\omega}| = \# \{ j \in \{1, \ldots, N-2\} \mid \omega_j \omega_{j+1} = -1 \} .$$

In particular, if $\boldsymbol{\omega} = (1, 1, \ldots, 1)$, then $|\boldsymbol{\omega}| = 0$ and $\boldsymbol{z}_{\boldsymbol{\omega}}$ traces a circle.

As shown in Examples 3.9 and 3.10, if N is odd, then the periodic orbits $\boldsymbol{z}_{\boldsymbol{\omega}}$ for $\boldsymbol{\omega} \in \Omega_N$ are drawn based on 1-solid and 1-dotted horizontal line, $(N-1)/2$-solid and $(N-1)/2$-dotted vertical lines (Figure 9(1)). When N is even, the orbits are drawn based on 2-solid horizontal lines, $(N-2)/2$-solid, and $N/2$-dotted vertical lines (Figure 10(1)). The trajectory $\boldsymbol{z}_{\boldsymbol{\omega}}([(i/2), (i+1)/2])$ consists of N oriented curves and jumps from solid to dotted from $t = i/2$ to $t = (i+1)/2$ if i is even (see Figures 9(2) and 10(2)) and from dotted to solid if i is odd (see Figures 9(3) and 10(3)).

Remark 3.11. We provide remarks on the figure-eight and the super-eight.

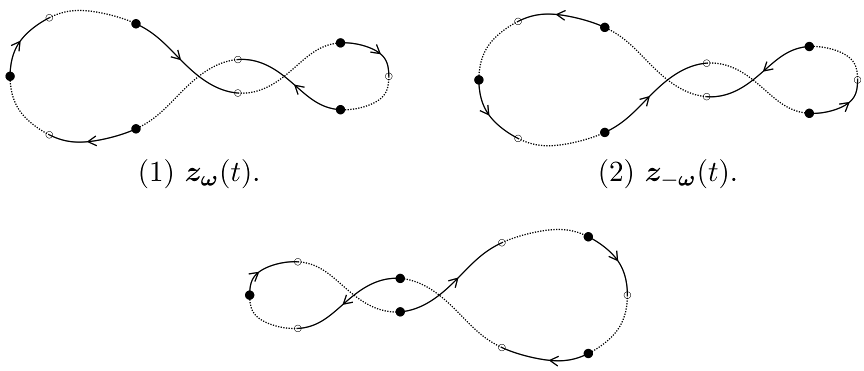
(1) The figure-eight, a simple choreography for the 3-body problem with equal masses, was discovered by Moore [Moo93] as a numerical solution and later its existence was proven mathematically by Chenciner and Montgomery [CM00] using variational methods in a function space with a certain symmetry.

(2) The existence of the super-eight, another simple choreography for the 4-body problem with equal masses, was established by several researchers. Gerver first discovered it numerically, Kapela and Zgliczyński [KZ03] provided a computer-assisted proof for its existence, and the third author [Shi14] later gave a variational proof.

(3) The reason why we write 'like the figure-eight' and 'like the super-eight' in Examples 3.9 and 3.10 is that the obtained orbits in Theorem 3.6 only have symmetry with respect to the x-axis (see (3.6)), whereas the figure-eight and super-eight have symmetry with respect to both the x- and y-axis. Moreover, it is not clear whether the vertical solid and dotted lines are evenly spaced. However, by considering additional symmetries, we obtain the same periodic solutions as the figure-eight and super-eight. See [Yu17, Section 3] for more details.

To clarify the number $2^{N-3} + 2^{\lfloor (N-3)/2 \rfloor}$ in Theorem 3.1, we define elements $-\boldsymbol{\omega}, \widehat{\boldsymbol{\omega}} \in \Omega_N$ for each $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{N-1}) \in \Omega_N$ as follows.

$$-\boldsymbol{\omega} := (-\omega_1, -\omega_2, \dots, -\omega_{N-1}), \text{ and}$$
$$\widehat{\boldsymbol{\omega}} := (\omega_{N-1}, \omega_{N-2}, \dots, \omega_1).$$



(1) $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$. (2) $\boldsymbol{z}_{-\boldsymbol{\omega}}(t)$. (3) $\boldsymbol{z}_{\widehat{\boldsymbol{\omega}}}(t)$.

FIGURE 11. Case $N = 5$, $\boldsymbol{\omega} = (1, 1, -1, 1)$.

We say that elements $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \Omega_N$ are *equivalent* and write $\boldsymbol{\omega} \sim \boldsymbol{\omega}'$ if $\boldsymbol{\omega}' \in \{\pm\boldsymbol{\omega}, \pm\widehat{\boldsymbol{\omega}}\}$. Figure 11 shows examples of simple choreographies $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ corresponding to equivalent elements $\boldsymbol{\omega}$, $-\boldsymbol{\omega}$ and $\widehat{\boldsymbol{\omega}}$. The number of elements in Ω_N up to the equivalence relation $\sim$ is given by

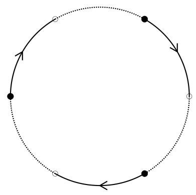
$$2^{N-3} + 2^{\lfloor (N-3)/2 \rfloor}.$$

Thus, this number represents the number of simple choreographies up to the equivalence relation $\sim$ obtained in Theorem 3.6. See Figures 12, 13, 14 and 15 in the case $N = 3, 4, 5$ and 6, respectively. Moreover, Theorem 3.1 follows immediately from Theorem 3.6.

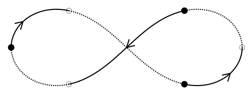
Figure 4 shows particular examples of simple choreographies $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ for $\boldsymbol{\omega} = \boldsymbol{\omega}_{\min}, \boldsymbol{\omega}_{\max}$ in the case $N = 19$.

Lemma 3.12. *Suppose that $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are equivalent. If the braid $\alpha_{\boldsymbol{\omega}}$ is pseudo-Anosov, then $\alpha_{\boldsymbol{\omega}'}$ is also pseudo-Anosov. Moreover, $\alpha_{\boldsymbol{\omega}}$ and $\alpha_{\boldsymbol{\omega}'}$ have the same stretch factor.*

Proof. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{N-1})$. Suppose that $\boldsymbol{\omega}' = -\boldsymbol{\omega}$. The braid $\alpha_{-\boldsymbol{\omega}}$ is the mirror of $\alpha_{\boldsymbol{\omega}}$, i.e., $\alpha_{-\boldsymbol{\omega}}$ is obtained from $\alpha_{\boldsymbol{\omega}}$ by changing the sign of each crossing in $\alpha_{\boldsymbol{\omega}}$. Then the assertion holds, since the pseudo-Anosov property and the stretch factor are preserved under the mirror.

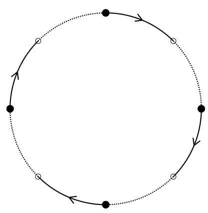


(1) $\boldsymbol{\omega} = (1,1)$.

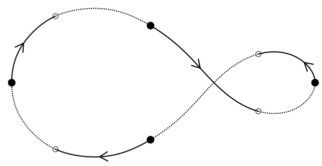


(2) $\boldsymbol{\omega} = (1,-1)$.

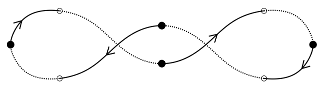
FIGURE 12. Simple choreographies $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ up to the equivalence $\sim$ in the case $N = 3$.



(1) $\boldsymbol{\omega} = (1,1,1)$.



(2) $\boldsymbol{\omega} = (1,1,-1)$.



(3) $\boldsymbol{\omega} = (1,-1,1)$.

FIGURE 13. Simple choreographies $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ up to the equivalence $\sim$ in the case $N = 4$.

Suppose that $\boldsymbol{\omega}' = -\widehat{\boldsymbol{\omega}} = (-\omega_{N-1}, -\omega_{N-2}, \dots, -\omega_1)$. Then

$$\alpha_{-\widehat{\boldsymbol{\omega}}} = \sigma_1^{-\omega_{N-1}} \sigma_2^{-\omega_{N-2}} \cdots \sigma_{N-1}^{-\omega_1}.$$

On the other hand, the inverse of $\alpha_{\boldsymbol{\omega}} = \sigma_1^{\omega_1}\sigma_2^{\omega_2} \dots \sigma_{N-1}^{\omega_{N-1}}$ is given by

$$\alpha_{\boldsymbol{\omega}}^{-1} = \sigma_{N-1}^{-\omega_{N-1}} \sigma_{N-2}^{-\omega_{N-2}} \cdots \sigma_1^{-\omega_1}.$$

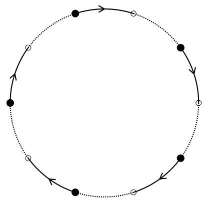
Then $\Delta \alpha_{\boldsymbol{\omega}}^{-1} \Delta^{-1} = \alpha_{-\widehat{\boldsymbol{\omega}}}$, that is $\alpha_{-\widehat{\boldsymbol{\omega}}}$ is conjugate to $\alpha_{\boldsymbol{\omega}}^{-1}$. Clearly, if $\alpha_{\boldsymbol{\omega}}$ is a pseudo-Anosov braid, then $\alpha_{\boldsymbol{\omega}}^{-1}$ is also a pseudo-Anosov braid with the same stretch factor as that of $\alpha_{\boldsymbol{\omega}}$. Hence, the assertion follows since $\alpha_{-\widehat{\boldsymbol{\omega}}}$ is conjugate to $\alpha_{\boldsymbol{\omega}}^{-1}$.

Finally, we suppose that $\boldsymbol{\omega}' = \widehat{\boldsymbol{\omega}}$. Since $\alpha_{-\widehat{\boldsymbol{\omega}}}$ is the mirror of $\alpha_{\widehat{\boldsymbol{\omega}}}$, the assertion follows from the above two cases. This completes the proof. $\square$

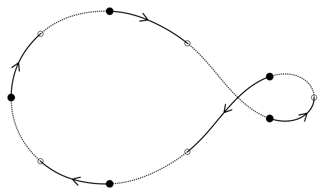
4. PROOFS

Let $Z_{\boldsymbol{\omega}}$ be the braid type of the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ for each $\boldsymbol{\omega} \in \Omega_N$. For the proof of Theorem 1.2, we first prove the following result which tells us a representative of $Z_{\boldsymbol{\omega}}$.

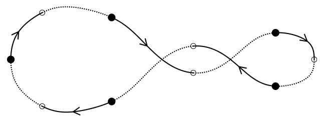
Theorem 4.1. *For each $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{N-1}) \in \Omega_N$, the primitive braid type of the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ is given by $\alpha_{-\boldsymbol{\omega}} = \sigma_1^{-\omega_1}\sigma_2^{-\omega_2} \cdots \sigma_{N-1}^{-\omega_{N-1}}$. In particular, the braid type $Z_{\boldsymbol{\omega}}$ of $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ is represented by $(\alpha_{-\boldsymbol{\omega}})^N$.*



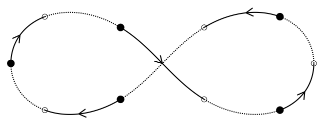
(1) $\boldsymbol{\omega} = (1, 1, 1, 1)$.



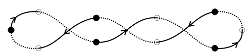
(2) $\boldsymbol{\omega} = (1, 1, 1, -1)$.



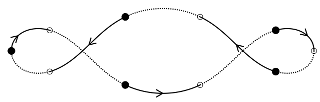
(3) $\boldsymbol{\omega} = (1, 1, -1, 1)$.



(4) $\boldsymbol{\omega} = (1, 1, -1, -1)$.



(5) $\boldsymbol{\omega} = (1, -1, 1, -1)$.



(6) $\boldsymbol{\omega} = (1, -1, -1, 1)$.

FIGURE 14. Simple choreographies $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ up to the equivalence $\sim$ in the case $N = 5$.

Proof. The simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ associated with $\boldsymbol{\omega} \in \Omega_N$ has the period N and the primitive period 1. Choosing a small $\epsilon > 0$, we consider a primitive braid $b(\boldsymbol{z}_{\boldsymbol{\omega}}([\epsilon, \epsilon + 1]))$ of $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ (see Section 2.3). Figure 16 illustrates $b(\boldsymbol{z}_{\boldsymbol{\omega}}([\epsilon, \epsilon + 1]))$ in the case $\boldsymbol{\omega} = (1, -1)$. For simplicity, we write

$$b_{[\epsilon,1+\epsilon]} := b(\boldsymbol{z}_{\boldsymbol{\omega}}([\epsilon, \epsilon + 1])).$$

We now prove that the braid type $\langle b_{[\epsilon,1+\epsilon]} \rangle$ is given by $\alpha_{-\boldsymbol{\omega}}$ (see (2.5) for the braid $\alpha_{-\boldsymbol{\omega}}$). By (3.3) and (3.4), it holds

$$z_j(0) = \bar{z}_{N-1-j}(1) = \bar{z}_{N-j}(0) \text{ for } j \in \{0, 1, \ldots, N-1\}.$$

Hence we have

$$\mathrm{Re}(z_j(0)) = \mathrm{Re}(z_{N-j}(0)) \text{ for } j \in \{0, 1, \ldots, N-1\}.$$

If $\epsilon > 0$ is small, then $\mathrm{Re}(z_j(\epsilon)) \neq \mathrm{Re}(z_{N-j}(\epsilon))$ by (3.5). Moreover by the properties of the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ (see Section 3), if $i, j \in \{0, \ldots, N-1\}$ with $i \neq j$, then $\mathrm{Re}(z_i(\epsilon)) \neq \mathrm{Re}(z_j(\epsilon))$.

Let us set $\mathrm{Re}(z_i(\epsilon)) = p_i$ and let $a_i = (p_i, 0) \in \mathbb{R} \times \{0\}$ be the projection of $z_i(\epsilon) \in \mathbb{C} \simeq \mathbb{R}^2$ on the first component. Then $\{a_0, \ldots, a_{N-1}\} \subset \mathbb{R} \times \{0\}$ is a set of N points, which is denoted by A_N. Consider the projection of the braid $b_{[\epsilon,1+\epsilon]}$ onto the xt-plane. Note that the projection contains at most double points, since the periodic orbits is a simple choreography and the closed curve $z_0([0, N])$ on which N particles lie contain at most double

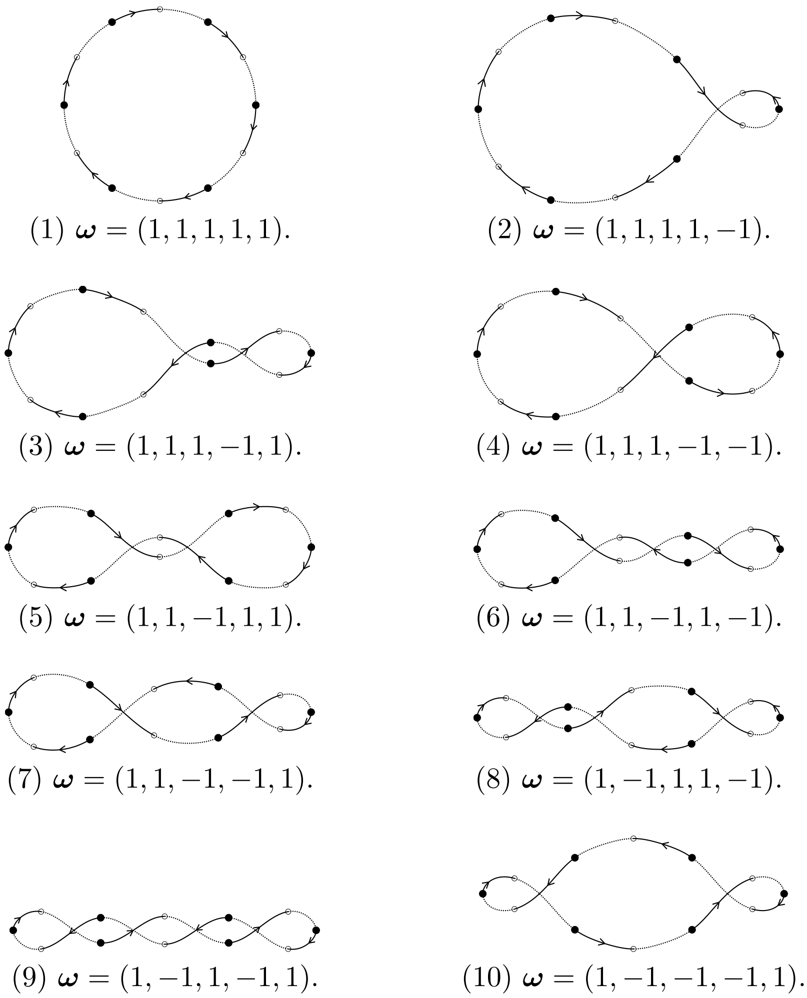


FIGURE 15. Simple choreographies $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ up to the equivalence $\sim$ in the case $N = 6$.

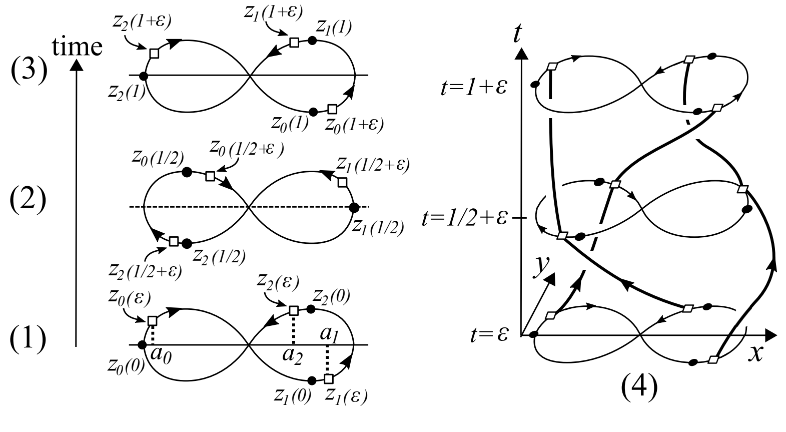


FIGURE 16. Case $N = 3$, $\boldsymbol{\omega} = (1, -1)$. (1) $\boldsymbol{z}_{\boldsymbol{\omega}}(\epsilon)$. ($a_i$ is the projection of $z_i(\epsilon)$ for $i = 0, 1, 2$.) (2) $\boldsymbol{z}_{\boldsymbol{\omega}}(\frac{1}{2} + \epsilon)$. (3) $\boldsymbol{z}_{\boldsymbol{\omega}}(1 + \epsilon) = \boldsymbol{z}_{\boldsymbol{\omega}}(\epsilon)$. (4) Braid $b_{[\epsilon, 1+\epsilon]} = b(\boldsymbol{z}_{\boldsymbol{\omega}}([\epsilon, \epsilon + 1]))$.

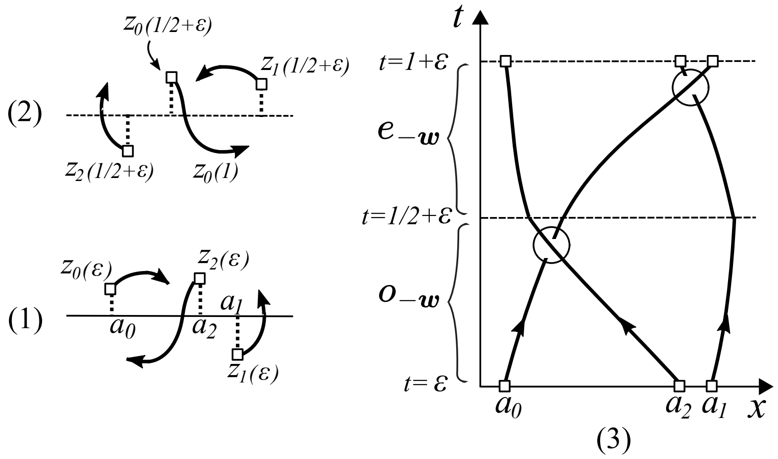


FIGURE 17. Case $N = 3$, $\boldsymbol{\omega} = (1, -1)$. (1) $\boldsymbol{z_\omega}([\epsilon, \frac{1}{2} + \epsilon])$. (2) $\boldsymbol{z_\omega}([\frac{1}{2} + \epsilon, 1 + \epsilon])$. (3) Projection $\overline{b_{[\epsilon,1+\epsilon]}}$ on the xt-plane. $\overline{b_{[\epsilon,1+\epsilon]}} = e_{-\boldsymbol{\omega}} \cdot o_{-\boldsymbol{\omega}} = \sigma_2 \cdot \sigma_1^{-1}$ in this case. Small circles indicate the double points.

points. At each double point of the projection, we indicate the over/under crossings determined by the braid $b_{[\epsilon,1+\epsilon]}$, as shown in Figure 17(3). Then the result is a braid (with base points A_N) denoted by $\overline{b_{[\epsilon,1+\epsilon]}} = \overline{b(\boldsymbol{z_\omega}([\epsilon, \epsilon + 1]))}$. We also call the braid $\overline{b_{[\epsilon,1+\epsilon]}}$ the *projection* of $b_{[\epsilon,1+\epsilon]}$. Note that $\overline{b_{[\epsilon,1+\epsilon]}}$ has the same braid type as $b_{[\epsilon,1+\epsilon]}$.

Claim. For each $\boldsymbol{\omega} \in \Omega_N$ the braids $\overline{b_{[\epsilon,1+\epsilon]}}(= \overline{b(\boldsymbol{z_\omega}([\epsilon, \epsilon + 1]))})$ and $\alpha_{-\boldsymbol{\omega}}$ are conjugate in B_N.

Proof of Claim. We consider the trajectory $\boldsymbol{z_\omega}([0, \frac{N}{2}])$. See Figure 18. We focus on the motion of 0th particle $z_0(t)$. Recall that $z_0(t)$ satisfies the two conditions (1) and (2) in Theorem 3.6. Between $t = 0$ and $t = \frac{N}{2}$, the 0th particle passes by all other $N - 1$ particles. More precisely, for each $j = 1, \ldots, N-1$, when $z_0(t)$ and the $(N-j)$th particle $z_{N-j}(t)$ pass each other, they lie on the same vertical line at $t = \frac{j}{2}$, i.e., $\mathrm{Re}(z_0(j/2)) = \mathrm{Re}(z_{N-j}(j/2))$. The $\boldsymbol{\omega}$-topological constraints (Definition 3.5) tell us the following inequality about the imaginary part:

$$\begin{array}{lcll} \mathrm{Im}(z_0(\frac{j}{2})) & > & \mathrm{Im}(z_{N-j}(\frac{j}{2})) & \text{if } \omega_j = 1, \\ \mathrm{Im}(z_0(\frac{j}{2})) & < & \mathrm{Im}(z_{N-j}(\frac{j}{2})) & \text{if } \omega_j = -1. \end{array}$$

This means that $z_0(t)$ passes over $z_{N-j}(t)$ at $t = \frac{j}{2}$ if $\omega_j = 1$. Otherwise, it passes under $z_{N-j}(t)$ at $t = \frac{j}{2}$. These together with the choreographic condition (3.3) enables us to read off the braid word from $\overline{b_{[\epsilon,1+\epsilon]}}$, as we will see now.

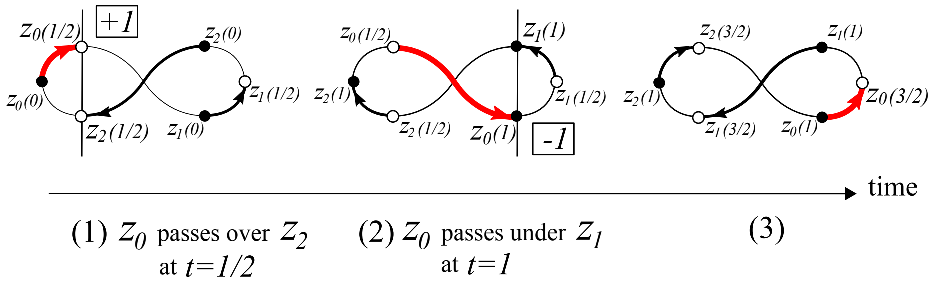


FIGURE 18. $\boldsymbol{z}_{\boldsymbol{\omega}}([0, \frac{N}{2}])$ when $N = 3$, $\boldsymbol{\omega} = (1, -1)$. In this case, the 0th particle $z_0(t)$ passes over $z_2(t)$ at $t = \frac{1}{2}$. Then it passes under $z_1(t)$ at $t = 1$. (1) $\boldsymbol{z}_{\boldsymbol{\omega}}([0, \frac{1}{2}])$. (2) $\boldsymbol{z}_{\boldsymbol{\omega}}([\frac{1}{2}, 1])$. (3) $\boldsymbol{z}_{\boldsymbol{\omega}}([1, \frac{3}{2}])$.

Recall that $o_{\boldsymbol{\omega}}$, $e_{\boldsymbol{\omega}}$ are braids for $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{N-1}) \in \Omega_N$ as in (2.6). We claim that $\overline{b_{[\epsilon,1+\epsilon]}}$ is written by

$$\overline{b_{[\epsilon,1+\epsilon]}} = e_{-\boldsymbol{w}} \cdot o_{-\boldsymbol{w}} = \prod_{\substack{i \in \{1,\dots,N-1\} \\ i \text{ even}}} \sigma_i^{-\omega_i} \prod_{\substack{j \in \{1,\dots,N-1\} \\ j \text{ odd}}} \sigma_j^{-\omega_j}.$$

See (2.6) for braids $e_{-\boldsymbol{w}}$ and $o_{-\boldsymbol{w}}$. In fact, if j is odd and $\omega_j = 1$ (resp. $\omega_j = -1$), then the braid word σ_j^{-1} (resp. σ_j^{+1}) in $o_{-\boldsymbol{w}}$ appears in $\overline{b_{[\epsilon,1+\epsilon]}}$ by the fact that $z_0(t)$ and $z_{N-j}(t)$ meet at the same vertical line at $t = \frac{j}{2}$ with the inequality $\mathrm{Im}(z_0(\frac{j}{2})) > \mathrm{Im}(z_{N-j}(\frac{j}{2}))$ (resp. $\mathrm{Im}(z_0(\frac{j}{2})) < \mathrm{Im}(z_{N-j}(\frac{j}{2}))$) Similarly, if i is even and $\omega_i = 1$ (resp. $\omega_i = -1$), then the braid word σ_i^{-1} (resp. σ_i^{+1}) in $e_{-\boldsymbol{\omega}}$ appears in $\overline{b_{[\epsilon,1+\epsilon]}}$. See Figures 17(3).

By Lemma 2.5, $e_{-\boldsymbol{w}} o_{-\boldsymbol{w}} (= \overline{b_{[\epsilon,1+\epsilon]}})$ is conjugate to $\alpha_{-\boldsymbol{\omega}}$. This completes the proof of Claim.

By Claim, the primitive braid type of the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ is given by $\alpha_{-\boldsymbol{\omega}}$. Thus the braid type $Z_{\boldsymbol{\omega}}$ of $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$ is given by $(\alpha_{-\boldsymbol{\omega}})^N$. This completes the proof of Theorem 4.1. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Given $\boldsymbol{\omega} \in \Omega_N$, we consider the simple choreography $\boldsymbol{z}_{-\boldsymbol{\omega}}(t)$ corresponding to $-\boldsymbol{\omega} \in \Omega_N$. By Theorem 4.1, $\alpha_{-(-\boldsymbol{\omega})} = \alpha_{\boldsymbol{\omega}}$ (resp. the Nth power $(\alpha_{\boldsymbol{\omega}})^N$) represents the primitive braid type (resp. braid type $Z_{-\boldsymbol{\omega}}$) of the solution $\boldsymbol{z}_{-\boldsymbol{\omega}}(t)$. This together with Theorem 3.6 implies the former statement of Theorem 1.2.

We turn to the latter statement. By Lemma 3.12, we may assume that $\boldsymbol{\omega} = (1, \omega_2, \dots, \omega_{N-1}) \in \Omega_N^+$. Suppose that $\boldsymbol{\omega} = (1, 1, \dots, 1)$. In this case, by Example 2.3(2) and Lemma 2.7, the braid type $Z_{\boldsymbol{\omega}}$ is periodic.

Suppose that there exists $i \in \{2, \dots, N-1\}$ such that $\omega_i = -1$. Let $\Theta : \Psi_{N-1} \to \Omega_N^+$ be the bijection given in Lemma 2.2. Then $\Theta^{-1}(\boldsymbol{\omega}) \in \Psi_{N-1}$ is a composition of $N-1$ corresponding to the $(k+1)$-tuple of positive integers with $k > 0$. Recall that $\beta_{\Theta^{-1}(\boldsymbol{\omega})} = \alpha_{\boldsymbol{\omega}}$ by Lemma 2.9, and this is a pseudo-Anosov braid by Theorem 2.8.

The Nth power $(\alpha_{\boldsymbol{\omega}})^N$ that represents the braid type $Z_{\boldsymbol{\omega}}$ is also pseudo-Anosov, since the pseudo-Anosov property is preserved under the power. This completes the proof. $\square$

We recall the elements $\boldsymbol{\omega}_{\max}, \boldsymbol{\omega}_{\min} \in \Omega_N$ defined in Section 1. The element $\boldsymbol{\omega}_{\min}$ can be written by $\boldsymbol{\omega}_{\min} = \Theta(\boldsymbol{m})$ by using the bijection $\Theta : \Psi_{N-1} \to \Omega_N^+$, where $\boldsymbol{m} = (n,n) \in \Psi_{N-1}$ if $N = 2n+1$ and $\boldsymbol{m} = (n, n-1) \in \Psi_{N-1}$ if $N = 2n$.

Proof of Theorem 1.3. Suppose that $\boldsymbol{\omega} \in \Omega_N$ is neither $(1,1,\dots,1)$ nor $(-1,-1,\dots,-1)$. Then the braid $\alpha_{\boldsymbol{\omega}} \in B_N$ is pseudo-Anosov by Lemmas 2.9 and 3.12. By Lemma 3.12 again, $\alpha_{\boldsymbol{\omega}}$ and $\alpha_{-\boldsymbol{\omega}}$ have the same stretch factor. By definitions of $\boldsymbol{\omega}_{\max}$ and $\boldsymbol{\omega}_{\min}$, we have $\alpha_{\boldsymbol{\omega}_{\max}} = \beta_{\boldsymbol{1}_{N-1}}$. Moreover $\alpha_{\boldsymbol{\omega}_{\min}} = \beta_{(n,n)}$ when $N = 2n+1$ (resp. $\alpha_{\boldsymbol{\omega}_{\min}} = \beta_{(n,n-1)}$ when $N = 2n$). The assertion follows from Theorems 2.10 and 4.1. $\square$

Example 4.2. Suppose that $\boldsymbol{\omega} = (1,-1) \in \Omega_3$. The braid type of the figure-eight solution of the planar Newtonian 3-body problem is the same as that of the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$. See Remark 3.11(1)(3). Hence, $(\alpha_{-\boldsymbol{\omega}})^3 = (\sigma_1^{-1}\sigma_2)^3$ is a representative of the braid type of the figure-eight solution. By (2.3), its stretch factor $\lambda((\sigma_1^{-1}\sigma_2)^3)$ equals $(\lambda(\sigma_1^{-1}\sigma_2))^3 = (\frac{3+\sqrt{5}}{2})^3$.

Corollary 4.3. *The super-eight solution of the planar Newtonian* 4*-body problem has the pseudo-Anosov braid type with the stretch factor* $(2+\sqrt{3})^4$.

Proof. We take $\boldsymbol{\omega} = (1,-1,1) \in \Omega_4$. Figure 19 illustrates the braid $\overline{b_{[\epsilon,1+\epsilon]}}$ corresponding to the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$. The (primitive) braid type of the super-eight solution for the planar 4-body problem is the same as that of $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$. See Remark 3.11(2)(3). Thus the braid type of the super-eight is pseudo-Anosov by Theorem 1.2. The primitive braid type of the super-eight is given by $\alpha_{-\boldsymbol{\omega}} = \sigma_1^{-1}\sigma_2\sigma_3^{-1}$. By Lemma 3.12, braids $\alpha_{-\boldsymbol{\omega}}$ and $\alpha_{\boldsymbol{\omega}} = \beta_{(1,1,1)}$ have the same stretch factor that is equal to $2+\sqrt{3}$ (see Example 2.14(1)). Therefore, by (2.3), the braid type of the super-eight has the stretch factor $(2+\sqrt{3})^4$. $\square$

Proof of Corollary 1.4. We consider the case $N = 4$, $\boldsymbol{\omega} = (1,-1,-1)$. Figure 20 illustrates the braid $\overline{b_{[\epsilon,1+\epsilon]}}$ corresponding to $\boldsymbol{z}_{\boldsymbol{\omega}}(t)$. See also Figure 2. The primitive braid type of the simple choreography $\boldsymbol{z}_{\boldsymbol{\omega}}$ is given by $\alpha_{-\boldsymbol{\omega}} = \sigma_1^{-1}\sigma_2\sigma_3$. If we take $-\boldsymbol{\omega} = (-1,1,1) \in \Omega_4$, the braid $\alpha_{-(-\boldsymbol{\omega})} = \alpha_{\boldsymbol{\omega}} = \sigma_1\sigma_2^{-1}\sigma_3^{-1}$ represents the primitive braid type of the simple choreography $\boldsymbol{z}_{-\boldsymbol{\omega}}(t)$ corresponding to $-\boldsymbol{\omega}$. Since by Lemma 3.12, braids $\alpha_{-\boldsymbol{\omega}}$ and $\alpha_{\boldsymbol{\omega}} = \beta_{(1,2)}$ have the same stretch factor, Theorem 2.8 tells us that $\lambda(\alpha_{-\boldsymbol{\omega}}) = \lambda(\beta_{(1,2)}) \approx 2.2966$ is the largest real root of

$$F_{(1,2)}(t) = (t+1)(t^4 - 2t^3 - 2t + 1),$$

that is the largest real root of the degree 4 polynomial $t^4 - 2t^3 - 2t + 1$. Thus, the simple choreography $\boldsymbol{z}_{-\boldsymbol{\omega}}(t)$ has the desired properties. $\square$

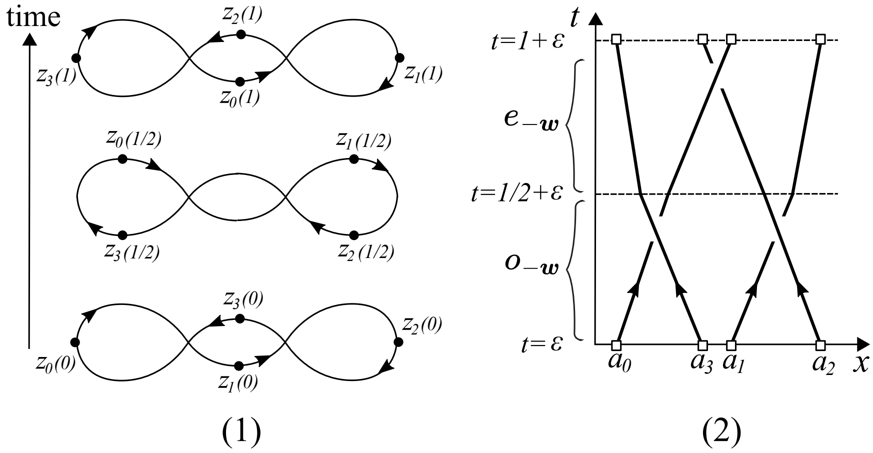


FIGURE 19. Case $N = 4$, $\boldsymbol{\omega} = (1, -1, 1)$. (1) From the bottom to the top: $\boldsymbol{z}_{\boldsymbol{\omega}}(0)$, $\boldsymbol{z}_{\boldsymbol{\omega}}(\frac{1}{2})$ and $\boldsymbol{z}_{\boldsymbol{\omega}}(1) = \boldsymbol{z}_{\boldsymbol{\omega}}(0)$. (2) $\overline{b_{[\epsilon,1+\epsilon]}} = e_{-\boldsymbol{\omega}} \cdot o_{-\boldsymbol{\omega}} = \sigma_2 \cdot \sigma_1^{-1}\sigma_3^{-1}$ in this case.

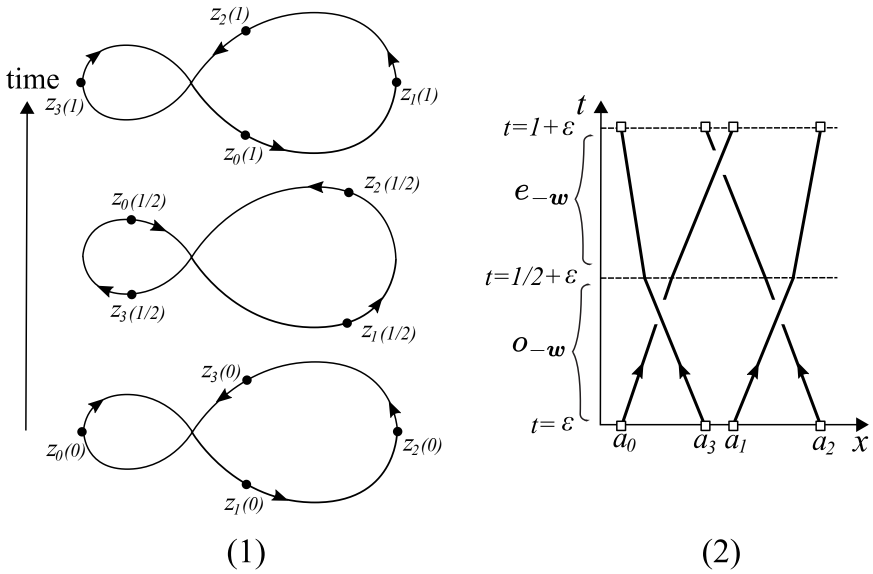


FIGURE 20. Case $N = 4$, $\boldsymbol{\omega} = (1, -1, -1)$. (1) From the bottom to the top: $\boldsymbol{z}_{\boldsymbol{\omega}}(0)$, $\boldsymbol{z}_{\boldsymbol{\omega}}(\frac{1}{2})$ and $\boldsymbol{z}_{\boldsymbol{\omega}}(1) = \boldsymbol{z}_{\boldsymbol{\omega}}(0)$. (2) $\overline{b_{[\epsilon,1+\epsilon]}} = e_{-\boldsymbol{\omega}} \cdot o_{-\boldsymbol{\omega}} = \sigma_2 \cdot \sigma_1^{-1}\sigma_3$ in this case.

5. CONCLUSION

We notice that if $b \in B_N$ is a primitive braid of some simple choreography of the planar N-body problem, then the permutation $\hat{s}(b) \in S_N$ is cyclic (see (2.1) for the definition of the homomorphism $\hat{s} : B_N \to S_N$). The following question is a choreographic version of Question 1.1.

Question 5.1. *Let b be a braid with N strands whose permutation $\hat{s}(b)$ is cyclic. Is the braid type $\langle b\rangle$ given by a primitive braid of a simple choreography of the planar Newtonian N-body problem?*

Theorem 1.2 tells us that Question 5.1 is true if a braid b is of the form $b = \alpha_{\boldsymbol{\omega}}$ for $\boldsymbol{\omega} \in \Omega_N$. We are far from solving Questions 5.1 and 1.1, yet we present an interesting example in this work. In fact, the simple choreography of the planar 4-body problem which satisfies the statement of Corollary 1.4 is intriguing in the sense that the braid $\sigma_1\sigma_2^{-1}\sigma_3^{-1}$ reaches the minimal stretch factor among all pseudo-Anosov braids with 4 strands [SKL02].

Table 1 in Section 2.6 shows that $\min_{\beta\in Y_5\setminus\{\beta_{(4)}\}} \lambda(\beta) = \lambda(\beta_{(2,2)}) \approx 2.01536$. We note that there exists a pseudo-Anosov 5-braid whose stretch factor is smaller than that of the 5-braid $\beta_{(2,2)}$. Ham-Song [HS07] proved that $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2$ is a pseudo-Anosov braid which realizes the minimal stretch factor $\delta_5 \approx 1.7220$ among all pseudo-Anosov 5-braids, and δ_5 is the largest real root of $t^4 - t^3 - t^2 - t + 1$.

We finally ask the following question.

Question 5.2. *Does there exist a simple choreography of the planar Newtonian 5-body problem whose primtive braid type is given by a pseudo-Anosov 5-braid with the minimal stretch factor $\delta_5 \approx 1.7220$?*

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