VECTOR VALUED SIEGEL MODULAR FORMS OF SYMMETRIC TENSOR WEIGHT OF SMALL DEGREES

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1. Introduction

In this paper, we give explicit generators of the module given by the direct sum over k of vector valued Siegel modular forms of degree two of level 1 of weight $\det^k Sym(j)$ for $j = 2, 4, 6$. The results have been announced in [12] and [13] and also a version of preprint was quoted in [7], but this is the first version containing precise proofs. Vector valued Siegel modular forms seem to attract more attention nowadays in many respects, like in Harder's conjecture, cohomology of local systems, or in some liftings or lifting conjectures (cf. [10], [7], [21], [14], [17] for example), and it seems worthwhile to publish these results now. More precise contents are as follows. We denote by $A_{k,j}(\Gamma_2)$ the linear space of Siegel modular forms of degree two of weight $\det^k Sym(j)$ where $Sym(j)$ is the symmetric tensor representation of degree j and Γ_2 is the full Siegel modular group of degree two. When $j = 0$, this is nothing but the space of scalar valued Siegel modular forms and we write $A_{k,0}(\Gamma_2) = A_k(\Gamma_2)$. We define $A^{even}_{sym(j)}(\Gamma_2) = \oplus_{k:even} A_{k,j}(\Gamma_2)$ and $A^{odd}_{sym(j)}(\Gamma_2) = \oplus_{k:odd} A_{k,j}(\Gamma_2)$. When $j = 0$, we write $A^{even}(\Gamma_2) = A^{even}_{sym(0)}(\Gamma_2)$. Then obviously $A^{even}_{sym(j)}(\Gamma_2)$ or $A^{odd}_{sym(j)}(\Gamma_2)$ is an $A^{even}(\Gamma_2)$ module. T. Satoh gave the structure of $A^{even}_{sym(2)}(\Gamma_2)$ as an $A^{even}(\Gamma_2)$ module in [22]. A rough content of our main theorem is as follows.

Theorem 1.1. *We have the following results as modules over* $A^{even}(\Gamma_2)$.
(1) $A^{odd}_{sym(2)}(\Gamma_2)$ *is spaned by four generators of determinant weight* 21, 23, 27, 29 *and there is one fundamental relation between generators.*
(2) $A^{even}_{sym(4)}(\Gamma_2)$ *is a free module over* $A^{even}(\Gamma_2)$ *spanned by five free generators of determinant weight* 8, 10, 12, 14, 16.
(3) $A^{odd}_{sym(4)}(\Gamma_2)$ *is a free module over* $A^{even}(\Gamma_2)$ *spanned by five free generators of determinant weight* 15, 17, 19, 21, 23.
(4) $A^{even}_{sym(6)}(\Gamma_2)$ *is a free module over* $A^{even}(\Gamma_2)$ *spanned by seven free generators of determinant weight* 6, 8, 10, 12, 14, 16, 18.
All these generators are given explicitly.

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Here by abuse of language we say that elements of $A_{k,j}(\Gamma_2)$ have determinant weight k. By the way, by T. Satoh it is known that $A^{even}_{sym(2)}(\Gamma_2)$ is spanned by 6 generators of determinant weight 10, 14, 16, 16, 18, 22 and there are three fundamental relations between generators. Some generalization for congruence sugroups of Γ_2 of the above result for $j = 2$ has been given by H. Aoki [1].

Precise construction of generators and structures will be given in the main text. Here we explain some technical points. There are at least three ways to construct vector valued Siegel modular forms.
(i) Eisenstein series.
(ii) Theta functions with harmonic polynomials.
(iii) Rankin-Cohen type differential operators.
Here (i) and (ii) are classical (cf. [3] for (i)). The Eisenstein series is defined only when k is even. (ii) is very powerful but sometimes we need a complicated computer calculation. The method (iii) is a way to construct new vector valued Siegel modular forms from known scalar valued Siegel modular forms. Forms of smaller determinant weight than those of given scalar valued forms cannot be constructed by this method, but this method is the easiest if it is available: easy to anticipate which kind of forms can be constructed, and easy to calculate large numbers of Fourier coefficients for applications, and so on. Actually in order to prove (4) of the above Theorem, we need all (i),(ii),(iii), but we mainly use (iii) for the other cases (1), (2), (3). For even determinant weight for sym(2), in [22] T. Satoh defined this kind of differential operators on a pair of scalar valued Siegel modular forms. We have already developped a general theory of this kind of operators in [11] and [5], and in the latter we gave certain explicit differential operators to increase weight by $sym(j)$. One of new points in this paper is to take derivatives of three scalar valued Siegel modular form of *even* weights to construct a vector valued Siegel modular forms of *odd* determiant weight. We already used this kind of trick to construct odd weight or Neben type forms in [2] (though the results in this paper had been obtained earlier). Rankin-Cohen type differential operators are very useful to give this kind of parity change.

We shortly write the content of each section. After reviewing elementary definitions and notation, we review a theory of Rankin-Cohen type differential operators and give some new results of their explicit shapes in section 2. If you are only interested in the structure theorems of vector valued Siegel modular forms, you can skip this section and proceed directly to later sections, where we can study odd deteminant weight of Sym(2) in section 3 (cf. Theorem 4.1), all weights of Sym(4) in section 4(cf. Theorem 5.1), and even deteminant weight of Sym(6) in section 5 (cf. Theorem 6.1). We also give Theorem 4.2 on a structure of the kernel of the Witt operator on $A_{sym(2)}(\Gamma_2)$ since we need it in another paper on Jacobi forms [16].

Of course we could continue a similar structure theory to higher j though it would be much more complicated. For example, from Tsushima's dimension formula, it seems that $A^{odd}_{sym(6)}(\Gamma_2)$ and $A_{sym(8)}(\Gamma_2)$ are also free $A^{even}(\Gamma_2)$ modules, and we see that $A_{sym(10)}(\Gamma_2)$ is not a free module. This obsrvation will be explained in section 7, together with some mysterious open problem.

Now we take all direct sum $A^{big} = \oplus_{k,j\geq 0} A_{k,j}(\Gamma_2)$. We have the irreducible decomposition of the tensor product of symmetric tensor representations as follows:

$$Sym(j) \otimes Sym(l) \cong \sum_{\substack{|j-l|\leq j+l-2\nu \\ 0\leq \nu}} \det{}^{\nu} Sym(j+l-2\nu).$$

This isomorphism is not canonical at all, but if we fix a linear isomorphism in the above for each pair (j,l), we can define a product of elements of A^{big} by taking the tensor as a product and identify it with an element of A^{big} through the above isomorphisms. We do not know if we can choose these isomorphisms so that the product is associative, but it would be interesting to ask generators of this big "ring". Since $A_{4,j}(\Gamma_2)$ never vanishes for big j, there should exist infinitely many "generators". But it would be also interesting to ask if there is any notion of "weak vector valued Siegel modular forms" A^{big}_{weak} as in the theory of Jacobi forms in [6] and if there are finitely many "generators" of A^{big}_{weak}. The structures of $A_{sym(j)}(\Gamma_2)$ for higher j and the tensor structures of the *big ring* is an open problem for future.

2. Definitions and a Lemma for small weights

We review definitions and notation first, then give a lemma on dimensions. We denote by H_n the Siegel upper half space of degree n. We denote by $Sp(n,\mathbb{R})$ the real symplectic group of size $2n$ and put $\Gamma_n = Sp(n,\mathbb{Z})$ (the full Siegel modular group of degree n). We denote by $(Sym(j), V_j)$ the symmetric tensor representation of $GL_n(\mathbb{C})$ of degree j. For any V_j-valued holomorphic function $F(Z)$ of $Z \in H_2$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2,\mathbb{R})$, we write

$$(F|_{k,j}[g])(Z) = \det(cZ+d)^{-k} Sym(j)(cZ+d)^{-1} F(gZ).$$

We say that a V_j-valued holomorphic function $F(Z)$ is a Siegel modular form of weight $\det^k Sym(j)$ of Γ_2 if we have $F|_{k,j}[\gamma] = F$ for any $\gamma \in \Gamma_2$. When $n = 2$, V_j is identified with homogeneous polynomials $P(u_1,u_2)$ in u_1, u_2 of degree j and the action is given by $P(u) \to P(uM)$ for $M \in GL_2(\mathbb{C})$, where $u = (u_1,u_2)$. Under this identification, $A_{k,j}(\Gamma_2)$ is the space of holomorphic functions $F(Z,u) = \sum_{\nu=0}^{j} F_\nu(Z) u_1^{j-\nu} u_2^{\nu}$ such that

$$F(\gamma Z, u) = \det(cZ+d)^k F(Z, u(cZ+d))$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2$. We say that F is a cusp form if $\Phi(F) := \lim_{\lambda\to\infty} F\begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix} = 0$, where $\tau \in H_1$. We denote the space of cusp forms by $S_{k,j}(\Gamma_2)$. When $j=0$, we simply write $A_k(\Gamma_2) = A_{k,0}(\Gamma_2)$. It is easy to see that we have $A_{k,j}(\Gamma_2) = 0$ for any odd j and $A_{k,j}(\Gamma_2) = S_{k,j}(\Gamma_2)$ for any odd k.

By Igusa [19], we have

$$\bigoplus_{k=0}^{\infty} A_k(\Gamma_2) = \mathbb{C}[\phi_4, \phi_6, \chi_{10}, \chi_{12}] \oplus \chi_{35}\mathbb{C}[\phi_4, \phi_6, \chi_{10}, \chi_{12}].$$

To fix a normalization, we review the definition of these Siegel modular forms. We define each ϕ_i to be the Eisenstein series of weight i whose constant term is 1. Each form χ_{10} or χ_{12} is the unique cusp form of weight 10 or 12 such that the coefficient at $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ is 1. We denote by χ_{35} the Siegel cusp form of weight 35 normalized so that the coefficient at $\begin{pmatrix} 3 & 1/2 \\ 1/2 & 2 \end{pmatrix}$ is -1. For $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$, we put

$$A_{35}(Z) = \begin{pmatrix} 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ \partial_1\phi_4 & \partial_1\phi_6 & \partial_1\chi_{10} & \partial_1\chi_{12} \\ \partial_2\phi_4 & \partial_2\phi_6 & \partial_2\chi_{10} & \partial_2\chi_{12} \\ \partial_3\phi_4 & \partial_3\phi_6 & \partial_3\chi_{10} & \partial_3\chi_{12} \end{pmatrix},$$

where we write

$$\partial_1 = (2\pi i)^{-1}\frac{\partial}{\partial\tau}, \qquad \partial_2 = (2\pi i)^{-1}\frac{\partial}{\partial z}, \qquad \partial_3 = (2\pi i)^{-1}\frac{\partial}{\partial\omega}.$$

Then as is shown in [2], we have $\chi_{35} = \det(A_{35}(Z))/(2^9\cdot 3^4)$.

Now we give some comments on dimensions which we use later. For $j>0$, the dimensions for $\dim A_{k,j}(\Gamma_2)$ is known for $k>4$ in [24]. Here we give a lemma for $\dim A_{k,j}(\Gamma_2)$ for small k and j for later use.

Lemma 2.1. *We have $A_{2,j}(\Gamma_2) = S_{2,j}(\Gamma_2)$. We have $S_{k,j}(\Gamma_2) = 0$ for all (k,j) with $0 \le k \le 4$ and $j \le 14$, and $A_{4,j}(\Gamma_2) = 0$ for all $j \le 6$.*

Proof. For any $F \in A_{k,j}(\Gamma_2)$, we denote by WF the restriction of F to the diagonal. Then the coefficient of u_1^j of WF is the tensor of modular forms of one variable of weight $k+j$ and k. When $k=2$, a modular form of weight 2 is zero. Since the Siegel Φ operator factors through W, we have $A_{2,j}(\Gamma_2) = S_{2,j}(\Gamma_2)$. Now as shown in [18], for $k \le 4$ we have $\dim A_{k,j}(\Gamma_2) \le \dim W(A_{k,j}(\Gamma_2))$. If we write $WF = \sum_{\nu=0}^{j} f_\nu(\tau,\omega)u_1^{j-\nu}u_2^{\nu}$, then for $F \in A_{k,j}(\Gamma_2)$, we have $f_\nu(\tau,\omega) = (-1)^k f_\nu(\omega,\tau)$ and this is in the tensor of $S_{j-\nu+k}(\Gamma_1)$ and $S_{\nu+k}(\Gamma_1)$ for $1 \le \nu \le j-1$ and in the tensor of $S_{j+k}(\Gamma_1)$ and $A_k(\Gamma_1)$ for $\nu=0$. If $F \in S_{k,j}(\Gamma_2)$, $f_0(\tau,\omega)$ is in the tensor of $S_{j+k}(\Gamma_1)$ and $S_k(\Gamma_1)$. Since

$S_m(\Gamma_1) = 0$ for $m < 12$, we see that the image $W(F)$ of $F \in S_{k,j}(\Gamma_2)$ is zero unless $j - \nu + k \geq 12$ and $\nu + k \geq 12$ for some ν. In this case, we have $2k + j \geq 24$ and this is not satisfied for $k \leq 4$ and $j \leq 14$, so $W(F) = 0$ and hence $F = 0$ in these cases. By virtue of Arakawa [3], for $F \in A_{k,j}(\Gamma_2)$, we have $\Phi(F) = f(\tau)u_1^j$ for some $f \in S_{k+j}(\Gamma_1)$. When $k \leq 4$ and $j \leq 6$, we have $S_{k+j}(\Gamma_1) = 0$, so we also have $A_{k,j}(\Gamma_2) = S_{k,j}(\Gamma_2)$, but we already have shown that the latters are zero for these k, j. □

By the way, by virtue of Freitag [8], we have always $A_{0,j}(\Gamma_2) = 0$. By vanishing of Jacobi forms of weight 1 by Skoruppa, we also have $A_{1,j}(\Gamma_2) = 0$ for any j. There are more cases such that we can show the vanishing in the similar ad hoc way as in the proof above (e.g. see [18]).

3. Review on differential operators

3.1. **General theory.** We review a characterization of the Rankin-Cohen type differential operators given in [11] restricting to the cases we need here (see also [5],[4]). We consider V_j valued linear homogeneous holomorphic differential operators $\mathbb{D}$ with constant coefficients acting on functions of $(Z_1, \ldots, Z_r) \in H_2 \times \cdots \times H_2$. For any $Z = (z_{ij}) \in H_2$, we write $\partial_Z = \left(\frac{1+\delta_{ij}}{2(2\pi i)}\frac{\partial}{\partial z_{ij}}\right)$. We denote 2×2 symmetric matrices of variable components by R_i. Then it is clear that we have $\mathbb{D} = Q_{\mathbb{D}}(\partial_{Z_1}, \ldots, \partial_{Z_r}, u)$ for some polynomial $Q_{\mathbb{D}}(R_1, \ldots, R_r, u)$ in components of R_i and homogeneous in u_i of degree j. We fix natural numbers k_i ($1 \leq i \leq r$) and k. We consider the following condition on $\mathbb{D}$.

Condition 3.1. *For any holomorphic functions $F_i(Z_i)$ on H_2,*

$$\begin{aligned} &Res_{(Z_i)=(Z)}(\mathbb{D}((F_1|_{k_1}[g])(Z_1)\cdots(F_r|_{k_r}[g])(Z_r))) \\ &= \left(Res_{(Z_i)=(Z)}\mathbb{D}(F_1(Z_1)\cdots F_r(Z_r))\right)|_{k_1+\cdots+k_r+k,j}[g] \end{aligned}$$

for any $g \in Sp(2,\mathbb{R})$, where Res is the restriction to replace all Z_i to the same $Z \in H_2$.

This condition means that if $F_i \in A_{k_i}(\Gamma_2)$, then we have

$$Res_{(Z_1,\ldots,Z_r)=(Z,\ldots,Z)}\Big(\mathbb{D}(F_1(Z_1)\cdots F_r(Z_r))\Big) \in A_{k_1+\cdots+k_r+k,j}(\Gamma_2).$$

We have given a characterization of such $\mathbb{D}$ by the associated polynomial Q in [11]. Indeed, consider the following conditions.

Condition 3.2. *(1) For any $A \in GL_2$, we have*

$$Q(AR_1\,{}^tA, \cdots, AR_r\,{}^tA,\, u) = \det(A)^k Q(R_1, \ldots, R_r,\, uA).$$

(2) For $2\times d_i$ matrices X_i of variables components for $1\le i\le r$, the polynomials $Q(X_1\,{}^tX_1,\cdots,X_r\,{}^tX_r,u)$ are pluri-harmonic with respect to $X=(X_1,\dots,X_r)=(x_{ij})$, i.e.

$$\sum_{\nu=1}^{2(k_1+\cdots+k_r)}\frac{\partial^2 Q}{\partial x_{i\nu}\partial x_{j\nu}}=0$$

for any $1\le i,j\le 2$.

For any such Q, we write $\mathbb{D}_Q=Q(\partial_{Z_1},\dots,\partial_{Z_r},u)$. If a polynomial Q satisfies the condition 3.2, then $\mathbb{D}_Q$ satisfies the condition 3.1. On the contrary, if $\mathbb{D}$ satisfies the condition 3.1, then there exists the unique $\mathbb{Q}_{\mathbb{D}}$ which satisfies the condition 3.2 such that $\mathbb{D}=Q_{\mathbb{D}}(\partial_{Z_1},\dots,\partial_{Z_r},u)$.

3.2. **Brackets of two forms.** For general j, in the case $r=2$ and $k=0$, the above Q satisfying Condition 3.2 is given explicitly in [5] p. 460 Prop. 6.1 for general degree, and in the case $k>0$ of degree 2 in [20]. The degree two case for $k=0$ is explained as follows. For the sake of notational simplicity, we put $R_1=R=(r_{ij})$ and $R_2=S=(s_{ij})$ in the previous section. Here R and S are 2×2 symmetric matrices. For any natural number k, l, m, we put

$$Q_{k,l,m}(x,y)=\sum_{i=0}^{m}(-1)^i\binom{m+l-1}{i}\binom{m+k-1}{m-i}x^iy^{m-i}.$$

If we put $r=r_{11}u_1^2+2r_{12}u_1u_2+r_{22}u_2^2$ and $s=s_{11}u_1+2s_{12}u_1u_2+s_{22}$, then the polynomial $Q_{k,l,m}(r,s)$ in r_{ij}, s_{ij}, u_1, u_2 satisfies Condition 3.2 for $k_1=k$, $k_2=l$, $j=2m$. In other words, we have the following results. We put

$$m_1=u_1^2\frac{\partial}{\partial\tau_1}+u_1u_2\frac{\partial}{\partial z_1}+u_2^2\frac{\partial}{\partial\omega_1},\quad m_2=u_1^2\frac{\partial}{\partial\tau_2}+u_1u_2\frac{\partial}{\partial z_2}+u_2^2\frac{\partial}{\partial\omega_2},$$

and

$$\mathbb{D}_{k,l,(k+l,j)}=Q_{k,l,j/2}(m_1,m_2),$$

where $Z_i=\begin{pmatrix}\tau_i & z_i\\ z_i & \omega_i\end{pmatrix}\in H_2$ ($i=1$, 2). For any $F\in A_k(\Gamma_2)$ and $G\in A_l(\Gamma_2)$, we define

$$\{F,G\}_{Sym(j)}(Z)=Res_{Z_1=Z_2=Z}\Big(\mathbb{D}_{k,l,(k+l,j/2)}(F(Z_1)G(Z_2))\Big).$$

Then we have $\{F,G\}_{Sym(j)}\in A_{k+l,j}(\Gamma_2)$. When $j=2$, this is nothing but the operator defined by T. Satoh[22] and given by

$$\begin{aligned}\{F,G\}_{Sym(2)}(Z) &= \left(kF\frac{\partial G}{\partial\tau}-lG\frac{\partial F}{\partial\tau}\right)u_1^2+\left(kF\frac{\partial G}{\partial z}-lG\frac{\partial F}{\partial z}\right)u_1u_2\\ &\quad+\left(kF\frac{\partial G}{\partial\omega}-lG\frac{\partial F}{\partial\omega}\right)u_2^2\end{aligned}$$

(up to the difference of the choice of the coordinate). For the readers' convenience, we give also explicit expression of brackets for $j=4$ which we use. For $F \in A_k(\Gamma_2)$ and $G \in A_l(\Gamma_2)$, we have

$$\{F,G\}_{Sym(4)} =$$

$$\begin{aligned}
&\left(\frac{l(l+1)}{2}\frac{\partial^2 F}{\partial\tau^2}G-(l+1)(k+1)\frac{\partial F}{\partial\tau}\frac{\partial G}{\partial\tau}+\frac{k(k+1)}{2}F\frac{\partial^2 G}{\partial\tau^2}\right)u_1^4\\
&+\left(l(l+1)\frac{\partial^2 F}{\partial\tau\partial z}G-(k+1)(l+1)\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial\tau}+\frac{\partial F}{\partial\tau}\frac{\partial G}{\partial z}\right)\right.\\
&\left.+k(k+1)F\frac{\partial^2 G}{\partial\tau\partial z}\right)u_1^3u_2+\left(\frac{l(l+1)}{2}\frac{\partial^2 F}{\partial z^2}G+l(l+1)\frac{\partial^2 F}{\partial\tau\partial\omega}G\right.\\
&-(k+1)(l+1)\frac{\partial F}{\partial\omega}\frac{\partial G}{\partial\tau}-(k+1)(l+1)\frac{\partial F}{\partial z}\frac{\partial G}{\partial z}\\
&\left.+\frac{k(k+1)}{2}F\frac{\partial^2 G}{\partial z^2}-(k+1)(l+1)\frac{\partial F}{\partial\tau}\frac{\partial G}{\partial\omega}+k(k+1)F\frac{\partial^2 G}{\partial\tau\partial\omega}\right)u_1^2u_2^2\\
&+\left(l(l+1)\frac{\partial^2 F}{\partial z\partial\omega}G-(k+1)(l+1)\left(\frac{\partial F}{\partial\omega}\frac{\partial G}{\partial z}+\frac{\partial F}{\partial z}\frac{\partial G}{\partial\omega}\right)\right.\\
&\left.+k(k+1)F\frac{\partial^2 G}{\partial z\partial\omega}\right)u_1u_2^3\\
&+\left(\frac{l(l+1)}{2}\frac{\partial^2 F}{\partial\omega^2}G-(k+1)(l+1)\frac{\partial F}{\partial\omega}\frac{\partial G}{\partial\omega}+\frac{k(k+1)}{2}F\frac{\partial^2 G}{\partial\omega^2}\right)u_2^4.
\end{aligned}$$

An explicit shape of $\{F,G\}_{sym(6)}$ can be given similarly but omit it here since it is lengthy and the general formula is already given above.

We give one more example from [5] p. 461. (Also note that a typo there is corrected in [20] p. 374.) For any even natural number k, l, j, we define a polynomial $Q_{k,l,(2,j)}(R,S,u)$ in r_{ij}, s_{ij}, u_1, u_2 as follows:

$$\begin{aligned}
Q_{k,l,(2,j)}(R,S,u) &= 4^{-1}Q_2(R,S)Q_{k+1,l+1,j/2}(r,s)\\
&\quad +2^{-1}((2l-1)\det(R)s-(2k-1)\det(S)r)\\
&\quad \times\left(\frac{\partial Q_{k+1,l+1,j/2}}{\partial x}(r,s)-\frac{\partial Q_{k+1,l+1,j/2}}{\partial y}(r,s)\right),
\end{aligned}$$

where r, s are defined as before and we put

$$\begin{aligned}
Q_2(R,S) &= (2k-1)(2l-1)\det(R+S)-(2k-1)(2k+2l-1)\det(S)\\
&\quad -(2l-1)(2k+2l-1)\det(R).
\end{aligned}$$

Then this $Q_{k,l,(2,j)}$ satisfies Condition 3.2. For $F \in A_k(\Gamma_2)$ and $G \in A_l(\Gamma_2)$, we put

$$\{F,G\}_{\det^2 Sym(j)} = Res_{(Z_i)=(Z)}\big(Q_{k,l,(2,j)}(\partial_{Z_1},\partial_{Z_2},u)F(Z_1)G(Z_2)\big).$$

Then we have $\{F,G\}_{\det^2 Sym(j)} \in A_{k+l+2,j}(\Gamma_2)$.

3.3. **Bracket of three forms.** In case of bracket of two forms, we cannot construct odd determinant weight from scalar valued Siegel modular forms of even weight. But if we take three forms, we can do such a thing. This is a crucial point for our construction. In order to construct vector valued Siegel modular forms of weight $\det^k Sym(2)$ and $\det^k Sym(4)$ for odd k, we define brackets in the following way. We consider three 2×2 symmetric matrix $R=(r_{ij})$, $S=(s_{ij})$, $T=(t_{ij})$ and we prepare two polynomials. For natural numbers k_1, k_2, k_3, first we put

$$Q_{\det Sym(2)}(R,S,T)=$$
$$\begin{vmatrix} r_{11} & s_{11} & t_{11} \\ 2r_{12} & 2s_{12} & 2t_{12} \\ k_1 & k_2 & k_3 \end{vmatrix} u_1^2 - 2\begin{vmatrix} r_{11} & s_{11} & t_{11} \\ k_1 & k_2 & k_3 \\ r_{22} & s_{22} & t_{22} \end{vmatrix} u_1u_2 + \begin{vmatrix} k_1 & k_2 & k_3 \\ 2r_{12} & 2s_{12} & 2t_{12} \\ r_{22} & s_{22} & t_{22} \end{vmatrix} u_2^2.$$

For $F\in A_{k_1}(\Gamma_2)$, $G\in A_{k_2}(\Gamma_2)$, $H\in A_{k_3}(\Gamma_2)$, we put

$$\{F,G,H\}_{\det Sym(2)}$$
$$= Res_{(Z_i)_{1\le i\le 3}=(Z)}(Q_{\det Sym(2)}(\partial_{Z_1},\partial_{Z_2},\partial_{Z_3})(F(Z_1)G(Z_2)H(Z_3))).$$

Then we have $\{F,G,H\}_{\det Sym(2)}\in A_{k_1+k_2+k_3+1,2}(\Gamma_2)$. More explicitly, for $Z=\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix}$, this can be written as

$$\{F,G,H\}_{\det Sym(2)}(Z,u)=$$
$$\begin{vmatrix} \partial_1 F & \partial_1 G & \partial_1 H \\ \partial_2 F & \partial_2 G & \partial_2 H \\ k_1F & k_2G & k_3H \end{vmatrix} u_1^2 - 2\begin{vmatrix} \partial_1 F & \partial_1 G & \partial_1 H \\ k_1F & k_2G & k_3H \\ \partial_3 F & \partial_3 G & \partial_3 H \end{vmatrix} u_1u_2 + \begin{vmatrix} k_1F & k_2G & k_3H \\ \partial_2 F & \partial_2 G & \partial_2 H \\ \partial_3 F & \partial_3 G & \partial_3 H \end{vmatrix} u_2^2.$$

Next we consider the case of Sym(4). We define the following polynomial

$$Q_{\det Sym(4)}(R,S,T,u)=\sum_{\nu=0}^{4} Q_\nu(R,S,T)u_1^{4-\nu}u_2^{\nu},$$

where $Q_\nu(R,S,T)$ are defined by

$$Q_0 = (k_2+1)\begin{vmatrix} (k_1+1)r_{11} & k_2 & k_3 \\ r_{11}^2 & s_{11} & t_{11} \\ r_{11}r_{12} & s_{12} & t_{12} \end{vmatrix} - (k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{11} & k_3 \\ r_{11} & s_{11}^2 & t_{11} \\ r_{12} & s_{11}s_{12} & t_{12} \end{vmatrix},$$

$$\begin{aligned} Q_1 = {} & 2(k_2+1)\begin{vmatrix} (k_1+1)r_{12} & k_2 & k_3 \\ r_{11}r_{12} & s_{11} & t_{11} \\ r_{12}^2 & s_{12} & t_{12} \end{vmatrix} - 2(k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{12} & k_3 \\ r_{11} & s_{11}s_{12} & t_{11} \\ r_{12} & s_{12}^2 & t_{12} \end{vmatrix} \\ & +(k_2+1)\begin{vmatrix} (k_1+1)r_{11} & k_2 & k_3 \\ r_{11}^2 & s_{11} & t_{11} \\ r_{11}r_{22} & s_{22} & t_{22} \end{vmatrix} - (k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{11} & k_3 \\ r_{11} & s_{11}^2 & t_{11} \\ r_{22} & s_{11}s_{22} & t_{22} \end{vmatrix}, \end{aligned}$$

$$Q_2 = 3(k_2+1)\begin{vmatrix}(k_1+1)r_{12} & k_2 & k_3\\ r_{11}r_{12} & s_{11} & t_{11}\\ r_{22}r_{12} & s_{22} & t_{22}\end{vmatrix} - 3(k_1+1)\begin{vmatrix}k_1 & (k_2+1)s_{12} & k_3\\ r_{11} & s_{11}s_{12} & t_{11}\\ r_{22} & s_{22}s_{12} & t_{22}\end{vmatrix},$$

$$\begin{aligned}Q_3 &= 2(k_2+1)\begin{vmatrix}(k_1+1)r_{12} & k_2 & k_3\\ r_{12}^2 & s_{12} & t_{12}\\ r_{12}r_{22} & s_{22} & t_{22}\end{vmatrix} - 2(k_1+1)\begin{vmatrix}k_1 & (k_2+1)s_{12} & k_3\\ r_{12} & s_{12}^2 & t_{12}\\ r_{22} & s_{12}s_{22} & t_{22}\end{vmatrix}\\ &\quad +(k_2+1)\begin{vmatrix}(k_1+1)r_{22} & k_2 & k_3\\ r_{11}r_{22} & s_{11} & t_{11}\\ r_{22}^2 & s_{22} & t_{22}\end{vmatrix} - (k_1+1)\begin{vmatrix}k_1 & (k_2+1)s_{22} & k_3\\ r_{11} & s_{11}s_{22} & t_{11}\\ r_{22} & s_{22}^2 & t_{22}\end{vmatrix},\end{aligned}$$

$$Q_4 = (k_2+1)\begin{vmatrix}(k_1+1)r_{22} & k_2 & k_3\\ r_{22}r_{12} & s_{12} & t_{12}\\ r_{22}^2 & s_{22} & t_{22}\end{vmatrix} - (k_1+1)\begin{vmatrix}k_1 & (k_2+1)s_{22} & k_3\\ r_{12} & s_{22}s_{12} & t_{12}\\ r_{22} & s_{22}^2 & t_{22}\end{vmatrix}.$$

Taking F, G, H as before, we define

$$\{F,G,H\}_{\det Sym(4)} = Res_{(Z_i)=(Z)}\left(Q_{\det Sym(4)}(\partial_{Z_1},\partial_{Z_2},\partial_{Z_3})(F(Z_1)G(Z_2)H(Z_3))\right).$$

Then we have $\{F,G,H\}_{\det Sym(4)} \in A_{k_1+k_2+k_3+1,4}(\Gamma_2)$. Explicit expression of $\{F,G,H\}_{\det Sym(4)}$ by concrete derivatives is similarly obtained as in $\{F,G,H\}_{\det Sym(2)}$ but we omit it here since it is obvious but lengthy.

4. Structure in case $Sym(2)$

In this section, we prove the following two theorems.

Theorem 4.1. *We have*

$$\begin{aligned}A^{odd}_{sym(2)}(\Gamma_2) &= A^{even}(\Gamma_2)\{\phi_4,\phi_6,\chi_{10}\}_{\det Sym(2)} + A^{even}(\Gamma_2)\{\phi_4,\phi_6,\chi_{12}\}_{\det Sym(2)}\\ &\quad + A^{even}(\Gamma_2)\{\phi_4,\chi_{10},\chi_{12}\}_{\det Sym(2)} + A^{even}(\Gamma_2)\{\phi_6,\chi_{10},\chi_{12}\}_{\det Sym(2)}\end{aligned}$$

with the following fundamental relation

$$\begin{aligned}&4\phi_4\{\phi_6,\chi_{10},\chi_{12}\}_{\det Sym(2)} - 6\phi_6\{\phi_4,\chi_{10},\chi_{12}\}_{\det Sym(2)}\\ &\quad + 10\chi_{10}\{\phi_4,\phi_6,\chi_{12}\}_{\det Sym(2)} - 12\chi_{12}\{\phi_4,\phi_6,\chi_{10}\}_{\det Sym(2)} = 0.\end{aligned}$$

For any holomorphic function $F : H_2 \to V_2$, we define the Witt operator W by the restriction to the diagonals $H_1 \times H_1$ given by

$$(WF)(\tau,\omega) = F\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix},$$

where τ, $\omega \in H_1$. For $\epsilon = even$ or odd, we write

$$A^{\epsilon,0}_{sym(2)}(\Gamma_2) = \{F \in A^{\epsilon}_{sym(2)}(\Gamma_2); WF = 0\}.$$

Theorem 4.2. *The modules* $A^{even,0}_{sym(2)}(\Gamma_2)$ *and* $A^{odd,0}_{sym(2)}(\Gamma_2)$ *are free* $A^{even}(\Gamma_2)$ *modules and given by*

$$\begin{aligned} A^{even,0}_{sym(2)}(\Gamma_2) &= A^{even}(\Gamma_2)\{\phi_4,\chi_{10}\}_{Sym(2)} \oplus A^{even}(\Gamma_2)\{\phi_6,\chi_{10}\}_{Sym(2)} \\ &\quad \oplus A^{even}(\Gamma_2)\{\chi_{10},\chi_{12}\}_{Sym(2)}, \\ A^{odd,0}_{sym(2)}(\Gamma_2) &= A^{even}(\Gamma_2)\{\phi_4,\phi_6,\chi_{10}\}_{\det Sym(2)} \oplus A^{even}(\Gamma_2)\{\phi_4,\chi_{10},\chi_{12}\}_{\det Sym(2)} \\ &\quad \oplus A^{even}(\Gamma_2)\{\phi_6,\chi_{10},\chi_{12}\}_{\det Sym(2)}. \end{aligned}$$

4.1. **Module structure of odd determinant weight.** Theorem 4.1 can be proved in various ways but here we use the Fourier Jacobi expansion. For $F \in A_{k_1,2}(\Gamma_2)$, $G \in A_{k_2,2}(\Gamma_2)$, $H \in A_{k_3,j}(\Gamma_2)$, we write

$$\begin{aligned} F(Z) &= f_0(\tau) + f_1(\tau,z)q' + O(q'^2), \\ G(Z) &= g_0(\tau) + g_1(\tau,z)q' + O(q'^2), \\ H(Z) &= h_0(\tau) + h_1(\tau,z)q' + O(q'^2), \end{aligned}$$

where we write $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$ and $q' = e^{2\pi i\omega}$. Here f_0, g_0, or h_0 is an elliptic modular form of weight k_1, k_2 or k_3 and f_1, g_1, or h_1 is a Jacobi form of index 1 of weight k_1, k_2, or k_3. We write $\partial_1 = (2\pi i)^{-1}\frac{\partial}{\partial\tau}$, $\partial_2 = (2\pi i)^{-1}\frac{\partial}{\partial z}$ and $\partial_3 = (2\pi i)^{-1}\frac{\partial}{\partial\omega}$ as before. For any elliptic modular forms $f(\tau)$ of weight k and $g(\tau)$ of weight l, we put

$$\begin{aligned} \{f,g\}_2 &= kf\partial_1 g - lg\partial_1 f, \\ \{f,g\}_4 &= 2^{-1}k(k+1)f\partial_1^2 g - (k+1)(l+1)(\partial_1 f)(\partial_1 g) + 2^{-1}l(l+1)g\partial_1^2 f. \end{aligned}$$

This is the usual Rankin-Cohen bracket and for each $i = 2$ or 4, $\{f,g\}_i$ is an elliptic modular form of weight $k+l+i$. Also for Jacobi forms $\phi(\tau,z)$ of weight k of index 1 and $\psi(\tau,z)$ of weight l of index 1, we put

$$\{\phi,\psi\}_{jac} = \psi\partial_2\phi - \phi\partial_2\psi.$$

Then $\{\phi,\psi\}_{jac}$ is a Jacobi form of weight $k+l+1$ of index 2 (cf. [6] Th. 9.5). We can define many similar differential operators of this sort. For example, for an elliptic modular form f of weight k and a Jacobi form ϕ of weight l of index m, we put

$$\{f,\phi\}^* = kf(\partial_1 - (4m)^{-1}\partial_2^2)\phi - (l-1/2)\phi\partial_1 f.$$

Then we have $\{f,\phi\}^*$ is a Jacobi form of weight $k+l+2$ of index m. This operator is used implicitly in some calculations later in section 5 or 6 without explanation. The details will be omitted.
By definition, we have

$$\begin{aligned} &\{F,G,H\}_{\det Sym(2)} = \\ &\quad (\{f_0,g_0\}_2\partial_2 h_1 - \{f_0,h_0\}_2\partial_2 g_1 + \{g_0,h_0\}_2\partial_2 f_1)q' + O(q'^2))u_1^2 \\ &\quad +(\{f_0,g_0\}_2 h_1 - \{f_0,h_0\}_2 g_1 + \{g_0,h_0\}_2 f_1)q' + q'^2)u_1u_2 \\ &\quad +(k_1 f_0\{g_1,h_1\}_{jac} - k_2 g_0\{f_1,h_1\}_{jac} + k_3 h_0\{f_1,g_1\}_{jac})q'^2 + O(q'^3))u_2^2. \end{aligned}$$

We apply these formulas to concrete cases. We denote by $E_k(\tau)$ the Eisenstein series of Γ_1 of weight k whose constant term is one and by Δ the Ramanujan Delta function. It is well known that

$$\begin{aligned}\phi_4(Z) &= E_4(\tau) + 240E_{4,1}(\tau,z)q' + O(q'^2),\\ \phi_6(Z) &= E_6(\tau) - 504E_{6,1}(\tau,z)q' + O(q'^2),\\ \chi_{10}(Z) &= \phi_{10,1}(\tau,z)q' + O(q'^2),\\ \chi_{12}(Z) &= \phi_{12,1}(\tau,z)q' + O(q'^2),\end{aligned}$$

where $q' = \exp(2\pi i\omega)$. Here we are using the same notation and normalization as in [6] p. 38 for Jacobi forms. In particular, we have

$$\begin{aligned}E_{4,1}(\tau,z) &= 1 + (126 + 56(\zeta+\zeta^{-1}))q + O(q^2),\\ E_{6,1}(\tau,z) &= 1 - (330 + 88(\zeta+\zeta^{-1}))q + O(q^2),\\ \phi_{10,1}(\tau,z) &= (144)^{-1}(E_6E_{4,1} - E_4E_{6,1}) = (2\pi i)^2z^2\Delta(\tau) + O(z^4),\\ &= (-2+\zeta+\zeta^{-1})q + (36 - 16(\zeta+\zeta^{-1}) - 2(\zeta^2+\zeta^{-2}))q^2 + O(q^3),\\ \phi_{12,1}(\tau,z) &= (144)^{-1}(E_4^2E_{4,1} - E_6E_{6,1}) = 12\Delta(\tau) + O(z^2),\\ &= (10+\zeta+\zeta^{-1})q + (-132 - 88(\zeta+\zeta^{-1}) + 10(\zeta^2+\zeta^{-2}))q^2 + O(q^3),\end{aligned}$$

where $q = e(\tau)$, $\zeta = e(z)$. We have $\{E_4, E_6\}_2 = -3456\Delta$, $\{E_4, E_4\}_4 = 4800\Delta$, $\{E_4, E_{6,1}\}^* = -264\phi_{12,1}$, $\{E_6, E_{4,1}\}^* = 252\phi_{12,1}$. If we put $\phi_{23,2} = \{\phi_{10,1}, \phi_{12,1}\}_{jac}$, then we have

$$\phi_{23,2}(\tau,z) = -24(2\pi i)\Delta(\tau)^2z + O(z^2)$$

and in particular this is not zero. If we put $\phi_{11,2} = \{E_{4,1}, E_{6,1}\}_{jac}/144$ as in [6] p. 112, then we have $\phi_{23,2} = 12\Delta\phi_{11,2}$. We also have

$$\begin{aligned}\{\phi_4,\phi_6,\chi_{10}\}_{\det Sym(2)} &= (-3456\Delta(\tau))(\partial_2\phi_{10,1}(\tau,z)u_1^2 + \phi_{10,1}(\tau,z)u_1u_2)q' + O(q'^2),\\ \{\phi_4,\phi_6,\chi_{12}\}_{\det Sym(2)} &= -3456\Delta(\tau)(\partial_2\phi_{12,1}(\tau,z)u_1^2 + \phi_{12,1}(\tau,z)u_1u_2)q' + O(q'^2),\\ \{\phi_4,\chi_{10},\chi_{12}\}_{\det Sym(2)} &= O(q'^2)u_1^2 + O(q'^2)u_1u_2 + (4E_4\phi_{23,2}q'^2 + O(q'^3))u_2^2.\end{aligned}$$

The determinant $B(Z)$ of the 3×3 matrix whose components are coefficients of u_1^2, u_1u_2 and u_2^2 of the above three forms is equal to

$$4\cdot 12^2\cdot 3456^2\Delta^4E_4\phi_{11,2}^2q'^4 + O(q'^5) \neq 0.$$

So, $\{\phi_4,\phi_6,\chi_{10}\}_{\det Sym(2)}$, $\{\phi_4,\phi_6,\chi_{12}\}_{\det Sym(2)}$, $\{\phi_4,\chi_{10},\chi_{12}\}_{\det Sym(2)}$ are linearly independent over $A^{even}(\Gamma_2)$. Actually we can have more direct expression of this determinant. For any $n\times n$ matrix $A = (a_{ij})$ $(1\leq i,j\leq n)$, we write $\widetilde{a}_{ij}$ the (i,j)-cofactor of A, that is, $(-1)^{i+j}$ times the determinant of matrix subtracting the i-th row and j-th column from A. Then an elementary linear algebra tells us that $\det((\widetilde{a}_{i,j})_{2\leq i,j\leq n}) = a_{11}\det(A)^{n-2}$. Applying this to the matrix $A_{35}(Z)$, we can show that $\det(B(Z)) = 4\phi_4\det(A_{35}(Z)^2) = 4(2^93^4)^2\phi_4\chi_{35}^2$ which is not zero.

Now we see the relation. For the sake of simplicity, we put

$$\begin{aligned} F_{21,2} &= \{\phi_4, \phi_6, \chi_{10}\}_{\det Sym(2)}, \\ F_{23,2} &= \{\phi_4, \phi_6, \chi_{12}\}_{\det Sym(2)}, \\ F_{27,2} &= \{\phi_4, \chi_{10}, \chi_{12}\}_{\det Sym(2)}, \\ F_{29,2} &= \{\phi_6, \chi_{10}, \chi_{12}\}_{\det Sym(2)}. \end{aligned}$$

By definition, the coefficient of $4\phi_4 F_{29,2} - 6\phi_6 F_{27,2} + \chi_{10} F_{23,2} - \chi_{12} F_{21,2}$ of u_1^2, u_1u_2 or u_2^2 is given by

$$\begin{aligned} \begin{vmatrix} 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ \partial_1\phi_4 & \partial_1\phi_6 & \partial_1\chi_{10} & \partial_1\chi_{12} \\ \partial_2\phi_4 & \partial_2\phi_6 & \partial_2\chi_{10} & \partial_2\chi_{12} \\ 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \end{vmatrix} &= - \begin{vmatrix} 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ \partial_1\phi_4 & \partial_1\phi_6 & \partial_1\chi_{10} & \partial_1\chi_{12} \\ 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ \partial_3\phi_4 & \partial_3\phi_6 & \partial_3\chi_{10} & \partial_3\chi_{12} \end{vmatrix} \\ &= \begin{vmatrix} 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ \partial_2\phi_4 & \partial_2\phi_6 & \partial_2\chi_{10} & \partial_2\chi_{12} \\ \partial_3\phi_4 & \partial_3\phi_6 & \partial_3\chi_{10} & \partial_3\chi_{12} \end{vmatrix} = 0. \end{aligned}$$

So we have the relation in the theorem. Now fix an odd natural number k and assume that

$$F_1 F_{21,2} + F_2 F_{23,2} + F_3 F_{27,2} + F_4 F_{29,2} = 0$$

for some $F_1 \in A_{k-21}(\Gamma_1)$, $F_2 \in A_{k-23}(\Gamma_2)$, $F_3 \in A_{k-27}(\Gamma_2)$ and $F_4 \in A_{k-29}(\Gamma_2)$. Then by the above relation and the linear independence of $F_{21,2}$, $F_{23,2}$, $F_{27,2}$, we have

$$(4\phi_4 F_1 + \chi_{12} F_4) F_{21,2} + (4\phi_4 F_2 - 10\chi_{10} F_4) F_{23,2} + (4\phi_4 F_3 + 6\phi_6 F_4) F_{27,2} = 0.$$

So we have $4\phi_4 F_1 + \chi_{12} F_4 = 4\phi_4 F_2 - 10\chi_{10} F_4 = 4\phi_4 F_3 + 6\phi_6 F_4 = 0$. Since $\mathbb{C}[\phi_4, \phi_6, \chi_{10}, \chi_{12}]$ is a weighted polynomial ring, we have $F_4 = 4\phi_4 F_5$ for some $F_5 \in A_{k-33}(\Gamma_2)$ and $F_1 = -\chi_{12} F_5$, $F_2 = 10\chi_{10} F_5$, $F_3 = -6\phi_6 F_5$. So we have

$$\begin{aligned} & F_1 F_{21,2} + F_2 F_{23,2} + F_3 F_{27,2} + F_4 F_{29,2} \\ = \ & F_5 (4\phi_4 F_{29,2} - 6\phi_6 F_{27,2} + 10\chi_{10} F_{21,2} - 12\chi_{12} F_{29,2}). \end{aligned}$$

So the relation in the theorem is the fundamental relation. Finally we must show that these generates the whole $A^{odd}_{sym(2)}(\Gamma_2)$. By the dimension formula of Tsushima [24] and Lemma 2.1, we have

$$\sum_{k:odd} \dim A_{k,2}(\Gamma_2) t^k = \frac{t^{21} + t^{23} + t^{27} + t^{29} - t^{33}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

By the result we proved in the above, the sum below is a direct sum.

$$A^* = A^{even}(\Gamma_2) F_{21,2} \oplus A^{even}(\Gamma_2) F_{23,2} \oplus A^{even}(\Gamma_2) F_{27,2} \oplus \mathbb{C}[\phi_6, \chi_{10}, \chi_{12}] F_{29,2}.$$

So it is obvious that

$$\sum_{k:odd}^{\infty} \dim(A^* \cap A_{k,2}(\Gamma_2))t^k = \frac{t^{21}+t^{23}+t^{27}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} + \frac{t^{29}}{(1-t^6)(1-t^{10})(1-t^{12})}.$$

But this is equal to the generating function of $A_{k,2}(\Gamma_2)$ for odd k, so we have $A^* = A^{odd}_{sym(2)}(\Gamma_2)$. q.e.d.

4.2. **The kernel of the Witt operator.** We first prove the even weight case. By [22], $A^{even}_{sym(2)}(\Gamma_2)$ is spanned by 6 Rankin-Cohen type bracket for a pair among ϕ_4, ϕ_6, χ_{10}, χ_{12}. Since $\chi_{10} = O(z^2)$, we have $W(\partial_i\chi_{10}) = 0$ for $i = 1, 2, 3$, so we have $W(\{\phi_4, \chi_{10}\}_{Sym(2)}) = W(\{\phi_6, \chi_{10}\}_{Sym(2)}) = W(\{\chi_{10}, \chi_{12}\}_{Sym(2)}) = 0$. Then by the structure theorem of [22], we see that

$$W(A^{even}_{sym(2)}(\Gamma_2)) = W(A^{even}(\Gamma_2))W(\{\phi_4,\phi_6\}_{Sym(2)}) + W(A^{even}(\Gamma_2))W(\{\phi_4,\chi_{12}\}_{Sym(2)}) + \mathbb{C}[W(\phi_6), W(\chi_{12})]W(\{\phi_6,\chi_{12}\}_{Sym(2)}).$$

It is obvious and well known that three functions $W(\phi_4)$, $W(\phi_6)$ and $W(\chi_{12})$ are algebraically independent. Assume that

$$F_1W(\{\phi_4,\phi_6\}_{Sym(2)}) + F_2W(\{\phi_4,\chi_{12}\}_{Sym(2)}) + F_3W(\{\phi_6,\chi_{12}\}_{Sym(2)}) = 0$$

for F_1, $F_2 \in W(A^{even}(\Gamma_2))$ and $F_3 \in \mathbb{C}[W(\phi_6), W(\chi_{12})]$. By the relation

$$4\phi_4\{\phi_6,\chi_{12}\}_{Sym(2)} - 6\phi_6\{\phi_4,\chi_{12}\}_{Sym(2)} + 12\chi_{12}\{\phi_4,\phi_6\}_{Sym(2)} = 0,$$

we have

$$4W(\phi_4)W(\{\phi_6,\chi_{12}\}_{Sym(2)}) - 6W(\phi_6)W(\{\phi_4,\chi_{12}\}_{Sym(2)}) + 12W(\chi_{12})W(\{\phi_4,\phi_6\}_{Sym(2)}) = 0.$$

So we have

$$(4W(\phi_4)F_1 - 12W(\chi_{12})F_3)W(\{\phi_4,\phi_6\}_{Sym(2)}) + (4W(\phi_4)F_2 + 6W(\phi_6)F_3)W(\{\phi_4,\chi_{12}\}_{Sym(2)}) = 0.$$

We have

$$W(\{\phi_4,\phi_6\}_{Sym(2)}) = 1278(\Delta(\tau)E_4(\omega)E_6(\omega)u_1^2 + E_4(\tau)E_6(\tau)\Delta(\omega)u_2^2),$$

$$W(\{\phi_4,\chi_{12}\}_{Sym(2)}) = -240(E_6(\tau)\Delta(\tau)E_4(\omega)\Delta(\omega)u_1^2 + E_4(\tau)\Delta(\tau)E_6(\omega)\Delta(\omega)u_2^2),$$

and

$$\begin{vmatrix} \Delta(\tau)E_4(\omega)E_6(\omega) & E_4(\tau)E_6(\tau)\Delta(\omega) \\ E_6(\tau)\Delta(\tau)E_4(\omega)\Delta(\omega) & E_4(\tau)\Delta(\tau)E_6(\omega)\Delta(\omega) \end{vmatrix} = E_4(\tau)\Delta(\tau)E_4(\omega)E_6(\omega)(\Delta(\tau)E_6(\omega)^2 - E_6(\tau)^2\Delta(\omega)) \neq 0.$$

So we have $4W(\phi_4)F_1 = 12W(\chi_{12})F_3$ and $4W(\phi_4)F_2 = -6W(\chi_6)F_3$. Since $F_3 \in \mathbb{C}[W(\phi_6), W(\chi_{12})]$, we have $F_3 = 0$ and $F_1 = F_2 = 0$. So we prove the case of even determinant weight. When the determinant weight is odd, then we see that $W(\{F,G,H\}_{\det Sym(2)}) = 0$ when F, G, or H is χ_{10}. We see easily that $W(\{\phi_4,\phi_6,\chi_{12}\}_{Sym(2)}) \neq 0$. But $\chi_{10}\{\phi_4,\phi_6,\chi_{12}\}_{Sym(2)}$ is contained in the modules spanned by the other generators, so we are done. q.e.d.

We can give an alternative proof of this theorem by using the surjectivity of the Witt operator (cf. [18]) and the dimension formulas.

5. Structure in case $Sym(4)$

In this section, we prove the following theorem.

Theorem 5.1. *The two modules $A^{even}_{sym(4)}(\Gamma_2)$ and $A^{odd}_{sym(4)}(\Gamma_2)$ are free $A^{even}(\Gamma_2)$ module and explicitly given by*

$$\begin{aligned} A^{even}_{sym(4)}(\Gamma_2) =& A^{even}(\Gamma_2)\{\phi_4,\phi_4\}_{Sym(4)} \oplus A^{even}(\Gamma_2)\{\phi_4,\phi_6\}_{Sym(4)} \\ &\oplus A^{even}(\Gamma_2)\{\phi_4,\phi_6\}_{\det{}^2 Sym(4)} \oplus A^{even}(\Gamma_2)\{\phi_4,\chi_{10}\}_{Sym(4)} \\ &\oplus A^{even}(\Gamma_2)\{\phi_6,\chi_{10}\}_{Sym(4)}, \end{aligned}$$

and

$$\begin{aligned} &A^{odd}_{sym(4)}(\Gamma_2) = \\ &\quad A^{even}(\Gamma_2)\{\phi_4,\phi_4,\phi_6\}_{\det Sym(4)} \oplus A^{even}(\Gamma_2)\{\phi_4,\phi_6,\phi_6\}_{\det Sym(4)} \\ &\quad \oplus A^{even}(\Gamma_2)\{\phi_4,\phi_4,\chi_{10}\}_{\det Sym(4)} \oplus A^{even}(\Gamma_2)\{\phi_4,\phi_4,\chi_{12}\}_{\det Sym(4)} \\ &\quad \oplus A^{even}(\Gamma_2)\{\phi_4,\phi_6,\phi_{12}\}_{\det Sym(4)}. \end{aligned}$$

By Tsushima's dimension formula in [24] and Lemma 2.1, we have

$$\sum_{k=0}^{\infty} \dim A_{k,4}(\Gamma_2)t^k = \frac{(1+t^7)(t^8+t^{10}+t^{12}+t^{14}+t^{16})}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

By seeing this, it is obvious that to prove the assertion of Theorem 5.1, all we should do is to prove the linear independence of generators over $A^{even}(\Gamma_2)$. We sometimes idenfity $F(Z) = \sum_{i=0}^4 f_i(\mathbb{Z})u_1^{4-i}u_2^i \in A_{k,4}(\Gamma_2)$ with a vector ${}^t(f_0(Z), f_1(Z), \ldots, f_4(Z))$. Since there are 5 generators for each of $A^{even}_{sym(4)}$ or $A^{odd}_{sym(4)}$, we have two 5×5 matrix $B^{even}(Z)$ whose column vectors are $\{\phi_4,\phi_4\}_{Sym(4)}$, $\{\phi_4,\phi_6\}_{Sym(4)}$, $\{\phi_4,\phi_6\}_{\det{}^2 Sym(4)}$, $\{\phi_4,\chi_{10}\}_{Sym(4)}$, $\{\phi_4,\chi_{12}\}_{Sym(4)}$ in this order and $B^{odd}(Z)$ whose columns are generators in the order as appearing in the theorem. We must show that $\det(B^{even}(Z))$ and $\det(B^{odd}(Z))$ are not identically zero as holomorphic functions. The proof of this fact is quite technical and maybe there are several ways to show this. We sketch one proof. We first consider $B^{even}(Z)$. We have $\{\phi_4,\phi_4\}_{sym(4)} = 4800\Delta u_1^4 + O(q')$, where $q' = e^{2\pi i\omega}$. All the other generators are divisible by q', so we have $\det(B^{even}(Z)) = 4800\Delta(B^{even}(Z))_{11} + O(q'^5)$ where $(*)_{11}$ denotes the

$(1,1)$ cofactor for any matrix $*$. By definition, $\{\phi_4, \phi_6\}_{Sym(4)}/840$ is given modulo q'^2 by q' times the following vector

$$\begin{pmatrix} * \\ 12(E_6\partial_1\partial_2 E_{4,1} - E_4\partial_1\partial_2 E_{6,1}) - 10\partial_1 E_6 \partial_2 E_{4,1} + 21\partial_1 E_4 \partial_2 E_{6,1} \\ 6E_6(\partial_2^2 E_{4,1} + 2\partial_1 E_{4,1}) + 21\partial_1 E_4 E_{6,1} - 10\partial_1 E_6 E_{4,1} - 6E_4(\partial_2^2 E_{6,1} + 2\partial_1 E_{6,1}) \\ 12(E_6\partial_2 E_{4,1} - E_4\partial_2 E_{6,1}) \\ 6(E_6 E_{4,1} - E_4 E_{6,1}) \end{pmatrix},$$

where $*$ is a certain function of τ and ω. We write $E_i' = \partial_1 E_i$ and $E_i'' = \partial_1^2 E_i$ for $i = 4$, 6. By using the relations

$$\begin{aligned} 144\partial_2^i \phi_{10,1} &= E_6\partial_2^i E_{4,1} - E_4\partial_2^i E_{6,1}, \\ 144\partial_1\partial_2\phi_{10,1} &= E_6\partial_1\partial_2 E_{4,1} - E_4\partial_1\partial_2 E_{6,1} + E_6'\partial_2 E_{4,1} - E_4'\partial_2 E_{6,1}, \end{aligned}$$

the above vector is equal to

$$\begin{pmatrix} * \\ 12\cdot 144\partial_1\partial_2\phi_{10,1} + 11(3E_4'\partial_2 E_{6,1} - 2E_6'\partial_2 E_{4,1}) \\ 6\cdot 144(\partial_2^2\phi_{10,1} + 2\partial_1\phi_{10,1}) + 11(3E_4' E_{6,1} - 2E_6' E_{4,1}) \\ 12\cdot 144\partial_2\phi_{10,1} \\ 6\cdot 144\phi_{10,1} \end{pmatrix}.$$

In the same way, we have

$$\{\phi_4, \chi_{10}\}_{Sym(4)} = q' \begin{pmatrix} * \\ -55E_4'\partial_2\phi_{10,1} + 20E_4\partial_1\partial_2\phi_{10,1} \\ -55E_4'\phi_{10,1} + 10E_4(\partial_2^2\phi_{10,1} + 2\partial_1\phi_{10,1}) \\ 20E_4\partial_2\phi_{10,1} \\ 10E_4\phi_{10,1} \end{pmatrix} + O(q'^2),$$

$$\{\phi_6, \chi_{10}\}_{Sym(4)} = q' \begin{pmatrix} * \\ -77E_6'\partial_2\phi_{10,1} + 42E_6\partial_1\partial_2\phi_{10,1} \\ -77E_6'\phi_{10,1} + 21E_6(\partial_2^2\phi_{10,1} + 2\partial_1\phi_{10,1}) \\ 42E_6\partial_2\phi_{10,1} \\ 21E_6\phi_{10,1} \end{pmatrix} + O(q'^2),$$

and

$$\{\phi_4, \phi_6\}_{\det^2 Sym(4)} = -32598720q' \times \begin{pmatrix} * \\ * \\ * \\ 2\partial_2\phi_{12,1} \\ \phi_{12,1} \end{pmatrix} + O(q'^2).$$

We can show easily that $\{\phi_4,\chi_{10}\}_{Sym(4)} - 10E_4\{\phi_4,\phi_6\}_{Sym(4)}/(840\times 6\times 144)$ is given by

$$\begin{pmatrix} * \\ -220\Delta\partial_2 E_{4,1} \\ -220\Delta E_{4,1} \\ 0 \\ 0 \end{pmatrix},$$

and $\{\phi_6,\chi_{10}\}_{Sym(4)} - 21E_6\{\phi_4,\phi_6\}_{Sym(4)}/(840\times 6\times 144)$ by

$$\begin{pmatrix} * \\ -462\Delta\partial_2 E_{6,1} \\ -462\Delta E_{6,1} \\ 0 \\ 0 \end{pmatrix}.$$

So we have

$$\begin{aligned} &\det(B^{even}(Z)) \\ = &\ c_1\Delta \begin{vmatrix} \Delta\partial_2 E_{4,1} & \Delta\partial_2 E_{6,1} \\ \Delta E_{4,1} & -\Delta E_{6,1} \end{vmatrix} \times \begin{vmatrix} \partial_2\phi_{10,1} & \partial_2\phi_{12,1} \\ \phi_{10,1} & \phi_{12,1} \end{vmatrix} {q'}^4 + O({q'}^5) \\ = &\ c_2\Delta^4\phi_{11,2}^2 {q'}^4 + O({q'}^5), \end{aligned}$$

where c_1, c_2 are certain non-zero constants. So we prove $\det B^{even}(Z)$ is not identically zero and theorem follows for $A^{even}_{sym(4)}(\Gamma_2)$. Now we sketch the proof of $\det B^{odd}(Z)\neq 0$. We use the following relations.

$$\begin{aligned} \{E_4,E_6\}_2 &= -3456\Delta, \\ \{E_6,E_6\}_2 &= 0, \\ \{E_4,E_4\}_4 &= 4800\Delta, \\ \{E_4,E_6\}_4 &= 0, \\ \{E_{4,1},E_{6,1}\}_{jac} &= 144\phi_{11,2}. \end{aligned}$$

By definition of the bracket, we have

$$\{\phi_4,\phi_4,\phi_6\}_{\det Sym(4)} = 4147200 \begin{pmatrix} * \\ * \\ -3\Delta\partial_2 E_{4,1} \\ -2\Delta E_{4,1} \\ 0 \end{pmatrix} q' + O({q'}^2),$$

$$\{\phi_4,\phi_6,\phi_6\}_{\det Sym(4)} = 4354560 \begin{pmatrix} * \\ * \\ 3\Delta\partial_2 E_{6,1} \\ 2\Delta E_{6,1} \\ 0 \end{pmatrix} q' + O({q'}^2),$$

$$\{\phi_4,\phi_4,\chi_{10}\}_{\det Sym(4)} = 4800\begin{pmatrix}\Delta\partial_2\phi_{10,1}\\ -2\Delta\phi_{10,1}\\ 0\\ 0\\ 0\end{pmatrix} q' + O(q'^2),$$

$$\{\phi_4,\phi_4,\chi_{12}\}_{\det Sym(4)} = 4800\begin{pmatrix}\Delta\partial_2\phi_{12,1}\\ -2\Delta\phi_{12,1}\\ 0\\ 0\\ 0\end{pmatrix} q' + O(q'^2),$$

$$\{\phi_4,\phi_6,\chi_{12}\}_{\det Sym(4)} = 5040\cdot 144\begin{pmatrix} *\\ *\\ *\\ *\\ -\phi_{12,1}\partial_2\phi_{10,1}+\phi_{10,1}\partial_2\phi_{12,1}\end{pmatrix} q'^2 + O(q'^3).$$

So by noting $\phi_{12,1}\partial_2\phi_{10,1}-\phi_{10,1}\partial_2\phi_{12,1}=12\Delta\phi_{11,2}$, we see that

$$\det B^{odd}(Z) = c\times\Delta^6\phi_{11,2}^3 q'^6 + O(q'^7)$$

for some non-zero large constant c, hence this is not zero. So we prove Theorem 5.1. It is plausible that each $\det B^{even}(Z)$ or $\det B^{odd}(Z)$ is a constant multiple of χ_{35}^2 and χ_{35}^3, since $\chi_{35}=-\Delta^2\phi_{11,2}q'^2+O(q'^3)$.

6. Structure in case $Sym(6)$

First we see dimension formulas. By Tsushima's dimension formula and Lemma 2.1, we have

$$\sum_{k=0}^{\infty}\dim A_{k,6}(\Gamma_2) = \frac{(1+t^5)(t^6+t^8+t^{10}+t^{12}+t^{14}+t^{16}+t^{18})}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

We have the following table of dimensions.

k	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\dim A_{k,6}$	0	1	0	1	0	2	1	3	1	4	2	6	3	9	4
$\dim S_{k,6}$	0	0	0	1	0	1	1	2	1	3	2	5	3	7	4

So we must construct a form in each $A_{6,6}(\Gamma_2)$, $A_{8,6}(\Gamma_2)$ or $A_{10,6}(\Gamma_2)$. There is no way to construct $A_{6,6}(\Gamma_2)$ by the Rankin-Cohen type bracket since the deteminant weight should be at least $4+4=8$ by such construction. Also we see by definition that $\{\phi_4,\phi_4\}_{Sym(6)}=0$ and $\{\phi_4,\phi_4\}_{\det^2 Sym(6)}=0$, so we cannot construct $A_{8,6}(\Gamma_2)$ or $A_{10,6}(\Gamma_2)$ by this method, and we need other constructions. We use Eisenstein series and theta functions with harmonic polynomials. First we define theta functions in general. For a natural number m and vectors $x=(x_i)$, $(y_i)\in\mathbb{C}^m$, we define $(x,y)=\sum_{i=1}^m x_iy_i$. We assume $m\equiv 0 \bmod 8$ and fix an even unimodular lattice $L \subset \mathbb{R}^m$ and an integer $k \geq 0$. For any $a, b \in \mathbb{C}^m$ such that $(a,a)=0$, $(b,b)=0$, $(a,b)=0$, we put

$$\theta_{L,a,b,k,\nu}(Z) = \sum_{x,y\in L} (x,a)^{j-\nu}(y,a)^{\nu} \begin{vmatrix} (x,a) & (y,a) \\ (x,b) & (y,b) \end{vmatrix}^k e^{\pi i((x,x)\tau+2(x,y)z+(y,y)\omega)}.$$

We define

$$\theta_{L,a,b,(k,j)}(Z) = \sum_{\nu=0}^{j} \binom{j}{\nu} \theta_{L,a,b,k,\nu}(Z) u_1^{j-\nu} u_2^{\nu}.$$

Then it is well-known and easy to see that we have $\theta_{L,a,b,(k,j)}(Z) \in A_{m/2+k,j}(\Gamma_2)$ (cf. [9]). When $k>0$, this is also a cusp form. Now we take the even unimodular lattice $E_8 \subset \mathbb{R}^8$ of rank 8 which is unique up to isometry. More explicitly, E_8 is given as follows as in [23].

$$E_8 = \{x = (x_i)_{1\leq i\leq 8} \in \mathbb{Q}^8; 2x_i \in \mathbb{Z}, x_i - x_j \in \mathbb{Z}, \sum_{i=1}^{8} x_i \in 2\mathbb{Z}\}.$$

We put

$$\begin{aligned} a &= (2,1,i,i,i,i,i,0), \\ b &= (1,-1,i,i,1,-1,-i.i). \end{aligned}$$

We define $X_{8,6} \in A_{8,6}(\Gamma_2)$ and $X_{10,6} \in A_{10,6}(\Gamma_2)$ by

$$\begin{aligned} X_{8,6}(Z) &= \theta_{E_8,a,b,(4,6)}(Z)/111456000, \\ X_{10,6}(Z) &= \theta_{E_8,a,b,(6,6)}(Z)/450252000. \end{aligned}$$

By computer calculation, we can show that both forms do not vanish identically. Here both $X_{8,6}(Z)$ and $X_{10,6}(Z)$ are cusp forms. Now we must construct $A_{6,6}(\Gamma_2)$ also. Since $\theta_{L,a,b,(k,j)}$ is a cusp form when $k>0$ and $\mathrm{rank}(L) \equiv 0 \bmod 8$, we cannot construct a form in $A_{6,6}(\Gamma_2)$ by theta functions. But by virtue of Arakawa [3] Prop. 1.2, for any $f \in S_{k+j}(\Gamma_2)$ with even $k \geq 6$ and even $j \geq 0$, we have the Klingen type Eisenstein series $E_{k,j}(f) \in A_{k,j}(\Gamma_2)$ such that $\Phi(E_{k,j}(f)) = f(\tau)u_1^6$. So for the Ramanujan Delta function $\Delta \in S_{12}(\Gamma_1)$, we have $E_{6,6}(\Delta) \in A_{6,6}(\Gamma_2)$. Now we can state our theorem.

Theorem 6.1. *The module $A^{even}_{sym(6)}(\Gamma_2)$ is a free $A^{even}(\Gamma_2)$ module and given explicitly by*

$$\begin{aligned} A^{even}_{sym(6)}(\Gamma_2) = {} & A^{even}(\Gamma_2)E_{6,6}(\Delta) \oplus A^{even}(\Gamma_2)X_{8,6} \oplus A^{even}(\Gamma_2)X_{10,6} \\ & \oplus A^{even}(\Gamma_2)\{\phi_4,\phi_6\}_{\det^2 Sym(6)} \oplus A^{even}(\Gamma_2)\{\phi_4,\chi_{10}\}_{Sym(6)} \\ & \oplus A^{even}(\Gamma_2)\{\phi_4,\chi_{12}\}_{Sym(6)} \oplus A^{even}(\Gamma_2)\{\phi_6,\chi_{12}\}_{Sym(6)}. \end{aligned}$$

The point of the proof is again to show the linear independence of the generators. We can show that the determinant of the 7×7 matrix $C(Z)$ whose columns consist of generators is equal to $c\Delta^6\phi_{11,2}^3 q'^6 + O(q'^7)$ with some non-zero constant c. We can show this by similar calculation as in the last section and we sketch the proof here. By definition, we have $E_{6,6}(\Delta) = \Delta u_1^6 + O(q')$. All the other generators are divisible by q', so we have $\det(C(Z)) = \Delta C(Z)_{11} + O(q'^7)$ where $C(Z)_{11}$ is the $(1,1)$ cofactor of $C(Z)$. So it is enough to show that $C(Z)_{11} = c(\tau,z)q'^6 + O(q'^7)$ for a function $c(\tau,z)$ which is not identically zero. To calculate $c(\tau,z)$, we need the first Fourier-Jacobi coefficients of 6 generators except for $E_{6,6}(\Delta)$, in particular the coefficients of $u_1^{6-i}u_2^i$ for $i > 0$. Except for $X_{8,6}$ and $X_{10,6}$, we can obtain these from the definition. As for $X_{8,6}$ and $X_{10,6}$, to determine the Fourier-Jacobi coefficient of index one, (i.e. the coefficient of q' in the Fourier expansion), we use a general theory of Fourier-Jacobi expansion of vector valued forms in [15]. Fourier-Jacobi coefficients of index m of a vector valued Siegel modular form F of weight $\det^k Sym(j)$ is a linear combination of usual Jacobi forms of weight $k+\nu$ $(0 \le \nu \le j)$ of index m and their derivatives (cf. [15] Theorem 2.1). Since the space of Jacobi forms of index one is known, we can obtain the Fourier-Jacobi coefficient of index one of F if we have enough Fourier coefficients of F. We omit the details of the calculation, but by this method we see that the coefficient of q' of $X_{8,6}$ is given by

$$\begin{pmatrix} * \\ -\frac{11}{19}\partial_2^3\phi_{10,1} + \frac{6}{19}\partial_1\partial_2\phi_{10,1} + \frac{5}{19}\partial_2\phi_{12,1} \\ -\frac{30}{19}\partial_2^2\phi_{10,1} + \frac{6}{19}\partial_1\phi_{10,1} + \frac{5}{19}\phi_{12,1} \\ -2\partial_2\phi_{10,1} \\ -\phi_{10,1} \\ 0 \\ 0 \end{pmatrix},$$

and that of $X_{10,6}$ is by

$$\begin{pmatrix} * \\ \frac{78}{133}\partial_2^5\phi_{10,1} - \frac{120}{133}\partial_2^3\partial_1\phi_{10,1} + \frac{30}{133}\partial_2\partial_1^2\phi_{10,1} + \frac{1045}{161}E_4\partial_2\phi_{10,1} - \frac{325}{437}\partial_2^3\phi_{12,1} + \frac{150}{437}\partial_2\partial_1\phi_{12,1} \\ \frac{1045}{161}E_4\phi_{10,1} - \frac{900}{19\cdot 23}\partial_2^2\phi_{12,1} + \frac{150}{19\cdot 23}\partial_1\phi_{12,1} + \frac{330}{133}\partial_2^4\phi_{10,1} + \frac{30}{133}\partial_1^2\phi_{10,1} - \frac{330}{133}\partial_1\partial_2^2\phi_{10,1} \\ -\frac{50}{19}\partial_2\phi_{12,1} + \frac{110}{19}\partial_2^3\phi_{10,1} - \frac{60}{19}\partial_1\partial_2\phi_{10,1} \\ -\frac{25}{19}\phi_{12,1} + \frac{150}{19}\partial_2^2\phi_{10,1} - \frac{30}{19}\partial_1\phi_{10,1} \\ 6\partial_2\phi_{10,1} \\ 2\phi_{10,1} \end{pmatrix},$$

where $*$ are certain functions of τ and z. For the sake of simplicity, we write $F_{12,6} = \{\phi_4, \phi_6\}_{\det^2 Sym(6)}/11642400$. Then the coefficient of q' of

$F_{12,6}$ is given by

$$\begin{pmatrix} * \\ c_1(\tau,z) \\ c_2(\tau,z) \\ \frac{126}{23}E_4\partial_2\phi_{10,1}+\frac{65}{23}\partial_2^3\phi_{12,1}-\frac{30}{23}\partial_1\partial_2\phi_{12,1} \\ \frac{63}{23}E_4\phi_{10,1}+\frac{90}{23}\partial_2^2\phi_{12,1}-\frac{15}{23}\partial_1\phi_{12,1} \\ 3\partial_2\phi_{12,1} \\ \phi_{12,1} \end{pmatrix},$$

where

$$\begin{aligned} c_1(\tau,z) &= \frac{35}{12}E_6\partial_2\phi_{10,1}-\frac{49}{60}E_4\partial_2\phi_{12,1}-\frac{14}{23}\partial_1(E_4\partial_2\phi_{10,1}) \\ &\quad +\frac{35}{23}E_4\partial_2^3\phi_{10,1}+\frac{63}{230}\partial_2^5\phi_{12,1}-\frac{42}{115}\partial_2^3\partial_1\phi_{12,1}+\frac{9}{115}\partial_2\partial_1^2\phi_{12,1}, \\ c_2(\tau,z) &= \frac{35}{12}E_6\phi_{10,1}-\frac{49}{60}E_4\phi_{12,1}-\frac{14}{23}\partial_1(E_4\phi_{10,1}) \\ &\quad +\frac{98}{23}E_4\partial_2^2\phi_{10,1}+\frac{273}{230}\partial_2^4\phi_{12,1}-\frac{117}{115}\partial_1\partial_2^2\phi_{12,1}+\frac{9}{115}\partial_1^2\phi_{12,1}. \end{aligned}$$

Coefficients of q' of each $\{\phi_4,\chi_{10}\}_{Sym(6)}$, $\{\phi_4,\chi_{12}\}_{Sym(6)}$ and $\{\phi_6,\chi_{12}\}_{Sym(6)}$ are given by

$$\begin{pmatrix} * \\ 396E_4''(\partial_2\phi_{10,1})-360E_4'(\partial_1\partial_2\phi_{10,1})+60E_4(\partial_1^2\partial_2\phi_{10,1}) \\ 396E_4''\phi_{10,1}-180E_4'(2\partial_1\phi_{10,1}+\partial_2^2\phi_{10,1})+60E_4(\partial_1^2\phi_{10,1}+\partial_1\partial_2^2\phi_{10,1}) \\ -360E_4'(\partial_2\phi_{10,1})+20E_4(6(\partial_1\partial_2\phi_{10,1})+\partial_2^3\phi_{10,1}) \\ -180E_4'\phi_{10,1}+60E_4(\partial_1\phi_{10,1}+\partial_2^2\phi_{10,1}) \\ 60E_4(\partial_2\phi_{10,1}) \\ 20E_4(\phi_{10,1}) \end{pmatrix},$$

$$\begin{pmatrix} * \\ 546E_4''(\partial_2\phi_{12,1})-420E_4'(\partial_1\partial_2\phi_{12,1})+60E_4(\partial_1^2\partial_2\phi_{12,1}) \\ 546E_4''\phi_{12,1}-210E_4'(2\partial_1\phi_{12,1}+\partial_2^2\phi_{12,1})+60E_4(\partial_1^2\phi_{12,1}+\partial_1\partial_2^2\phi_{12,1}) \\ -420E_4'(\partial_2\phi_{12,1})+20E_4(6(\partial_1\partial_2\phi_{12,1})+\partial_2^3\phi_{12,1}) \\ -210E_4'\phi_{12,1}+60E_4(\partial_1\phi_{12,1}+\partial_2^2\phi_{12,1}) \\ 60E_4(\partial_2\phi_{12,1}) \\ 20E_4(\phi_{12,1}) \end{pmatrix},$$

and

$$\begin{pmatrix} * \\ 728E_6''(\partial_2\phi_{12,1}) - 784E_6'(\partial_1\partial_2\phi_{12,1}) + 168E_6(\partial_1^2\partial_2\phi_{12,1}) \\ 728E_6''\phi_{12,1} - 392E_6'(2\partial_1\phi_{12,1} + \partial_2^2\phi_{12,1}) + 168E_6(\partial_1^2\phi_{12,1} + \partial_1\partial_2^2\phi_{12,1}) \\ -784E_6'(\partial_2\phi_{12,1}) + 56E_6(6(\partial_1\partial_2\phi_{12,1}) + \partial_2^3\phi_{12,1}) \\ -392E_6'\phi_{12,1} + 168E_6(\partial_1\phi_{12,1} + \partial_2^2\phi_{12,1}) \\ 168E_6(\partial_2\phi_{12,1}) \\ 56E_6(\phi_{12,1}) \end{pmatrix},$$

respectively. Now we put

$$\begin{aligned} G_{14,6} &= \{\phi_4, \chi_{10}\}_{Sym(6)} - 10E_4X_{10,6} + 60E_6X_{8,6}, \\ G_{16,6} &= \{\phi_4, \chi_{12}\}_{Sym(6)} - 20E_4F_{12,6} - 70E_4^2X_{8,6}, \\ G_{18,6} &= \{\phi_6, \chi_{12}\}_{Sym(6)} - 56E_6F_{12,6} - 196E_4E_6X_{8,6}. \end{aligned}$$

Then the coefficients of q' of $G_{14,6}$, $G_{16,6}$ and $G_{18,6}$ are given by

$$\begin{pmatrix} * \\ * \\ * \\ 20E_4\partial_2\phi_{12,1} \\ 10E_4\phi_{12,1} \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} * \\ * \\ * \\ 140E_6\partial_2\phi_{12,1} \\ 70E_6\phi_{12,1} \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} * \\ * \\ * \\ 392E_4^2\partial_2\phi_{12,1} \\ 196E_4^2\phi_{12,1} \\ 0 \\ 0 \end{pmatrix},$$

respectively, where $*$ are certain functions of τ and z. If we put $H_{16,6} = G_{16,6} - 7(E_6/E_4)G_{14,6}$ and $H_{18,6} = G_{18,6} - (196/10)E_4G_{14,6}$, then by computer calculation, we see that the coefficients of q' of $H_{16,6}$ and $H_{18,6}$ are given by

$$120960(E_4)^{-1}\begin{pmatrix} * \\ \Delta\partial_2\phi_{12,1} \\ \Delta\phi_{12,1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad 282240\begin{pmatrix} * \\ \Delta\partial_2\phi_{10,1} \\ \Delta\phi_{10,1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

respectively. So $\det(C(Z))$ is a non-zero constant times

$$\det(E_{6,6}(\Delta), X_{8,6}, X_{10,6}, F_{12,6}, G_{14,6}, H_{16,6}, H_{18,6})$$

and the (1,1) cofactor of this is a non-zero constant times

$$\begin{vmatrix} * & * & * & * & (E_4)^{-1}\Delta\partial_2\phi_{12,1} & \Delta\partial_2\phi_{10,1} \\ * & * & * & * & (E_4)^{-1}\Delta\phi_{12,1} & \Delta\phi_{10,1} \\ -2\partial_2\phi_{10,1} & * & * & 20E_4\partial_2\phi_{12,1} & 0 & 0 \\ -\phi_{10,1} & * & * & 10E_4\phi_{12,1} & 0 & 0 \\ 0 & 6\partial_2\phi_{10,1} & 3\partial_2\phi_{12,1} & 0 & 0 & 0 \\ 0 & 2\phi_{10,1} & \phi_{12,1} & 0 & 0 & 0 \end{vmatrix}.$$

This is a constant multiple of $\Delta^5\phi_{11,2}^3$. So $\det(C(Z))$ is a non-zero constant times $\Delta^6\phi_{11,2}^3$, so this is not identically zero. So we prove Theorem 6.1. It is plausible that $\det(C(Z))$ is a constant multiple of χ_{35}^3.

7. Concluding remarks

We give here two concluding remarks.

Remark 1. When are $A^{even}_{Sym(j)}(\Gamma_2)$ or $A^{odd}_{Sym(j)}(\Gamma_2)$ free?
By Tsushima's dimension formula, Satake's surjectivity of the Φ-operator, and by some ad-hoc arguments, we can calculate the generating functions of dimensions of $A_{k,j}(\Gamma_2)$ for small j. We can show

$$\begin{aligned} \sum_{k:odd} \dim A_{k,6}(\Gamma_2)t^k &= \frac{t^{11}+t^{13}+t^{15}+t^{17}+t^{19}+t^{21}+t^{23}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}, \\ \sum_{k:even} \dim A_{k,8}(\Gamma_2)t^k &= \frac{t^4+t^8+2t^{10}+2t^{12}+t^{14}+t^{16}+t^{18}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}, \\ \sum_{k:odd} \dim A_{k,8}(\Gamma_2)t^k &= \frac{t^9+t^{11}+t^{13}+2t^{15}+2t^{17}+t^{19}+t^{23}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}, \end{aligned}$$

$$\sum_{k:even} \dim A_{k,10}(\Gamma_2)t^k = \frac{t^6+t^8+2t^{10}+2t^{12}+3t^{14}+2t^{16}+t^{18}+t^{20}-t^{24}-t^{26}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

Since we have

$$\sum_{k:even} \dim A_k(\Gamma_2) = \frac{1}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

it is very likely that $A^{odd}_{Sym(6)}(\Gamma_2)$, $A^{even}_{Sym(8)}(\Gamma_2)$, $A^{odd}_{Sym(8)}(\Gamma_2)$ are also free $A^{even}(\Gamma_2)$ modules, each spanned by elements with determinant weights of the powers of t appearing in the numerator of the generating functions. On the other hand, as for $A^{even}_{sym(10)}(\Gamma_2)$, by the above generating function, we can see that there are no homogeneous free generators over $A^{even}(\Gamma_2)$. (Actually this means that $A^{even}_{sym(10)}(\Gamma_2)$ does not have either inhomogeneous free generators over $A^{even}(\Gamma_2)$. Indeed T. Hibi informed the author answering to his question that Juergen Herzog proved the following claim. Let R be a graded ring such that R_0 is a field and M a finitely generated graded module over R. If M has (not necessarily homogeneous) free generators over R, then we have homogeneous free generators of M over R.)

Remark 2. Problem on mysterious weights.
In general, if we take $f_i \in A_{k_i,Sym(j)}(\Gamma_2)$ for $i = 1, \ldots, j+1$ and identify f_i as a $j+1$ dimensional vector of functions on H_2, then $\det(f_1, \ldots, f_{j+1})$ is a scalar valued Siegel modular form of Γ_2 of weight

$$k_1 + k_2 + \cdots + k_{j+1} + j(j+1)/2.$$

We already note that when we take free generators of $A^{even}_{sym(4)}(\Gamma_2)$, $A^{odd}_{sym(4)}(\Gamma_2)$ and $A^{even}_{sym(6)}(\Gamma_2)$, then the above weight of the deteminant of generators is $70 = 35 \times 2$, $70 = 35 \times 2$, or $105 = 35 \times 3$, respectively. Also by judging from the first non-vanishing Fourier Jacobi coefficients, it is very plausible that these are constant multiples of χ_{35}^2, χ_{35}^2 and χ_{35}^3, respectively. Now let's believe the conjecture on free modules in Remark 1. Then surprisingly the similar weights of the determinant are all multiples of 35. Indeed we have $11+13+15+17+19+21+23+21 = 140 = 35 \times 4$ for $A^{odd}_{sym(6)}(\Gamma_2)$, $4+8+2\times 10+2\times 12+14+16+18+36 = 140 = 35\times 4$ for $A^{even}_{sym(8)}(\Gamma_2)$ and $9+11+13+2\times 15+2\times 17+19+23+36 = 175 = 35\times 5$ for $A^{odd}_{sym(8)}(\Gamma_2)$. It is natural to ask if all these determinants are constant mutiples of powers of χ_{35}. We can make a bolder guess including the case when $A_{sym(j)}(\Gamma_2)$ itself is not free. Assume that $f_1 \in A_{k_1,j}(\Gamma_2)$, $\ldots$, $f_{j+1} \in A_{k_{j+1},j}(\Gamma_2)$ are free over $A^{even}(\Gamma_2)$ and $k_i \mod 2$ are the same. Assume that

$$k_1 + \cdots + k_{j+1} + j(j+1)/2 = 35q + r$$

with $0 \le r < 35$. Then, is the determinant $\det(f_1, \ldots, f_{j+1})$ divisible by χ_{35}^q? We can show that this is true at least for generators of $A_{sym(2)}(\Gamma_2)$.

8. Appendix

Fourier coefficients of the generators which appear in the previous sections are easily calculated by Fourier coefficients of the ring generators of $A^{even}(\Gamma_2)$ except for $X_{8,6}$, $X_{10,6}$. The ring generators ϕ_4, ϕ_6, χ_{10}, χ_{12} of $A^{even}(\Gamma)$ are all Saito-Kurokawa lifts and their Fourier coefficients are very easy to calculate. Here we give tables of Fourier coefficients for the remaining cases, i.e. for $X_{8,6}$ and $X_{10,6}$, which we used implicitly in the calculation of section 6. For any half integral matrix $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, we denote by (a,c,b,i) the coefficients of $u_1^{6-i}u_2^i$ at T of $X_{8,6}$ or $X_{10,6}$. Then we have

(a,c,b,i)	$X_{8,6}$	$X_{10,6}$
(1,1,0,0)	0	−4
(1,1,0,1)	0	0
(1,1,0,2)	2	−10
(1,1,0,3)	0	0
(1,1,0,4)	2	−10
(1,1,0,5)	0	0
(1,1,0,6)	0	−4
(1,1,1,0)	0	2
(1,1,1,1)	0	6
(1,1,1,2)	−1	5
(1,1,1,3)	−2	0
(1,1,1,4)	−1	5
(1,1,1,5)	0	6
(1,1,1,6)	0	2
(2,1,0,0)	24	504
(2,1,0,1)	0	0
(2,1,0,2)	−12	−2940
(2,1,0,3)	0	0
(2,1,0,4)	−36	60
(2,1,0,5)	0	0
(2,1,0,6)	0	72
(2,1,1,0)	−16	−224
(2,1,1,1)	−24	1464
(2,1,1,2)	−8	1600
(2,1,1,3)	32	240
(2,1,1,4)	16	40
(2,1,1,5)	0	−96
(2,1,1,6)	0	−32
(2,1,2,0)	4	−28
(2,1,2,1)	12	−84
(2,1,2,2)	14	−130
(2,1,2,3)	8	−120
(2,1,2,4)	2	−70
(2,1,2,5)	0	−24
(2,1,2,6)	0	−4
(3,1,0,0)	−208	12368
(3,1,0,1)	0	0
(3,1,0,2)	−64	26480
(3,1,0,3)	0	0
(3,1,0,4)	272	320
(3,1,0,5)	0	0
(3,1,0,6)	0	−544
(3,1,1,0)	132	−3906
(3,1,1,1)	372	−9906
(3,1,1,2)	273	−11865
(3,1,1,3)	−198	−3720
(3,1,1,4)	−99	−1365
(3,1,1,5)	0	594
(3,1,1,6)	0	198

(a,c,b,i)	$X_{8,6}$	$X_{10,6}$
(3,1,2,0)	−24	−2304
(3,1,2,1)	−168	−5208
(3,1,2,2)	−228	−1500
(3,1,2,3)	−144	1680
(3,1,2,4)	−36	1140
(3,1,2,5)	0	432
(3,1,2,6)	0	72
(3,1,3,0)	−4	26
(3,1,3,1)	−12	78
(3,1,3,2)	−13	125
(3,1,3,3)	−6	120
(3,1,3,4)	−1	65
(3,1,3,5)	0	18
(3,1,3,6)	0	2
(4,1,0,0)	0	−172992
(4,1,0,1)	0	0
(4,1,0,2)	576	480
(4,1,0,3)	0	0
(4,1,0,4)	−1056	−2880
(4,1,0,5)	0	0
(4,1,0,6)	0	2112
(4,1,1,0)	0	44880
(4,1,1,1)	−2280	−33240
(4,1,1,2)	−2040	−21600
(4,1,1,3)	480	22800
(4,1,1,4)	240	10200
(4,1,1,5)	0	−1440
(4,1,1,6)	0	−480
(4,1,2,0)	0	38624
(4,1,2,1)	960	50976
(4,1,2,2)	1568	20240
(4,1,2,3)	1088	−9600
(4,1,2,4)	272	−7840
(4,1,2,5)	0	−3264
(4,1,2,6)	0	−544
(4,1,3,0)	0	2992
(4,1,3,1)	120	4872
(4,1,3,2)	184	1120
(4,1,3,3)	96	−1200
(4,1,3,4)	16	−920
(4,1,3,5)	0	−288
(4,1,3,6)	0	−32
(5,1,0,0)	6000	640800
(5,1,0,1)	0	0
(5,1,0,2)	600	−825000
(5,1,0,3)	0	0
(5,1,0,4)	1800	−3000
(5,1,0,5)	0	0
(5,1,0,6)	0	−3600

(a,c,b,i)	$X_{8,6}$	$X_{10,6}$
(5,1,1,0)	−3040	−116690
(5,1,1,1)	6072	451242
(5,1,1,2)	6325	421135
(5,1,1,3)	506	−60720
(5,1,1,4)	253	−31625
(5,1,1,5)	0	−1518
(5,1,1,6)	0	−506
(5,1,2,0)	−480	−173280
(5,1,2,1)	−3072	2112
(5,1,2,2)	−5760	14880
(5,1,2,3)	−4224	30720
(5,1,2,4)	−1056	28800
(5,1,2,5)	0	12672
(5,1,2,6)	0	2112
(5,1,3,0)	480	−29970
(5,1,3,1)	−72	−46098
(5,1,3,2)	−915	−22305
(5,1,3,3)	−594	720
(5,1,3,4)	−99	4575
(5,1,3,5)	0	1782
(5,1,3,6)	0	198
(5,1,4,0)	40	−460
(5,1,4,1)	72	−1128
(5,1,4,2)	50	−1210
(5,1,4,3)	16	−720
(5,1,4,4)	2	−250
(5,1,4,5)	0	−48
(5,1,4,6)	0	−4
(6,1,0,0)	−25200	−123888
(6,1,0,1)	0	0
(6,1,0,2)	−17016	3705480
(6,1,0,3)	0	0
(6,1,0,4)	1464	85080
(6,1,0,5)	0	0
(6,1,0,6)	0	−2928
(6,1,1,0)	8400	237312
(6,1,1,1)	−120	−1055784
(6,1,1,2)	−2856	−1057920
(6,1,1,3)	−5472	1200
(6,1,1,4)	−2736	14280
(6,1,1,5)	0	16416
(6,1,1,6)	0	5472
(6,1,2,0)	8400	−190800
(6,1,2,1)	8400	−1683600
(6,1,2,2)	11400	−897000
(6,1,2,3)	7200	−84000
(6,1,2,4)	1800	−57000
(6,1,2,5)	0	−21600
(6,1,2,6)	0	−3600

(a,c,b,i)	$X_{8,6}$	$X_{10,6}$
$(6,1,3,0)$	-3600	21120
$(6,1,3,1)$	-3960	22680
$(6,1,3,2)$	840	86400
$(6,1,3,3)$	1440	39600
$(6,1,3,4)$	240	-4200
$(6,1,3,5)$	0	-4320
$(6,1,3,6)$	0	-480
$(6,1,4,0)$	-600	-5688
$(6,1,4,1)$	-1200	3984
$(6,1,4,2)$	-876	15780
$(6,1,4,3)$	-288	12000
$(6,1,4,4)$	-36	4380
$(6,1,4,5)$	0	864
$(6,1,4,6)$	0	72
$(7,1,0,0)$	32256	-2458624
$(7,1,0,1)$	0	0
$(7,1,0,2)$	58112	-5608960
$(7,1,0,3)$	0	0
$(7,1,0,4)$	-12544	-290560

(a,c,b,i)	$X_{8,6}$	$X_{10,6}$
$(7,1,0,5)$	0	0
$(7,1,0,6)$	0	25088
$(7,1,1,0)$	10656	-3184776
$(7,1,1,1)$	-40896	-1507464
$(7,1,1,2)$	-36612	-1298700
$(7,1,1,3)$	8568	408960
$(7,1,1,4)$	4284	183060
$(7,1,1,5)$	0	-25704
$(7,1,1,6)$	0	-8568
$(7,1,2,0)$	-40752	3663744
$(7,1,2,1)$	-28176	7733712
$(7,1,2,2)$	-8232	4172040
$(7,1,2,3)$	5856	281760
$(7,1,2,4)$	1464	41160
$(7,1,2,5)$	0	-17568
$(7,1,2,6)$	0	-2928
$(7,1,3,0)$	10116	661318
$(7,1,3,1)$	21252	974226
$(7,1,3,2)$	9361	26455

(a,c,b,i)	$X_{8,6}$	$X_{10,6}$
$(7,1,3,3)$	1518	-212520
$(7,1,3,4)$	253	-46805
$(7,1,3,5)$	0	-4554
$(7,1,3,6)$	0	-506
$(7,1,4,0)$	3888	88592
$(7,1,4,1)$	8448	11712
$(7,1,4,2)$	6464	-96400
$(7,1,4,3)$	2176	-84480
$(7,1,4,4)$	272	-32320
$(7,1,4,5)$	0	-6528
$(7,1,4,6)$	0	-544
$(7,1,5,0)$	-36	434
$(7,1,5,1)$	-60	1050
$(7,1,5,2)$	-37	1085
$(7,1,5,3)$	-10	600
$(7,1,5,4)$	-1	185
$(7,1,5,5)$	0	30
$(7,1,5,6)$	0	2

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