REPRESENTATIONS OF $GSp(4,\mathbf{R})$ WITH EMPHASIS ON DISCRETE SERIES

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Abstract. In this note, we survey some elementary aspects of the representation theory of the real reductive groups $GSp(4,\mathbf{R})$ or $Sp(4,\mathbf{R})$. We also discuss automorphic forms on $Sp(4,\mathbf{R})$ which generate discrete series representations. *Caution:* Some terminologies such as "tempered" are used before we recall their definitions. We use some technical terms without giving detailed explanation, for which we refer the reader to standard textbooks such as [Kn], [Wa-1] or the survey article [N].

Contents

§1. The Langlands classification
§2. Harish-Chandra's parameterization of discrete series representations
§3. Automorphic forms on $Sp(4,\mathbf{R})$ of discrete series type

§1. The Langlands classification

In this section, we shall explain the classification theory of irreducible admissible representations of a reductive linear Lie group G, which is known as *Langlands Classification*. Before discussing the general theory, let us look at the simplest example $G = SL(2,\mathbf{R})$.

$\boxed{\textbf{Ex. 1.}\ G = SL(2,\mathbf{R})}$ We fix an Iwasawa decomposition $G = NAK$ of G as follows:

$$N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} | x \in \mathbf{R} \right\}, \qquad A := \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} | y > 0 \right\},$$

$$K := \left\{ r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} | \theta \in \mathbf{R} \right\}.$$

We introduce two kinds of irreducible representations of G – (i) (limits of) discrete series representations and (ii) (non-unitary) principal series representations.

(i) (limits of) discrete series. As a $\mathbf{C}$-basis of $\mathfrak{sl}_2(\mathbf{C})$, we take

$$H := \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad X_\pm := \frac{1}{2} \begin{pmatrix} 1 & \pm\sqrt{-1} \\ \pm\sqrt{-1} & -1 \end{pmatrix}.$$

Then $\{H, X_+, X_-\}$ is an sl_2-triplet. For each $k \in \mathbf{C}$, consider the Verma module

$$M(k) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} \mathbf{C}_k \qquad \text{with} \qquad \bar{\mathfrak{b}} = \mathbf{C}\cdot H \oplus \mathbf{C}\cdot X_-.$$

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Here $\mathbf{C}_k = \mathbf{C}v_k$ is a one dimensional $\bar{\mathfrak{b}}$-module characterized by $H \cdot v_k = kv_k$ and $X_- \cdot v_k = 0$. Then it is well-known that $M(k)$ is unitarizable if $k \in \mathbf{Z}_{>0}$. That is, there exists an irreducible unitary representation D_k^+ of G such that its underlining $(\mathfrak{g}, K)$-module is $M(k)$. We denote by D_k^- the contragredient representation of D_k^+, which has the highest weight $-k$. If $k \geq 2$, every matrix coefficient of D_k^+ belongs to $L^2(G)$. The representation $D_k^\pm$ $(k \geq 2)$ is called the discrete series representation with Blattner parameter $\pm k$. The representations $D_1^\pm$ are the limits of discrete series representations. For a congruence subgroup Γ of G and $k \geq 1$, it is a standard fact (see §3) that

$$M_k(\Gamma) \cong \mathrm{Hom}_{\mathfrak{g},K}(D_k^+, \mathcal{A}(\Gamma\backslash G));$$
$$S_k(\Gamma) \cong \mathrm{Hom}_{\mathfrak{g},K}(D_k^+, \mathcal{A}^{cusp}(\Gamma\backslash G)) \cong \mathrm{Hom}_G(D_k^+, L^2(\Gamma\backslash G)).$$

Here $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) is the space of modular forms (resp. cusp forms) on $\mathbf{H}_1 := \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ with respect to Γ.

(ii) (non-unitary) principal series. We set

$$B := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G \right\} = MAN \text{ with } M := \{\pm I_2\} \qquad \text{(Borel subgroup)} .$$

For each $\epsilon : M \to \mathbf{C}^\times$ and $\nu \in \mathbf{C}$, the (non-unitary) principal series representation of G is given by

$$I(\epsilon, \nu) := \{f : G \overset{C^\infty}{\to} \mathbf{C} \mid f(\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} g) = \epsilon(a/|a|)|a|^{\nu+1} f(g),\ \forall \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \in B, \forall g \in G\}.$$

Then we have the following exact sequence:

$$(1.1) \qquad 0 \to D_k^+ \oplus D_k^- \to I((-1)^k, k-1) \to F_{k-2} \to 0, \quad k \geq 2.$$

Moreover we have

$$I(-, 0) \cong D_1^+ \oplus D_1^-.$$

Here for a non-negative integer $m \geq 0$, $F_m = \mathrm{Sym}^m(\mathbf{C}^2)$ stands for the m-th symmetric tensor representation of the natural (tautological) two dimensional representation of G. Here is the classification:

Classification for $SL(2, \mathbf{R})$. Let $\widehat{G}_{adm}$ be the set of (infinitesimal equivalent classes of) irreducible admissible representations of G. The set of irreducible tempered representations of G (resp. irreducible discrete series representations of G) is denoted by $\widehat{G}_{temp}$ (resp. $\widehat{G}_{DS}$). Note that

$$\widehat{G}_{DS} \subset \widehat{G}_{temp} \subset \widehat{G}_{adm}.$$

Then we have

(i) $\widehat{G}_{adm} \setminus \widehat{G}_{temp} = \{I(+;\nu) \mid \mathrm{Re}(\nu) > 0, \nu \neq 1, 3, 5, \cdots\} \cup \{I(-;\nu) \mid \mathrm{Re}(\nu) > 0, \nu \neq 2, 4, 6, \cdots\} \cup \{F_m \mid m \geq 0\}$;

(ii) $\widehat{G}_{temp} \setminus \widehat{G}_{DS} = \{D_1^+, D_1^-\} \cup \{I(+;\nu) \mid \nu \in \sqrt{-1}\mathbf{R}\} \cup \{I(-;\nu) \mid \nu \in \sqrt{-1}\mathbf{R} \setminus \{0\}\}$;

(iii) $\widehat{G}_{DS} = \{D_k^+, D_k^- \mid k \geq 2\}$.

Moreover the set $\widehat{G}_u$ of unitary equivalence classes of irreducible unitary representations of G is given by $\widehat{G}_u = \widehat{G}_{temp} \cup \{I(+;\nu) \mid 0 < \nu < 1\} \cup \{F_0\}$. The family $\{I(+;\nu) \mid 0 < \nu < 1\}$

of irreducible non-tempered unitary representations is called the *complementary series.*

Remarks (i) *Translation functor* ([Kn, Ch X, §9]). We can "jump" from a representation D_k^+ to another D_l^+ in the following sense. For $k \geq 2$ we have

$$D_k^+ \otimes F_1 \cong D_{k+1}^+ \oplus D_{k-1}^+.$$

Hence we have

$$D_{k-1}^+ = \{v \in D_k^+ \otimes F_1 \mid \Omega v = \{(k-2)^2 - 1\}v\}, \quad (\Omega\text{: Casimir element}).$$

That is, we obtain D_{k-1}^+ from D_k^+ considering the tensor product with a finite dimensional representation of G. Hence $D_1^\pm$ is the "limit" of $\{D_k^\pm \mid k \geq 2\}$.

(ii) *Selberg's* $1/4$ *conjecture.* Let $f : \Gamma\backslash G/K \to \mathbf{C}$ be a wave cusp form with respect to a congruence subgroup Γ. Then f generates an irreducible unitary representation $\pi_f \cong I(+,\nu)$ with $\nu \in \sqrt{-1}\mathbf{R} \cup (0,1)$. This gives

$$\Delta f(z) \equiv -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)f(z) = \frac{1}{4}(1-\nu^2)f(z).$$

The conjecture asserts that $\frac{1}{4}(1-\nu^2) \geq 1/4$ should hold, which is equivalent to that π_f should be tempered (i.e. $\nu \in \sqrt{-1}\mathbf{R}$). Note that this is an archimedean analogue of the Ramanujan conjecture.

General case The classification in the general case goes as follows:

$$\widehat{G}_{adm} := \{\text{irreducible admissible representations of } G\}/\sim$$

$$\Downarrow \text{Step1: Theorem 1.2}$$

$$\widehat{M}_{temp} := \{\text{irreducible tempered representations of } M\}/\sim$$

$$P = MAN(\subseteq G) : \text{a parabolic subgroup of } G$$

$$\Downarrow \text{Step2: Theorem 1.3}$$

$$\widehat{M}_{1,DS} := \{\text{discrete series representations of } M_1\}/\sim$$

$$P_1 = M_1A_1N_1(\subseteq M) : \text{a parabolic subgroup of } M.$$

Def-Prop 1.1. *Let* $(\pi, H_\pi) \in \widehat{G}_u$ *be an irreducible unitary representation of* G.
(i) π *is tempered* $:\Leftrightarrow$ *the global character* Θ_π *of* π *can be extended to the Schwartz space* $\mathcal{C}(G)$. $\Leftrightarrow$ *Every* K*-finite matrix coefficient of* π *belongs to* $\cap_{\epsilon>0}L^{2+\epsilon}(G)$.
(ii) π *belongs to the discrete series* $:\Leftrightarrow$ $\exists$ *a non-zero matrix coefficient of* G *belongs to* $L^2(G)$. $\Leftrightarrow$ *Every matrix coefficient of* G *belongs to* $L^2(G)$.

Let P be a parabolic subgroup of G with Langlands decomposition $P = MAN$. For each $\mu \in \mathfrak{a}_{\mathbf{C}}^*$, we denote the corresponding quasi-character of A by a^μ. We define $\rho \equiv \rho_P \in \mathfrak{a}_{\mathbf{C}}^*$ by

$$a^\rho = \det(\mathrm{Ad}(a)|_{\mathrm{Lie}(N)})^{1/2}.$$

Theorem 1.2 (Langlands, c.f. [L],[Wa, 5.4.4])**.** *For* $\forall\pi \in \widehat{G}_{adm}$, $\exists P = MAN \subseteq G$, $\exists\sigma \in \widehat{M}_{temp}$, $\exists\nu \in \mathfrak{a}_{\mathbf{C}}^*$ *with* $\mathrm{Re}(\nu)$ *belonging to the open Weyl Chamber defined from* N *such that* π *is infinitesimally equivalent to the unique irreducible quotient of* $I(P;\sigma,\nu) \equiv \mathrm{Ind}_P^G(\sigma \otimes$

$a^{\nu+\rho} \otimes 1_N)$, *which we denote by* $J(P;\sigma,\nu)$. *Moreover the triplet* (P,σ,ν) *is unique up to conjugation.*

Theorem 1.3 (cf. [Wa, 5.2.5]). *For* $\forall\sigma \in \widehat{M}_{temp}$, $\exists P_1 = M_1A_1N_1 \subseteq M$, $\exists\sigma \in \widehat{M}_{1,DS}$, $\exists\nu \in \sqrt{-1}\mathfrak{a}^*$ *such that* σ *is unitarily equivalent to a direct summand of* $I(P_1;\sigma_1,\nu)$.

Note that there exists a parabolic subgroup P' of G with $P' \subseteq P \subseteq G$ such that $P' \cap M = P_1$. Hence we have

Corollary 1.4. *For* $\forall\pi \in \widehat{G}_{adm}$, $\exists P = MAN \subseteq G$, $\exists\sigma \in \widehat{M}_{DS}$, $\exists\nu \in \mathfrak{a}^*_{\mathbf{C}}$ *with* $\mathrm{Re}(\nu)$ *belonging to the closed Weyl Chamber defined from* N *such that* π *is infinitesimally equivalent to an irreducible quotient of* $I(P;\sigma,\nu)$.

We also note

Theorem 1.5 (c.f. [Wa, 3.8]). *(Casselman's subrepresentation theorem).*
Let $P_0 = M_0A_0N_0 \subset G$ *be a minimal parabolic subgroup of* G. *For* $\forall\pi \in \widehat{G}_{adm}$, $\exists(\sigma,\nu) \in \widehat{M}_{0,u} \times \mathfrak{a}^*_{0,\mathbf{C}}$ *such that* π *can be embedded into* $I(P_0;\sigma,\nu)$ *as* $(\mathfrak{g},K)$*-modules.*

The classification of irreducible *unitary* representations of general G is unknown except for a few cases. This is a big open problem in the representation theory of real reductive groups.

$\boxed{\textbf{Ex. 2.}\ G = Sp(4,\mathbf{R})}$ Up to conjugation, $Sp(4,\mathbf{R})$ has 3 proper parabolic subgroups: P_0, P_S, and P_J.

(i) The minimal parabolic subgroup. $P_0 = M_0A_0N_0$.

$$M_0 = \{\mathrm{diag}(\epsilon_1,\epsilon_2,\epsilon_1,\epsilon_2) \mid \epsilon_i = \pm 1\}; \quad A_0 = \{\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1}) \mid a_i > 0\};$$

$$N_0 = \left\{ n(x_0,x_1,x_2,x_3) := \left(\begin{array}{cc|cc} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & x_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array}\right) \in G \right\}.$$

(ii) The Siegel parabolic subgroup. $P_S = M_SA_SN_S$.

$$M_S = \left\{ \left(\begin{array}{c|c} m & \\ \hline & {}^t m^{-1} \end{array}\right) | m \in SL^{\pm}(2,\mathbf{R}) \right\}; \quad A_S = \{\mathrm{diag}(t,t,t^{-1},t^{-1}) \mid t > 0\};$$

$$N_S = \left\{ \left(\begin{array}{c|c} 1_2 & x \\ \hline & 1_2 \end{array}\right) \mid {}^t x = x \in M_2(\mathbf{R}) \right\}.$$

(iii) The Jacobi (or Klingen) parabolic subgroup. $P_J = M_JA_JN_J$.

$$M_J = \left\{ \left(\begin{array}{cc|cc} \epsilon & & & \\ & a & & b \\ \hline & & \epsilon & \\ & c & & d \end{array}\right) \Big| \epsilon = \pm 1, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R}) \right\};$$

$$A_J = \left\{ \mathrm{diag}(a_1,1,a_1^{-1},1) \mid a_1 > 0 \right\}; \quad N_J = \{n(x_0,x_1,x_2,0) \mid x_i \in \mathbf{R}\}.$$

• As an example, we consider the *degenerate principal series* representation $I(P_S; \text{triv.}, \nu)$. The representation space of $I(P_S; \text{triv.}, \nu)$ is given by

$$\{f : G \overset{C^\infty}{\to} \mathbf{C} \mid f(\left(\begin{array}{c|c} m & \\ \hline & {}^t m^{-1} \end{array}\right) g) = |\det(m)|^{\nu+3} f(g), \forall (m,g) \in GL(2,\mathbf{R}) \times G\},$$

on which G acts by right translation. It follows from the exact sequence (1.1) that the representation $I(P_S; \text{triv.}, \nu)$ is a quotient of

$$\begin{aligned} &I(P_0; \text{triv.}, \tfrac{\nu+1}{2}, \tfrac{\nu-1}{2}) \\ :=&\{f : G \overset{C^\infty}{\to} \mathbf{C} \mid f(mang) = a_1^{(\nu+1)/2+2} a_2^{(\nu-1)/2+1} f(g), \ \ \forall (m,a,n,g) \in M_0 \times A_0 \times N_0 \times G\}. \end{aligned}$$

For ν in a general position, $I(P_S; \text{triv.}, \nu)$ is irreducible. Hence, if such ν satisfies $\mathrm{Re}(\nu) > 0$, then we have

$$I(P_S; \text{triv.}, \nu) \cong J(P_0; \text{triv.}, \tfrac{\nu+1}{2}, \tfrac{\nu-1}{2}).$$

The degenerate principal series representation $I(P_S; \text{triv.}, \nu)$ and its p-adic counterparts are the local components of the Siegel Eisenstein series. Partly because of this, the degenerate principal series representations are studied by many authors.

In summary, the Langlands classification reduces the classification of irreducible admissible representations of a reductive linear Lie group G to that of discrete series representations of a semisimple part M of a parabolic subgroup $P = MAN$ of G. In the next section we shall take up the latter problem.

§2. Harish-Chandra's parameterization of discrete series

The basic setting of this section is as follows:

• G: a connected semisimple algebraic group $/\mathbf{R}$.
• $G := \mathsf{G}_{\mathbf{R}}$: the group of $\mathbf{R}$-valued points of G.
Assume that G is simply connected. (This implies that G is a connected Lie group.)
• $K \subset G$: a fixed maximal compact subgroup of G.

The passage to the case of reductive G is not difficult. The first important fact is the following:

Theorem 2.1 (Harish-Chandra, c.f. [Kn, Theorems 12.20])**.** *The set $\widehat{G}_{DS}$ is not empty if and only if* $\mathrm{rk}(G) = \mathrm{rk}(K)$.

From now on, we assume that $\mathrm{rk}(G) = \mathrm{rk}(K)$. Then a Cartan subgroup T of K is also a Cartan subgroup of G. We fix some notation:
• $\Delta = \Delta(G,T)$: the root system;
• W_G: the Weyl group of Δ;
• $\Delta_c = \Delta(K,T) = \Delta_c^+ \sqcup \Delta_c^-$: the set of compact roots;
• W_K: the Weyl group of Δ_c;
• $\tau_\mu \in \widehat{K}$: the irreducible representation of K with highest weight $\mu \in \sqrt{-1}\mathfrak{t}^*$ w.r.t Δ_c^+;

• $P(\Delta) := \{\lambda \in \mathfrak{t}_{\mathbf{C}}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbf{Z}\}$: the weight lattice of Δ.

Now we can state the following parameterization theorem of discrete series representations:

Theorem 2.2 (Harish-Chandra, c.f. [Kn, Theorems 12.21]). *The set $\widehat{G}_{DS}$ is parameterized by the set*

$$\Xi := \{\lambda \in P(\Delta) \mid \langle \lambda, \alpha^\vee \rangle \neq 0 \ (\forall \alpha \in \Delta), \ \textit{and} \ \langle \lambda, \alpha^\vee \rangle > 0 \ (\forall \alpha \in \Delta_c^+)\}.$$

The discrete series representation π_λ ($\lambda \in \Xi$) is characterized by the following properties:
(i) *The infinitesimal character of π is given by $\lambda \in \mathfrak{t}_{\mathbf{C}}^* / \sim_{W_G}$.*
(ii) *Let Δ_λ^+ be the (unique) positive system of Δ such that $\langle \lambda, \alpha^\vee \rangle > 0$ ($\forall \in \Delta_\lambda^+$). Put*

$$\Lambda = \lambda + \frac{1}{2} \sum_{\alpha \in \Delta_\lambda^+} \alpha - \sum_{\alpha \in \Delta_c^+} \alpha.$$

Then the irreducible representation τ_Λ of K with highest weight Λ occurs in π_λ with multiplicity one. The specific K-type τ_Λ of π_λ is called **the minimal K-type of** *π_λ.*
(iii) *If an irreducible representation τ_μ of K occurs in π_λ, then we have*

$$\mu = \Lambda + \sum_{\alpha \in \Delta_\lambda^+ \setminus \Delta_c^+} m_\alpha \alpha, \qquad \textit{for some } m_\alpha \in \mathbf{Z}_{\geq 0}.$$

Some terminologies and remarks:
(1) λ (resp. Λ) is called the **Harish-Chandra parameter** (resp. the **Blattner parameter**) of π_λ. Some authors interchange the letters λ and Λ.
(2) **Infinitesimal character.** By Schur's lemma, the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ acts on each irreducible admissible representation (π, H_π) as scalar multiplication. That is, we have

$$\pi(z)v = \chi_\pi(z)v, \quad (\forall v \in H_\pi, \forall z \in Z(\mathfrak{g}))$$

for some $\mathbf{C}$-algebra homomorphism $\chi_\pi : Z(\mathfrak{g}) \to \mathbf{C}$. On the other hand, there is a $\mathbf{C}$-algebra isomorphism (the Harish-Chandra isomorphism):

$$q : Z(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{t}_{\mathbf{C}})^{W_G}.$$

Hence there exists $\nu \in \mathfrak{t}_{\mathbf{C}}^* / \sim_{W_G}$ such that

$$\chi_\pi(z) = \langle \nu, q(z) \rangle, \qquad \forall z \in Z(\mathfrak{g}).$$

We call χ_π or $\nu \in \mathfrak{t}_{\mathbf{C}}^*$ the infinitesimal character of π. The theorem implies that there are $\sharp(W_G/W_K)$ numbers of discrete series representations having the same infinitesimal character.

(3) **Global character.** Let (π, H_π) be an irreducible admissible Hilbert representation of G. Fix an arbitrary complete orthonormal system $\{v_i\}$ of H_π. We define a distribution Θ_π on G by

$$\Theta_\pi(f) := \sum_i \int_G (f(g)v_i, v_i) dg, \qquad f \in C_c^\infty(G),$$

which is independent of the choice of $\{v_i\}$. The distribution Θ_π is called the global character of π. It is known that Θ_π is real analytic on the regular set G^{reg} of G. On $T_{reg} := G^{reg} \cap T = \{t \in T \mid \alpha(t) \neq 1 (\forall \alpha \in \Delta)\}$, we have

$$\Theta_{\pi_\lambda}(t) = (-1)^{\frac{1}{2}\dim(G/K)} \frac{\sum_{w \in W_K} \operatorname{sgn}(w) t^{w(\lambda)}}{\prod_{\alpha \in \Delta_\lambda^+} (t^{\alpha/2} - t^{-\alpha/2})}.$$

This gives another characterization of π_λ. We can formally write $\Theta_{\pi_\lambda}(t)$ as

$$\Theta_{\pi_\lambda}(t) = \sum_{\mu:\text{dominant}} n(\pi_\lambda;\mu) \mathrm{char}_K(\tau_\mu)(t), \qquad n(\pi_\lambda;\mu) \in \mathbf{Z}_{\geq 0}.$$

Here $\mathrm{char}_K(\tau_\mu)$ is the character of the irreducible finite dimensional K-module τ_μ. Then the **Blattner formula** ([He-Sch]) asserts that

$$\dim_{\mathbf{C}} \mathrm{Hom}_K(\tau_\mu, \pi_\lambda) = n(\pi_\lambda; \nu).$$

(4) There are several ways of constructing discrete series representations. For this interesting topic, we refer the reader to the survey article [Sch].

$\boxed{\textbf{Ex. 1. } G = SL(2,\mathbf{R})}$ The Blattner parameter of $D_k^{\pm}$ $(k \geq 2)$ is $\pm k$, while its Harish-Chandra parameter is $\pm(k-1)$.

$\boxed{\textbf{Ex. 2. } G = Sp(4,\mathbf{R})}$ As a maximal compact subgroup, we take

$$K := G \cap O(4) = \left\{ k_{A,B} := \left(\begin{array}{c|c} A & B \\ \hline -B & A \end{array} \right) \in G \right\}.$$

Fix an isomorphism $\kappa : U(2) \cong K$ by $A + \sqrt{-1}B \mapsto k_{A,B}$. As a Cartan subgroup T of K, we can take

$$T = \left\{ t = \kappa(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}) \mid t_i \in \mathbf{C}^{(1)} \right\}.$$

The character group $\widehat{T}$ of T is given by

$$\widehat{T} = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2, \qquad e_i(\kappa(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix})) = t_i \ (i = 1, 2).$$

We frequently write (a_1, a_2) in place of $a_1 e_1 + a_2 e_2 \in \widehat{T}$ $(a_i \in \mathbf{Z})$. Then we have

$$\Delta = \Delta(G,T) = \{\pm(2,0), \pm(1,1), \pm(0,2), \pm(1,-1)\}, \quad \Delta_c = \{\pm(1,-1)\}$$

$$W_G \cong (\mathbf{Z}/2\mathbf{Z})^2 \rtimes S_2, \quad W_K \cong S_2.$$

We fix a positive system Δ_c^+ of Δ_c as $\Delta_c^+ = \{(1,-1)\}$. The parameter set Ξ is a disjoint union of $\Xi^{p,q}$ $(p, q \geq 0, p + q = 3)$ below:

$$\Xi^{3,0} := \{\lambda = (\lambda_1, \lambda_2) \in \widehat{T} \mid \lambda_1 > \lambda_2 > 0\};$$

$$\Xi^{2,1} := \{\lambda = (\lambda_1, \lambda_2) \in \widehat{T} \mid \lambda_1 > -\lambda_2 > 0\};$$

$$\Xi^{1,2} := \{\lambda = (\lambda_1, \lambda_2) \in \widehat{T} \mid -\lambda_2 > \lambda_1 > 0\};$$

$$\Xi^{0,3} := \{\lambda = (\lambda_1, \lambda_2) \in \widehat{T} \mid 0 > \lambda_1 > \lambda_2\}.$$

The Blattner parameter Λ is given by

$$\lambda \in \Xi^{3,0} \Rightarrow \Lambda = \lambda + (1,2); \qquad \lambda \in \Xi^{2,1} \Rightarrow \Lambda = \lambda + (1,0);$$
$$\lambda \in \Xi^{1,2} \Rightarrow \Lambda = \lambda + (0,-1); \qquad \lambda \in \Xi^{0,3} \Rightarrow \Lambda = \lambda + (-2,-1).$$

For $\lambda \in \Xi$, we set $\lambda^\vee := (-\lambda_2, -\lambda_1)$. Then the contragredient representation $\pi_\lambda^\vee$ of π_λ is given by $\pi_{\lambda^\vee}$. Note that if $\lambda \in \Xi^{3,0}$ (resp. $\Xi^{2,1}$) then $\lambda^\vee \in \Xi^{0,3}$ (resp. $\lambda^\vee \in \Xi^{1,2}$). The discrete series representations of $G = Sp(4, \mathbf{R})$ are divided into two classes:

- $\lambda \in \Xi^{3,0} \cup \Xi^{0,3} \Rightarrow \pi_\lambda$ is called an **(anti-)holomorphic** discrete series representation.
- $\lambda \in \Xi^{2,1} \cup \Xi^{1,2} \Rightarrow \pi_\lambda$ is called a **large** discrete series representation.

One important property of large discrete series representations π_λ is that they have Whittaker models. That is, for a maximal unipotent subgroup N_0 of G and its non-degenerate character ψ, the intertwining space $\mathrm{Hom}_{\mathfrak{g},K}(\pi_\lambda, C_\psi^\infty(N_0\backslash G))$ is non-zero, where we set

$$C_\psi^\infty(N_0\backslash G) := \{W : G \overset{C^\infty}{\to} \mathbf{C} \mid W(ng) = \psi(n)W(g), \quad \forall (n,g) \in N_0 \times G\}.$$

A realization of π_λ in $C_\psi^\infty(N_0\backslash G)$ is called a Whittaker model of π_λ. On the other hand, (anti-)holomorphic discrete series representations do not have Whittaker models.
If $\lambda \in \Xi^{p,q}$, then it is easy to check that

$$\mathrm{H}^{(p,q)}(\mathfrak{g}, K, \pi_\lambda \otimes E) \cong \mathbf{C} \quad \text{ and } \quad \mathrm{H}^{(r,s)}(\mathfrak{g}, K, \pi_\lambda \otimes E) = 0 \ (\forall (r,s) \neq (p,q)).$$

Here E is the irreducible finite dimensional representation of G having the infinitesimal character $-\lambda$. This explains the superscripts appearing in $\Xi^{p,q}$. For the definition of $(\mathfrak{g}, K)$-cohomology groups $\mathrm{H}^{(r,s)}(\mathfrak{g}, K, \bullet)$ and their role in the theory of automorphic forms, we refer the reader to [B-W].

Langlands parameters. ([B],[La]). Let G be a connected reductive algebraic group $/\mathbf{R}$. Assume G is split over $\mathbf{R}$, for simplicity. We set $G := \mathsf{G}_\mathbf{R}$. Let $W_\mathbf{R} = \mathbf{C}^\times \sqcup \mathbf{C}^\times \cdot j$ ($jz = \bar{z}j$, $j^2 = -1$) be the Weil group of $\mathbf{R}$. We denote by $\mathsf{G}^\vee$ the dual group of G. We set

$$\Phi(G) := \{\phi : W_\mathbf{R} \to \mathsf{G}^\vee \mid \text{ continuous hom. }\}/\sim_{\mathsf{G}^\vee}.$$

Theorems 1.2 and 1.3 can be expressed by a "natural" surjection

$$\widehat{G}_{adm} \twoheadrightarrow \Phi(G).$$

For each $\phi \in \Phi(G)$, the inverse image $\Pi_\phi(G) \subset \widehat{G}_{adm}$ of ϕ is a finite set, which is called an L-packet.
Examples. Define two dimensional representations $\phi_{\mu,N} : W_\mathbf{R} \to GL(2, \mathbf{C})$ ($\mu \in \mathbf{C}, N \in \mathbf{Z}_{\geq 0}$) as follows:

$$\phi_{\mu,N}(re^{\sqrt{-1}\theta}) = \begin{pmatrix} r^{2\mu-N}e^{-\sqrt{-1}N\theta} & 0 \\ 0 & r^{2\mu-N}e^{+\sqrt{-1}N\theta} \end{pmatrix}, \qquad \phi_{\mu,N}(j) = \begin{pmatrix} 0 & (-1)^N \\ 1 & 0 \end{pmatrix}.$$

The representation $\phi_{\mu,N}$ is irreducible if $N > 0$. The L- and ϵ-factors corresponding to the representation $\phi_{\mu,N}$ ($N > 0$) are defined by

$$L(s, \phi_{\mu,N}) := \Gamma_\mathbf{C}(s + \mu), \quad \epsilon(s, \phi_{\mu,N}, \psi_\infty) := (\sqrt{-1})^{N+1}.$$

Here ψ_∞ is the character of $\mathbf{R}$ given by $\psi_\infty(t) = e^{2\pi\sqrt{-1}t}$ ($t \in \mathbf{R}$).

$\boxed{G = GL(2,\mathbf{R})}$ In this case, $\mathsf{G}^\vee \cong GL(2,\mathbf{C})$. We denote by $\pi = D_k[c]$ $(k \geq 1, c \in \mathbf{C})$ the irreducible admissible representation of $GL(2,\mathbf{R})$ characterized by $\pi|_{SL(2,\mathbf{R})} \cong D_k^+ \oplus D_k^-$ and $\pi(zI_2) = z^c \times \mathrm{id}$ $(z > 0)$. Then we have

$$\Pi_{\phi_{\mu,N}}(GL(2,\mathbf{R})) = \{D_k[c]\}, \quad \text{with } \mu = (c+k-1)/2,\ N = k-1.$$

$\boxed{G = GSp(4,\mathbf{R})}$ In this case, $\mathsf{G}^\vee \cong GSp(4,\mathbf{C})$. Let $\pi_\lambda[c]$ be the representation of $GSp(4,\mathbf{R})$ with $\pi_\lambda[c]|_{Sp(4,\mathbf{R})} \cong \pi_\lambda \oplus \pi_\lambda^\vee$ and $\pi(zI_4) = z^c \times \mathrm{id}$. $(z > 0)$. For $\lambda_1 > \lambda_2 > 0$, we consider the Langlands parameter $\phi : W_\mathbf{R} \to \mathsf{G}^\vee = GSp(4,\mathbf{C})$ defined by

$$\phi(w) = \left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ \hline c_1 & & d_1 & \\ & c_2 & & d_2 \end{array}\right) \text{ with } \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \phi_{\mu_i,N_i}(w) \quad (i = 1,2),$$

where we set

$$\mu_1 = (c+\lambda_1-\lambda_2)/2, \quad N_1 = \lambda_1 - \lambda_2; \quad \mu_2 = (c+\lambda_1+\lambda_2)/2, \quad N_2 = \lambda_1 + \lambda_2.$$

Then the corresponding L-packet consists of two elements:

$$\Pi_\phi(GSp(4,\mathbf{R})) = \{\pi_{(\lambda_1,\lambda_2)}[c], \pi_{(\lambda_1,-\lambda_2)}[c]\}.$$

• spinor L-function: Let $r : \mathsf{G}^\vee \hookrightarrow GL(4,\mathbf{C})$ be a natural inclusion. For $\pi \in \Pi_\phi(GSp(4,\mathbf{R}))$, we have

$$\begin{aligned} L(s, r\circ\phi) &= \Gamma_\mathbf{C}(s + \frac{c+\lambda_1+\lambda_2}{2})\Gamma_\mathbf{C}(s + \frac{c+\lambda_1-\lambda_2}{2}); \\ \epsilon(s, r\circ\phi, \psi_\infty) &= (-1)^{\lambda_1+1}. \end{aligned}$$

Remark. (i) $\phi(W_\mathbf{R})$ is not contained in any proper Levi subgroups of $\mathsf{G}^\vee$.
$\Leftrightarrow$ $\forall\pi \in \Pi_\phi(G)$ belongs to the discrete series (modulo center).
(ii) $\phi(W_\mathbf{R})$ is bounded $\Leftrightarrow$ $\Pi_\phi(G) \subset \widehat{G}_{temp}$.

§3. Automorphic forms on $G = Sp(4,\mathbf{R})$ of discrete series type

The space of automorphic forms. Let Γ be a congruence subgroup of $G = Sp(4,\mathbf{R})$. As usual, we define the space of automorphic forms on G w.r.t Γ by

$$\mathcal{A}(\Gamma\backslash G) := \left\{ f : G \xrightarrow{C^\infty} \mathbf{C} \mid \begin{array}{cc} \text{(i)} & f(\gamma g) = f(g) \quad (\gamma, g) \in \Gamma \times G; \\ \text{(ii)} & f \text{ is right } K\text{-finite and } Z(\mathfrak{g})\text{-finite;} \\ \text{(iii)} & f \text{ is of moderate growth.} \end{array} \right\}.$$

If $f \in \mathcal{A}(\Gamma\backslash G)$ satisfies

$$\int_{\Gamma\cap N_S\backslash N_S} f(ng)dn \equiv 0 \quad \text{and} \quad \int_{\Gamma\cap N_J\backslash N_J} f(ng)dn \equiv 0,$$

then we say that f is a **cusp form**. The totality of cusp forms on G w.r.t Γ is denoted by $\mathcal{A}^{cusp}(\Gamma\backslash G)$.

Let $\pi = \pi_\lambda$ be a discrete series representation of G with minimal K-type $\tau_\Lambda = (\tau, V_\tau)$. The unique K-homomorphism $\tau \hookrightarrow \pi$ yields

$$\mathrm{Hom}_{\mathfrak{g},K}(\pi, \mathcal{A}(\Gamma\backslash G)) \to \mathrm{Hom}_K(\tau, \mathcal{A}(\Gamma\backslash G)) \cong \{\mathcal{A}(\Gamma\backslash G) \otimes V_\tau^\vee\}^K.$$

More concretely, take a basis $\{v_i\}$ of τ and its dual basis $\{v_i^\vee\}$ of $\tau^\vee$. We attach to $\Phi \in \mathrm{Hom}_{\mathfrak{g},K}(\pi, \mathcal{A}(\Gamma\backslash G))$ the function

$$f(g) \equiv f_\Phi(g) := \sum_i \Phi(v_i)(g) \otimes v_i^\vee$$

Holomorphic case.

- $\mathbf{H}_2 := \{z \in M_2(\mathbf{C}) \mid {}^t z = z, \mathrm{Im}(z) \gg 0\}$: the Siegel upper half space of degree 2.
- The action of $G = Sp(4, \mathbf{R})$ on $\mathbf{H}_2$ is given by

$$g\langle z\rangle := (az+b)(cz+d)^{-1}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, z \in \mathbf{H}_2.$$

- Identify $X = G/K$ with $\mathbf{H}_2$ through $X \ni gK \mapsto g\langle\sqrt{-1}\rangle \in \mathbf{H}_2$. The holomorphic (resp. anti-holomorphic) part $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) of the complexified tangent space $T_{\bar{e}}(X) \otimes \mathbf{C}$ at $\bar{e} = eK \in X$ is given by

$$\mathfrak{p}_\pm = \left\{ \begin{pmatrix} A & \pm\sqrt{-1}A \\ \pm\sqrt{-1}A & A \end{pmatrix} \mid A = {}^t A \in M_2(\mathbf{C}) \right\}.$$

For $\Lambda = (\Lambda_1, \Lambda_2) \in \mathbf{Z}^2$ satisfying $\Lambda_1 \geq \Lambda_2 \geq 0$, we set

$$\begin{aligned} M_\Lambda^{holo}(\Gamma\backslash G) := \{f : \Gamma\backslash G \to V_\tau^\vee \mid C^\infty,\ & f(g;x) = 0,\ (\forall x \in \mathfrak{p}_-), \\ & f(gk) = \tau^\vee(k)^{-1} f(g)\ (\forall (g,k) \in G \times K)\}; \\ M_\Lambda(\Gamma\backslash X) := \{\phi : X \to V_\tau^\vee \mid \text{holomorphic},\ & \phi|_\gamma(z) = \phi(z),\ (\forall \gamma \in \Gamma)\}, \end{aligned}$$

where $\tau = \tau_\Lambda \in \widehat{K}$. Here we use the usual notation:

- $\phi|_\gamma(z) := \tau_\Lambda^\vee(\kappa({}^t j(\gamma, z)^{-1}))^{-1} \phi(\gamma\langle z\rangle)$;
- $j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) := cz + d, \qquad (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \in G \times \mathbf{H}_2,$
- $K_\mathbf{C}$: the analytic subgroup of $Sp(2, \mathbf{R})$ corresponding to $\mathfrak{k}_\mathbf{C}$;
- $\kappa : GL(2, \mathbf{C}) \cong K_\mathbf{C}$ is the holomorphic extension of $\kappa : U(2) \cong K$;
- $\tau^\vee : K_\mathbf{C} \to GL(V_\tau^\vee)$ is the holomorphic extension of $\tau^\vee : K \to GL(V_\tau^\vee)$;
- $f(g; x' + \sqrt{-1}x'') := \dfrac{d}{dt}_{|t=0} [f(g\exp(tx')) + \sqrt{-1} f(g \exp(tx''))], \quad x', x'' \in \mathfrak{g}.$

Theorem 3.1. *Suppose that $\Lambda_1 \geq \Lambda_2 \geq 3$. If we set $\lambda = (\Lambda_1 - 1, \Lambda_2 - 2)$, then we have*

$$\mathrm{Hom}_{\mathfrak{g},K}(\pi_\lambda, \mathcal{A}(\Gamma\backslash G)) \cong M_\Lambda^{holo}(\Gamma\backslash G) \cong M_\Lambda(\Gamma\backslash X).$$

Proof. The first isomorphism is given by $\Phi \mapsto f_\Phi$. The injectivity is clear from the irreducibility of π_λ. The surjectivity follows from the fact that π_λ is equivalent to the generalized Verma module $M(\Lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{k}_\mathbf{C} + \mathfrak{p}_-)} V_\tau$ and the Koecher principle (plus the second isomorphism).

The second isomorphism is well-known. For the sake of completeness, we review it here. We first note that a C^∞-function $f : G \to V_\tau^\vee$ satisfying $f(gk) = \tau^\vee(k)^{-1} f(g)$ defines a C^∞-section s_f of the homogeneous vector bundle $G \times_K V_\tau^\vee \to X$:

$$s_f : X \ni gK \mapsto [g, f(g)] \in G \times_K V_\tau^\vee.$$

We set $G_{\mathbf{C}} := Sp(4, \mathbf{C})$ and $P_\pm := \exp(\mathfrak{p}_\pm)$. Then it is well-known that $P_+ K_{\mathbf{C}} P_-$ is an open subset of $G_{\mathbf{C}}$ and that the multiplication map

$$P_+ \times K_{\mathbf{C}} \times P_- \ni (p_+, u, p_-) \mapsto p_+ u p_- \in P_+ K_{\mathbf{C}} P_-$$

is bi-holomorphic. We set $c_G := \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1}/2 & 1/2 \end{pmatrix} \in G_{\mathbf{C}}$. Then we have $c_G^{-1} G \subset P_+ K_{\mathbf{C}} P_-$. Through the map

$$G \times_K V_\tau^\vee \ni [g, \xi] \mapsto [c_G^{-1} g, \xi] \in P_+ K_{\mathbf{C}} P_- \times_{K_{\mathbf{C}} P_-} V_\tau^\vee,$$

the vector bundle $G \times_K V_\tau^\vee$ on X can be regarded as a pull-back of the *holomorphic* vector bundle $P_+ K_{\mathbf{C}} P_- \times_{K_{\mathbf{C}} P_-} V_\tau^\vee$ on P_+. Here P_- acts on $V_\tau^\vee$ trivially. Since $P_+ K_{\mathbf{C}} P_- \times_{K_{\mathbf{C}} P_-} V_\tau^\vee \cong P_+ \times V_\tau^\vee$, the C^∞-section s_f defined above gives rise to a C^∞-function $\phi_f : X \to V_\tau^\vee$:

$$\phi_f(g\langle\sqrt{-1}\rangle) := \tau^\vee((c_G^{-1} g)_0) f(g).$$

Here $(c_G^{-1} g)_0$ is the image of $c_G^{-1} g$ through the projection $P_+ K_{\mathbf{C}} P_- \to K_{\mathbf{C}}$. To compute $(c_G^{-1} g)_0$, it is convenient to introduce $P'_\pm = c_G P_\pm c_G^{-1}$ and $K'_{\mathbf{C}} = c_G K_{\mathbf{C}} c_G^{-1}$. It is easy to check that

$$P'_+ = \{ \begin{pmatrix} 1_2 & x \\ 0_2 & 1_2 \end{pmatrix} \mid x = {}^t x \in M_2(\mathbf{C}) \}, \quad P'_- = \{ \begin{pmatrix} 1_2 & 0_2 \\ x & 1_2 \end{pmatrix} \mid x = {}^t x \in M_2(\mathbf{C}) \}$$

$$K'_{\mathbf{C}} = \{ \begin{pmatrix} m & 0_2 \\ 0_2 & {}^t m^{-1} \end{pmatrix} \mid m \in GL(2, \mathbf{C}) \}.$$

By a simple computation, we have

$$g c_G^{-1} = \begin{pmatrix} 1_2 & g\langle\sqrt{-1}\rangle \\ 0_2 & 1_2 \end{pmatrix} \begin{pmatrix} {}^t(c\sqrt{-1} + d)^{-1} & 0_2 \\ 0_2 & c\sqrt{-1} + d \end{pmatrix} \begin{pmatrix} 1_2 & 0_2 \\ * & 1_2 \end{pmatrix}, \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By $c_G \kappa(m) c_G^{-1} = \begin{pmatrix} m & 0_2 \\ 0_2 & {}^t m^{-1} \end{pmatrix}$, we get $(c_G^{-1} g)_0 = \kappa({}^t j(g, \sqrt{-1})^{-1})$. Therefore we have

$$\phi_f(g\langle\sqrt{-1}\rangle) = \tau^\vee(\kappa({}^t j(g, \sqrt{-1})^{-1})) f(g). \tag{3.1}$$

Now it is easy to see that $\phi_f|_\gamma(z) = \phi_f(z)$ for all $\gamma \in \Gamma$. To complete the proof, we have to show that the holomorphy of $\phi_f(z)$ is equivalent to the condition $f(g; x) = 0$ $(\forall x \in \mathfrak{p}_-)$. Since the anti-holomorphic part of $T_{gK}(X) \otimes \mathbf{C}$ is given by $L_g(\mathfrak{p}_-)$, the function $\phi_f(z)$ is holomorphic at $gK \in X$ if and only if

$$\frac{d}{dt}_{|t=0} [\phi_f(g \exp(tx')K) + \sqrt{-1} \phi_f(g \exp(tx'')K)] = 0 \qquad \forall x = x' + \sqrt{-1} x'' \in \mathfrak{p}_-.$$

By using the holomorphy of $\tau^\vee$, we have

$$\begin{aligned}
&\frac{d}{dt}_{|t=0}[\phi_f(g\exp(tx')K)+\sqrt{-1}\phi_f(g\exp(tx'')K)]\\
=&\tau^\vee((c_G^{-1}g)_0)f(g;x'+\sqrt{-1}x'')\\
+&\frac{d}{dt}_{|t=0}[\tau^\vee(\left(c_G^{-1}g\exp(tx')\right)_0)f(g)+\sqrt{-1}\tau^\vee(\left(c_G^{-1}g\exp(tx'')\right)_0)f(g)]\\
=&\tau^\vee((c_G^{-1}g)_0)f(g;x)+\frac{d}{dt}_{|t=0}[\tau^\vee(\left(c_G^{-1}g\exp(tx)\right)_0)]f(g)\\
=&\tau^\vee((c_G^{-1}g)_0)f(g;x).
\end{aligned}$$

Hence we complete the proof of the second isomorphism. □

Remarks. (i) It is easy to check that $M_\Lambda(\Gamma\backslash X)$ is isomorphic to the following space

$$\Big\{\phi : X\to V_\tau \mid \text{holomorphic}, \quad \tau(\kappa(j(\gamma,z)))^{-1}\phi(\gamma\langle z\rangle)=\phi(z) \quad \forall\gamma\in\Gamma\Big\}.$$

(ii) the case of low weights. For $2\ge\Lambda_2\ge 0$, the above theorem holds if we replace π_λ with the generalized Verma module $M(\Lambda)$. We denote the unique irreducible quotient of $M(\Lambda)$ by $L(\Lambda)$. Then we have

$$\mathrm{Hom}_{\mathfrak{g},K}(M(\Lambda),\mathcal{A}^{cusp}(\Gamma\backslash G))\cong\mathrm{Hom}_{\mathfrak{g},K}(L(\Lambda),\mathcal{A}^{cusp}(\Gamma\backslash G))\cong S_\Lambda^{holo}(\Gamma\backslash G)\cong S_\Lambda(\Gamma\backslash X).$$

Here $S_\Lambda^{holo}(\Gamma\backslash G)$ (resp. $S_\Lambda(\Gamma\backslash X)$) is the space of cusp forms in $M_\Lambda^{holo}(\Gamma\backslash G)$ (resp. $M_\Lambda(\Gamma\backslash X)$). If $\Lambda_2=2$, then $L(\Lambda)$ is isomorphic to the **limit of discrete series representation.** For more details on the low weight case, see [Ha-Ja-1], [Ha-Ja-2].

(iii) N. Wallach's theorem. ([Wa-2]) For $\pi\in\widehat{G}_{temp}$, we have the following isomorphism

$$\mathrm{Hom}_{\mathfrak{g},K}(\pi_K,\mathcal{A}^{cusp}(\Gamma\backslash G))\cong\mathrm{Hom}_G(\pi,L^2(\Gamma\backslash G)).$$

(iv) Limit multiplicity formulae. $G=Sp(4,\mathbf{R})$. For $\lambda=(\lambda_1,\lambda_2)\in\Xi$, we set

$$m_{cusp}(\pi_\lambda,N):=\dim_{\mathbf{C}}\mathrm{Hom}_{\mathfrak{g},K}(\pi_\lambda,\mathcal{A}^{cusp}(\Gamma(N)\backslash G)).$$

G. Savin [Sa] proved that

$$m_{cusp}(\pi_\lambda,N)\sim\mathrm{vol}(\Gamma(N)\backslash G)\times\mathrm{degree}(\pi_\lambda),\qquad\text{as } N\to+\infty.$$

Here the formal degree $\deg(\pi_\lambda)$ of π_λ is proportional to $|\lambda_1\lambda_2(\lambda_1^2-\lambda_2^2)|$. This implies that for $\lambda_1>\lambda_2>0$, the "main term" of two quantities $m_{cusp}(\pi_{(\lambda_1,\lambda_2)},N)$ and $m_{cusp}(\pi_{(\lambda_1,-\lambda_2)},N)$ coincides. The multiplicity $m_{cusp}(\pi_{(\lambda_1,\lambda_2)},N)$ is nothing but the dimension of the space $S_\Lambda(\Gamma(N)\backslash X)$ of holomorphic cusp forms on $\mathbf{H}_2$ of wight Λ w.r.t $\Gamma(N)$. If λ is sufficiently regular, this dimension is known by R.Tsushima [T] (and S. Wakatsuki [Wak]). On the other hand, the multiplicity $m_{cusp}(\pi_{(\lambda_1,-\lambda_2)},N)$ of a large discrete series representation $\pi_{(\lambda_1,-\lambda_2)}$ is unknown.

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