PERIODS OF LIFTING

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1. Review of the Miyawaki lifting

In this article, we are going to discuss a conjecture on the Petersson norm of the Miyawaki liftings.

Let $f(\tau) \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, and $h(\tau) \in S^+_{k+(1/2)}(\Gamma_0(4))$ a Hecke eigenform corresponding to $f(\tau)$ by the Shimura correspondence. Here $S^+_{k+(1/2)}(\Gamma_0(4))$ is the Kohnen plus subspace. Put $L(s,f) = \sum_{N=1}^{\infty} a(N)N^{-s}$.

Let n, r be non-negative integers such that $n+r \equiv k \bmod 2$. In [16], we have constructed a Hecke eigenform $F(Z) \in S_{k+n+r}(\mathrm{Sp}_{2n+2r}(\mathbb{Z}))$ whose standard L-function is equal to

$$\zeta(s) \prod_{i=1}^{2n+2r} L(s+k+n+r-i, f).$$

In fact, we will make use of the linear version of the lifting

$$\begin{gathered} S^+_{k+(1/2)}(\Gamma_0(4)) \to S_{k+n+r}(\mathrm{Sp}_{2n+2r}(\mathbb{Z})) \\ h(\tau) \mapsto F(Z) \end{gathered}$$

constructed by Kohnen [19]. We shall call $F(Z)$ a Duke-Imamoglu lift of $f(\tau)$ (or $h(\tau)$) to degree $2n+2r$.

Let $g \in S_{k+r+n}(\mathrm{Sp}_r(\mathbb{Z}))$ be a Hecke eigenform. Then the Miyawaki lifting $\mathcal{F}_{h,g}(Z)$ is defined by the integral

$$\mathcal{F}_{h,g}(Z) = \int_{\mathrm{Sp}_r(\mathbb{Z})\backslash \mathfrak{h}_r} F\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right) \overline{g^c(W)} (\det \mathrm{Im} W)^{k+n-1} dW,$$

for $Z \in \mathfrak{h}_{2n+r}$. Here, $g^c(Z) = \overline{g(-\bar{Z})}$. Note that $\mathcal{F}_{h,g}$ is a cusp form, since $F(Z)$ is a cusp form. Then we have

Theorem 1.1. *Assume that $\mathcal{F}_{h,g}(Z)$ is not identically zero. Then the cusp form $\mathcal{F}_{h,g}(Z)$ is a Hecke eigenform whose standard L-function is equal to*

$$L(s, \mathcal{F}_{h,g}, \mathrm{st}) = L(s, g, \mathrm{st}) \prod_{i=1}^{2n} L(s+k+n-i, f).$$

This theorem is proved by local representation theory instead of the global unwinding technique.

2. L-values

Let $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. We put

$$\begin{aligned} \xi(s) &= \Gamma_{\mathbb{R}}(s)\zeta(s), \\ \Lambda(s,f) &= \Gamma_{\mathbb{C}}(s)L(s,f) \\ \Lambda(s,f,\mathrm{Ad}) &= \Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+2k-1)L(s,f,\mathrm{Ad}). \end{aligned}$$

Here, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then the following functional equations hold.

$$\begin{aligned} \xi(1-s) &= \xi(s), \\ \Lambda(2k-s,f) &= (-1)^k\Lambda(s,f) \\ \Lambda(1-s,f,\mathrm{Ad}) &= \Lambda(s,f,\mathrm{Ad}). \end{aligned}$$

We modify $\xi(s)$ and $\Lambda(s,f,\mathrm{Ad})$ as follows.

$$\begin{aligned} \tilde{\xi}(s) &= \Gamma_{\mathbb{R}}(s+1)\xi(s) = \Gamma_{\mathbb{C}}(s)\zeta(s), \\ \tilde{\Lambda}(s,f,\mathrm{Ad}) &= \Gamma_{\mathbb{R}}(s)\Lambda(s,f,\mathrm{Ad}) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+2k-1)L(s,f,\mathrm{Ad}). \end{aligned}$$

If i is a positive integer, $\tilde{\xi}(2i) = |B_{2i}|/2i \in \mathbb{Q}^\times$. It is well-known that $\tilde{\Lambda}(2i-1,f,\mathrm{Ad})/\langle f,f\rangle \in \mathbb{Q}(f)^\times$ for $1 \le i < k$.

For a Hecke eigenform $g \in S_{k+r+n}(\mathrm{Sp}_r(\mathbb{Z}))$, we will define the completed L-function $\Lambda(s,g,\mathrm{st})$ and the modified completed L-function $\tilde{\Lambda}(s,g,\mathrm{st})$ by

$$\begin{aligned} \Lambda(s,g,\mathrm{st}) =& \Gamma_{\mathbb{R}}(s+\epsilon_r)\prod_{i=1}^{r}\Gamma_{\mathbb{C}}(s+k+r+n-i)L(s,g,\mathrm{st}) \\ \tilde{\Lambda}(s,g,\mathrm{st}) =& \Gamma_{\mathbb{C}}(s)\prod_{i=1}^{r}\Gamma_{\mathbb{C}}(s+k+r+n-i)L(s,g,\mathrm{st}). \end{aligned}$$

Here, ϵ_r is 0, if r is even and 1 if r is odd. Then the functional equation $\Lambda(1-s,g,\mathrm{st}) = \Lambda(s,g,\mathrm{st})$ holds.

Let $L(s,\mathrm{st}(g)\boxtimes f)$ be the L-function defined by

$$L(s,\mathrm{st}(g)\boxtimes f) = \prod_p \det(\mathbf{1}_{4r+2} - A_p \otimes B_p \cdot p^{-s})^{-1},$$

where

$$L(s,f) = \prod_p \det(\mathbf{1}_2 - A_p \cdot p^{-s})^{-1}, \quad A_p \in \mathrm{GL}_2(\mathbb{C}),$$

$$L(s,g,\mathrm{st}) = \prod_p \det(\mathbf{1}_{2r+1} - B_p \cdot p^{-s})^{-1}, \quad B_p \in \mathrm{GL}_{2r+1}(\mathbb{C}).$$

The gamma factor of $L(s, \mathrm{st}(g) \boxtimes f)$ is given by

$$L_\infty(s, \mathrm{st}(g) \boxtimes f) = \Gamma_{\mathbb{C}}(s) \prod_{i=1}^{r} \Gamma_{\mathbb{C}}(s+n-k+i)\Gamma_{\mathbb{C}}(s+n+k+i-1).$$

We put $\Lambda(s, \mathrm{st}(g) \boxtimes f) = L_\infty(s, \mathrm{st}(g) \boxtimes f) L(s, \mathrm{st}(g) \boxtimes f)$. Then the expected functional equation should be

$$\Lambda(2k-s, \mathrm{st}(g) \boxtimes f) = (-1)^{k+r} \Lambda(s, \mathrm{st}(g) \boxtimes f)$$

3. A conjecture on the Petersson inner product

It is an interesting problem to determine when $\mathcal{F}_{h,g} \not\equiv 0$. Here we are going to give a conjecture on the Petersson inner product of $\mathcal{F}_{h,g}$.

Conjecture 3.1. Assume that $n < k$. Then there exists an integer $\alpha = \alpha(r,n,k)$ depending only on r, n, and k such that

$$\Lambda(k+n, \mathrm{st}(g) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(2i-1, f, \mathrm{Ad}) \tilde{\xi}(2i) = 2^\alpha \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle}{\langle g, g\rangle}.$$

In particular, $\mathcal{F}_{h,g}$ is non-zero if and only if $\Lambda(k+n, \mathrm{st}(g) \boxtimes f) \neq 0$.

Note that the left hand side does not vanish if $n = r = 1$.

When $\mathcal{F}_{h,g} \neq 0$, one can rewrite the right hand side in a more symmetric way. Namely, choose any non-zero $G \in \mathbb{C} \cdot \mathcal{F}_{h,g}$. Then

$$\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle = \frac{|\langle F|_{\mathfrak{h}_r \times \mathfrak{h}_{r+2n}}, g^c \times G\rangle|^2}{\langle G, G\rangle}$$

Here $\langle F|_{\mathfrak{h}_r \times \mathfrak{h}_{r+2n}}, g^c \times G\rangle$ is a Petersson inner product on $(\mathrm{Sp}_r(\mathbb{Z})\backslash \mathfrak{h}_r) \times (\mathrm{Sp}_{r+2n}(\mathbb{Z})\backslash \mathfrak{h}_{r+2n})$. Therefore the conjecture takes the form

$$\text{(C)} \qquad \Lambda(k+n, \mathrm{st}(g) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(2i-1, f, \mathrm{Ad}) \tilde{\xi}(2i) = 2^\alpha \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{|\langle F|_{\mathfrak{h}_r \times \mathfrak{h}_{r+2n}}, g^c \times G\rangle|^2}{\langle g, g\rangle \langle G, G\rangle}.$$

Remark 3.1. By some computer calculation (cf. Appendix), it seems the values of $\alpha = \alpha(r, n, k)$ are

$$\text{(a)} \qquad \alpha(0, n, k) = 2kn + 2n - k - 1,$$

$$\text{(b)} \qquad \alpha(r, 0, k) = r^2 + 2kr + r - k - 1,$$

$$\text{(c)} \qquad \alpha(r, n, k) = r^2 + 2kr + 2kn + 2rn + 2n + r - k - 2$$

for r, $n > 0$. As for the case $n = 0$, we will give some evidence for (C) in the next section.

Remark 3.2. Note that $s = k+n$ is a critical point for $\Lambda(s, \mathrm{st}(g) \boxtimes f)$ in the sense of Deligne [8]. In particular, the left hand side of (C) should be finite. Deligne's conjecture [8] implies the ratio RHS/LHS should belong to the field $\mathbb{Q}(f, g)$ under the assumption $n < k$. (cf. Yoshida [33]).

Example 3.1. When $r = n = 0$, we have $F(Z) = c(1)$. In this case, our conjecture is a special case of the result of Kohnen-Zagier [22]

$$\Lambda(k, f) = 2^{1-k} \frac{\langle f, f \rangle}{\langle h, h \rangle} |c(1)|^2.$$

It follows that our conjecture holds for $n = r = 0$ with $\alpha(0, 0, k) = 1-k$.

Example 3.2. When $r = 0$, $n = 1$, our conjecture is compatible with the Petersson inner product formula for the Saito-Kurokawa lift

$$\Lambda(k+1, f) = 3 \cdot 2^{-k+3} \frac{\langle F, F \rangle}{\langle h, h \rangle}$$

proved by Kohnen [20] and Kohnen and Skoruppa [21]. See also Krieg [23]. This is equivalent with

$$\Lambda(k+1, f) \tilde{\Lambda}(1, f, \mathrm{Ad}) \tilde{\xi}(2) = 2^{k+1} \frac{\langle f, f \rangle}{\langle h, h \rangle} \langle F, F \rangle,$$

since $\tilde{\Lambda}(1, f, \mathrm{Ad}) = 2^{2k} \langle f, f \rangle$. It follows that our conjecture holds for $(r, n) = (0, 1)$ with $\alpha(0, 1, k) = k + 1$.

4. Heuristics about the conjecture

We would like to explain how Conjecture 3.1 arose. In this section we write $A \sim_X B$ if there exists an "elementary" constant ω which depends only on X such that $A = \omega B$.

Recall that Kohnen's linear lifting map $h \mapsto F$ has an Eisenstein analogue. The image of the Cohen Eisentein series

$$\mathcal{H}_{k+(1/2)}(\tau) = \sum_{N \geq 0} H(k, N) q^N$$

can be thought of as the normalized Eisenstein series

$$\mathcal{E}^{(2r+2n)}_{k+r+n}(Z)=2^{-n-r}\zeta(1-k-r-n)\prod_{i=1}^{n+r}\zeta(1+2i-2k-2r-2n)\cdot E^{(2r+2n)}_{k+r+n}(Z).$$

We begin with the case $n=0$. Our starting point is Böcherer's theorem [3]. By the result of Böcherer, [3],

$$\begin{aligned}\int_{\mathrm{Sp}_r(\mathbb{Z})\backslash\mathfrak{h}_r} & E^{(2r+2n)}_{k+r+n}\left(\begin{pmatrix}Z & 0\\ 0 & W\end{pmatrix}\right)\overline{g^c(W)}(\det \mathrm{Im}W)^{k+n-1}dW\\ \sim_{k,r,n} & \left[\tilde{\xi}(k+r+n)\prod_{i=1}^{r}\tilde{\xi}(2k+2r+2n-2i)\right]^{-1}\\ & \times\tilde{\Lambda}(k+n,g,\mathrm{st})E^{(r+2n)}_{k+r+n}(g,Z).\end{aligned}$$

Here $E^{(r+2n)}_{k+r+n}(g,Z)$ is the Klingen Eisenstein series of g to degree $r+2n$.

When $n=0$, we have

$$\frac{\langle \mathcal{E}^{(2r)}_{k+r}|_{\mathfrak{h}_r\times\mathfrak{h}_r}, g^c\times g\rangle}{\langle g,g\rangle}\sim_{r,k}\tilde{\Lambda}(k,g,\mathrm{st}).$$

It follows that when $h_0=\mathcal{H}_{k+(1/2)}$, we have

$$\begin{aligned}\frac{\langle \mathcal{F}_{h_0,g},\mathcal{F}_{h_0,g}\rangle}{\langle g,g\rangle} & \sim_{r,k}\tilde{\Lambda}(k,g,\mathrm{st})^2\\ & \sim_{r,k}\tilde{\Lambda}(k,g,\mathrm{st})\tilde{\Lambda}(1-k,g,\mathrm{st})\\ & \sim_{r,k}\Lambda(k,\mathrm{st}(g)\boxtimes E_{2k})\end{aligned}$$

However, this is not satisfactory, because $\langle \mathcal{F}_{h,g},\mathcal{F}_{h,g}\rangle$ depends on h, but the RHS does not. Therefore we should consider

$$\frac{\langle \mathcal{F}_{h,g},\mathcal{F}_{h,g}\rangle}{\langle h,h\rangle\langle g,g\rangle}.$$

Again we do not have a good analogy for the Eisenstein case because $\langle h_0,h_0\rangle$ is not convergent. Now we consider

$$\frac{\langle f,f\rangle}{\langle h,h\rangle}\frac{\langle \mathcal{F}_{h,g},\mathcal{F}_{h,g}\rangle}{\langle g,g\rangle}.$$

This has a good Eisensetein analogy. Recall that the Kohnen-Zagier formula [22] says

$$\text{(KZ)}\qquad |c(|D|)|^2\frac{\langle f,f\rangle}{\langle h,h\rangle}=2^{k-1}|D|^{-1/2}\Lambda(k,f,\chi_D),$$

for any fundamental discriminant D such that $(-1)^kD>0$. Here,

$$\Lambda(s,f,\chi_D)=|D|^s\Gamma_{\mathbb{C}}(s)L(s,f,\chi_D).$$

Put $h_0(\tau) = \mathcal{H}_{k+(1/2)}$ and $f_0 = E_{2k}$. Then the Kohnen-Zagier formula suggests that the factor $\langle f_0, f_0\rangle\langle h_0, h_0\rangle^{-1}$ should be thought of as

$$2^{k-1}|D|^{-1/2}\frac{\Lambda(k, E_{2k}, \chi_D)}{|H(k, |D|)|^2} = (-1)^{k(k-1)/2}2^{k-1}.$$

Now we can expect that there might be a formula

$$\Lambda(k, \mathrm{st}(g) \boxtimes f) = (\text{constant}) \cdot \frac{\langle f, f\rangle}{\langle h, h\rangle}\frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle}{\langle g, g\rangle}.$$

A careful calculation shows the constant is equal to $2^{r^2+2kr+r-k+1}$ when $h = h_0$ and $f = f_0$, if we interpret $\langle f_0, f_0\rangle\langle h_0, h_0\rangle^{-1}$ as $(-1)^{k(k-1)/2}2^{k-1}$. This is the conjecture 3.1 for $n = 0$.

Now we consider the case $n > 0$. In this case, we cannot use Eisenstein analogy directly, because the Klingen Eisenstein seires $E^{(r+2n)}_{k+r+n}(g, Z)$ is no longer cuspidal and so $\langle \mathcal{F}_{h_0,g}, \mathcal{F}_{h_0,g}\rangle$ is not convergent. However, Böcherer's result still suggests that there might be a formula which relates

$$\frac{\langle f, f\rangle}{\langle h, h\rangle}\frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle}{\langle g, g\rangle}$$

and

$$\Lambda(k + n, \mathrm{st}(g) \boxtimes f) \times (\text{extra } L\text{-value}).$$

Let us denote this extra L-value by $X(n, r, f, g)$. To determine the factor $X(n, r, f, g)$, we consider the symmetry between g and $\mathcal{F}_{h,g}$.

Put $G = \mathcal{F}_{h,g}$ and assume that $G \neq 0$. We also assume that the multiplicity one property holds for $S_{k+r+n}(\mathrm{Sp}_r(\mathbb{Z}))$. Then we can show that $\mathcal{F}_{h,G}$ and g are proportional. Moreover, we have

$$\frac{\langle f, f\rangle}{\langle h, h\rangle}\frac{|\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle|^2}{\langle g, g\rangle} = \frac{\langle f, f\rangle}{\langle h, h\rangle}\frac{\langle F|_{\mathfrak{h}_r\times\mathfrak{h}_{r+2n}}, g^c \times G\rangle}{\langle g, g\rangle\langle G, G\rangle}.$$

This is symmetric with respect to g and G. Therefore it is natural to expect that

$$\begin{aligned}&\Lambda(k + n, \mathrm{st}(g) \boxtimes f)X(n, r, f, g)\\ &\qquad \sim_{k,r,n} \Lambda(k - n, \mathrm{st}(G) \boxtimes f)X(-n, r + 2n, f, G).\end{aligned}$$

Since

$$\frac{\Lambda(k-n,\mathrm{st}(G)\boxtimes f)}{\Lambda(k+n,\mathrm{st}(g)\boxtimes f)} \sim \prod_{i=1}^{2n}\Lambda(2k-i,f\times f)$$
$$\sim \prod_{i=1}^{2n}\xi(1-i)\Lambda(1-i,f,\mathrm{Ad})$$
$$\sim \prod_{i=1}^{2n}\xi(i)\Lambda(i,f,\mathrm{Ad})$$

(Here, $\xi(1)=\infty$ occurs, so we need a kind of regularization argument. See Proposition 4.1 below.) It is now natural to expect that $X(n,r,f,g)$ is a partial product of

$$\prod_{i=1}^{2n}\tilde{\xi}(i)\tilde{\Lambda}(i,f,\mathrm{Ad}).$$

Deligne's conjecture suggests that only critical L-values occur in this product (at least when $n<k$). Therefore it is now natural to expect

$$X(n,r,f,g)\sim\prod_{i=1}^{n}\tilde{\xi}(2i)\tilde{\Lambda}(2i-1,f,\mathrm{Ad}).$$

In fact, we can prove the following proposition, which guarantees the symmetry between g and G.

Proposition 4.1.

$$\Lambda(k-n,\mathrm{st}(G)\boxtimes f)\left[\prod_{i=1}^{n}\tilde{\Lambda}(s-2i+1,f,\mathrm{Ad})^{-1}\tilde{\xi}(s-2i+2)^{-1}\right]_{s=0}$$
$$=\Lambda(k+n,\mathrm{st}(g)\boxtimes f)\prod_{i=1}^{n}\tilde{\Lambda}(2i-1,f,\mathrm{Ad})\tilde{\xi}(2i).$$

Proof. By Theorem 1.1, $\Lambda(s+k-n,\mathrm{st}(G)\boxtimes f)$ is the product of

$$\prod_{i=1}^{2n}\Lambda(s+2k-i,f\times f)$$

and

$$\Lambda(s+k-n,\mathrm{st}(g)\boxtimes f)=(-1)^{k+r}\Lambda(-s+k+n,\mathrm{st}(g)\boxtimes f).$$

Since $\Lambda(s+2k-1, f\times f) = \Lambda(s, f, \mathrm{Ad})\xi(s)$, we have

$$\prod_{i=1}^{2n}\Lambda(s+2k-i, f\times f)\prod_{i=1}^{n}\tilde{\Lambda}(s-2i+1, f, \mathrm{Ad})^{-1}\tilde{\xi}(s-2i+2)^{-1}$$
$$=\prod_{i=1}^{n}\Gamma_{\mathbb{R}}(s-2i+1)^{-1}\Gamma_{\mathbb{R}}(s-2i+3)^{-1}$$
$$\times\prod_{i=1}^{n}\Lambda(-s+2i-1, f, \mathrm{Ad})\xi(-s+2i).$$

Now using $\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(-s+1) = \sin(\pi s/2)$, we have

$$\prod_{i=1}^{n}\Gamma_{\mathbb{R}}(-2i+1)^{-1}\Gamma_{\mathbb{R}}(-2i+3)^{-1} = (-1)^n\prod_{i=1}^{n}\Gamma_{\mathbb{R}}(2i-1)\Gamma_{\mathbb{R}}(2i+1).$$

Hence the lemma. □

5. Theta functions associated with Niemeier lattices

In this section, we write $M_k^{(n)} = M_k(\mathrm{Sp}_n(\mathbb{Z}))$ and $S_k^{(n)} = S_k(\mathrm{Sp}_n(\mathbb{Z}))$, for simplicity.

We recall the results of [28]. A Niemeier lattice is a positive definite even unimodular lattice of degree 24. The number of isomorphism classes of Niemeier lattices is 24. Let L_i $(1\le i\le 24)$ be Niemeier lattices, not isomorphic to each other.

Let V be a 24-dimensional vector space over $\mathbb{C}$ with a basis $\{\mathsf{e}_i \mid 1\le i\le 24\}$.

The theta function of degree n associated with L_i is denoted by $\Theta_{L_i}^{(n)}(Z)\in M_{12}^{(n)}$. By extending linearly, we obtain a linear map

$$\Theta^{(n)} : V \longrightarrow M_{12}^{(n)}$$
$$\sum_i c_i\mathsf{e}_i \mapsto \sum_i c_i\Theta_{L_i}^{(n)}(Z).$$

Let $V_n = \mathrm{Ker}(\Theta^{(n)})$. Then $\Theta^{(12)}$ is injective (cf. [12], [5]). If $n'+n''=n$, then the restriction of $\Theta_{L_i}^{(n)}(Z)$ to $\mathfrak{h}_{n'}\times\mathfrak{h}_{n''}$ is given by

$$\Theta_{L_i}^{(n)}\left(\begin{pmatrix} Z' & 0\\ 0 & Z''\end{pmatrix}\right) = \Theta_{L_i}^{(n')}(Z')\Theta_{L_i}^{(n'')}(Z'').$$

Following Nebe and Venkov, we define the Hermitian inner product $(\ ,\ )$ on V by

$$(\mathtt{e}_i, \mathtt{e}_j) = \begin{cases} (\#\mathrm{Aut}(L_i)), & i = j, \\ 0, & i \neq j, \end{cases}$$

and a multiplication on V by

$$\mathtt{e}_i \circ \mathtt{e}_j = \begin{cases} (\#\mathrm{Aut}(L_i))\mathtt{e}_i, & i = j \\ 0, & i \neq j. \end{cases}$$

Nebe and Venkov defined Hecke operators $K_{p,i}$, $(1 \leq i \leq 12)$ and $T(p)$ acting on V and calculated Hecke eigenvectors $\mathtt{d}_1, \mathtt{d}_2, \ldots, \mathtt{d}_{24}$.

We put

$$\mathtt{d}_i = \sum_j c_{ij} \mathtt{e}_j,$$

$$\mathtt{e}_i = \sum_j b_{ij} \mathtt{d}_j.$$

A table of coefficients c_{ij} $(i, j = 1, 2, \ldots, 24)$ can be found in [27]. Note that $c_{ij}, b_{ij} \in \mathbb{Q}$. As both $\{\mathtt{e}_1, \mathtt{e}_2, \ldots, \mathtt{e}_{24}\}$ and $\{\mathtt{d}_1, \mathtt{d}_2, \ldots, \mathtt{d}_{24}\}$ are orthogonal basis of V, we have

$$b_{ij} = (\mathtt{e}_i, \mathtt{e}_i)\, \overline{c_{ji}}\, (\mathtt{d}_j, \mathtt{d}_j)^{-1} = (\#\mathrm{Aut}(L_i))\, (\mathtt{d}_j, \mathtt{d}_j)^{-1}\, c_{ji}.$$

Nebe and Venkov showed that the degree n_i of $\mathtt{d}_i$ is as follows:

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}
0	1	2	3	4	4	5	5	6	6	6	7

n_{13}	n_{14}	n_{15}	n_{16}	n_{17}	n_{18}	n_{19}	n_{20}	n_{21}	n_{22}	n_{23}	n_{24}
8	7	8	7	8	8	–	9	–	10	11	12

For the definition of the degree, see [28]. Note that they have shown that $n_i = \min\{n \mid \Theta^{(n)}(\mathtt{d}_i) \neq 0\}$ in this case (See [28], Lemma 2.5). As for n_{19} and n_{21}, they have shown that $7 \leq n_{19} \leq 9$, $8 \leq n_{21} \leq 10$, but we do not use $\mathtt{d}_{19}$ or $\mathtt{d}_{21}$.

Note that the Petersson inner product $\langle \Theta^{(n_i)}(\mathtt{d}_i), \Theta^{(n_i)}(\mathtt{d}_j) \rangle$ vanishes for $i \neq j$, since the Hecke eigenvalues are different. We put $F_i = \Theta^{(n_i)}(\mathtt{d}_i) \in S_{12}^{(n_i)}$. Note that $F_i^c = F_i$ for $i = 1, 2, \ldots, 24$.

Lemma 5.1. *Let $\mathtt{d}_i$, $\mathtt{d}_j$, and $\mathtt{d}_k$ be Hecke eigenvectors of V. Then we have*

$$\langle \Theta^{(n_i+n_j)}(\mathtt{d}_k)|_{\mathfrak{h}_{n_i} \times \mathfrak{h}_{n_j}}, F_i \times F_j \rangle = \frac{\langle F_i, F_i \rangle \langle F_j, F_j \rangle}{(\mathtt{d}_i, \mathtt{d}_i)\ (\mathtt{d}_j, \mathtt{d}_j)} (\mathtt{d}_k, \mathtt{d}_i \circ \mathtt{d}_j).$$

In particular, $(\mathtt{d}_k, \mathtt{d}_i \circ \mathtt{d}_j) \neq 0$ *if and only if the left hand side is not zero.*

The proof of this lemma is a straightforward calculation of the theta function.

Nebe and Venkov [28] claimed that $F_{11} \in S_{12}^{(6)}$, $F_{13} \in S_{12}^{(8)}$, and $F_{24} \in S_{12}^{(12)}$ are the Duke-Imamoglu lift of $\phi_{18} \in S_{18}^{(1)}$, $\phi_{16} \in S_{16}^{(1)}$, and $\Delta \in S_{12}^{(1)}$, respectively. In fact this is easily verified by comparing the eigenvalue of $T(2)$ (See [27]). Nebe and Venkov [28] have shown that $(\mathtt{d}_{24}, \mathtt{d}_i \circ \mathtt{d}_j) \neq 0$ for

$$(i, j) = (2, 23), (3, 22), (4, 20), (5, 17), (6, 18), (7, 14), (8, 16).$$

Then it is easy to see s that F_j is the Miyawaki lift of F_i with respect to $F_{24} \in S_{12}^{(12)}$. Similarly, using the structure constants found in [27], one can prove that $F_8 \in S_{12}^{(5)}$ and $F_6 \in S_{12}^{(4)}$ are Miyawaki lift of $F_2 \in S_{12}^{(1)}$ and $F_3 \in S_{12}^{(2)}$, respectively. One can also prove that $F_{12} \in S_{12}^{(7)}$, $F_9 \in S_{12}^{(6)}$, and $F_7 \in S_{12}^{(5)}$ are the Miyawaki lift of $F_2 \in S_{12}^{(1)}$, $F_3 \in S_{12}^{(2)}$, and $F_4 \in S_{12}^{(3)}$ with respect to $F_{13} \in S_{12}^{(8)}$, respectively. We summarize these as Table A and Table B.

6. Numerical calculation

The following proposition follows from the result of Böcgerer [3].

Proposition 6.1. *Assume that* $k+r \equiv 2 \bmod 2$ *and* $g \in S_{k+r}(\mathrm{Sp}_r(\mathbb{Z}))$. *Then*

$$\left| \frac{\langle E_{k+r}^{(2r)}|_{\mathfrak{h}_r \times \mathfrak{h}_r}, g^c \times g \rangle}{\langle g, g \rangle} \right| = 2^{-(r^2 - r + 2kr - 2)/2} |\mathcal{A}_{r,k}|^{-1} \tilde{\Lambda}(k, g, \mathrm{st}).$$

Here $\mathcal{A}_{r,k} = \zeta(1-k-r) \prod_{i=1}^r \zeta(1-2k-2r+2i)$ *and* $\tilde{\Lambda}(s, g, \mathrm{st}) = \Gamma_{\mathbb{C}}(s) \prod_{i=1}^r \Gamma_{\mathbb{C}}(s+k+r-i) L(s, g, \mathrm{st})$.

We briefly explain how to calculate both sides of (C) by computers. For the calculation of various L-values, we have used a very useful program due to Dokchitser [9]. The Petersson norm $\langle f, f \rangle$ can be easily computed by $\tilde{\Lambda}(1, f, \mathrm{Ad}) = 2^{2k} \langle f, f \rangle$. Similarly, $\langle h, h \rangle$ can be computed by Kohnen-Zagier formula (KZ). The Petersson norm of g or G can be computed by Proposition 6.1 and Lemma 5.1. Finally, $\langle F|_{\mathfrak{h}_r \times \mathfrak{h}_{r+2n}}, g \times G \rangle$ is computed by Lemma 5.1. Note that the structure constants $(\mathtt{d}_k, \mathtt{d}_i \circ \mathtt{d}_j)$ are already computed by Nebe [27].

We discuss the case $f = \phi_{20} \in S_{20}^{(1)}$, $g = \Delta \in S_{12}^{(1)}$, and $G \in S_{12}^{(3)}$. We put

$$\begin{aligned} \mathtt{d}_1' &= \mathtt{d}_1/1027637932586061520960267, \\ \mathtt{d}_2' &= -\mathtt{d}_2/8104867379578640543040, \\ \mathtt{d}_4' &= \mathtt{d}_4/846305351287603200, \\ \mathtt{d}_5' &= -\mathtt{d}_5/212694241858560. \end{aligned}$$

We give a table of coefficients of $\mathtt{d}_2$, $\mathtt{d}_4$, and $\mathtt{d}_5$ below (See Nebe [27]). The coefficients of $\mathtt{d}_1$ can be found in [27] or [7], p. 413. Then $E_{12}^{(2r)} = \Theta^{(2r)}(\mathtt{d}_1')$, $F_2' = \Theta^{(1)}(\mathtt{d}_2') = \Delta \in S_{12}^{(1)}$, and $F_4' = \Theta^{(3)}(\mathtt{d}_4') \in S_{12}^{(3)}$ is the Miyawaki's cusp form [25]. Put $h = q - 56q^4 + 360q^5 - 13680q^8 + \cdots \in S_{21/2}^{+}(\Gamma(4))$. Then $F_5' = \Theta^{(4)}(\mathtt{d}_5') \in S_{12}^{(4)}$ is the Duke-Imamoglu lift of $h(\tau)$ to degree 4.

	$\mathtt{d}_2$	$\mathtt{d}_4$	$\mathtt{d}_5$
Leech	21625795628236800	−1992646656000	214592716800
A_1^{24}	21618140012108640000	−462916726272000	22783711104000
A_2^{12}	104595874904801280000	385220419584000	−56204746752000
A_3^8	−7569380452233600000	865252948560000	22644338640000
A_4^6	−66640754260236828672	−625041225768960	21173267275776
$A_5^4 D_4$	−37660962656647249920	−318497556529152	2319747268608
D_4^6	−861991027602705000	−7289830548000	4817683332000
A_6^4	−8962553548174786560	25632591249408	−23357975494656
$A_7^2 D_5^2$	−3844278424500433920	89124325640064	6074130446208
A_8^3	−400803255218995200	20932199608320	−1962418360320
$A_9^2 D_6$	−226886348300451840	20394416373760	168373460992
D_6^4	−40713248535359400	3659642586600	716314247880
$A_{11} D_7 E_6$	−22871209751470080	4366739579904	500824507392
E_6^4	−1056891465710080	201789491904	52888473792
A_{12}^2	−2655635220725760	675250266112	11615002624
D_8^3	−554584334604300	180878892480	32784927120
$A_{15} D_9$	−141086166819840	69909993856	8326316416
$D_{10} E_7^2$	−20420264058480	14273509536	4257598752
$A_{17} E_7$	−17203085475840	12024741888	2130518016
D_{12}^2	−426847644405	515734934	139737422
A_{24}	−30884364288	51875840	11128832
$D_{16} E_8$	−2482214625	6542775	2974851
E_8^3	−584290850	1540110	927894
D_{24}	−367740	2621	1601

We need the following computer calculations.

$$\begin{aligned} (\mathtt{d}_2', \mathtt{d}_2') &= 2^{31} \cdot 3^{10} \cdot 5^4 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 283^{-1} \cdot 617^{-1} \cdot 3617^{-1} \cdot 43867^{-1}, \\ (\mathtt{d}_4', \mathtt{d}_4') &= 2^{16} \cdot 3^{-1} \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 283 \cdot 617 \cdot 691^{-1} \cdot 3617^{-1}, \\ (\mathtt{d}_1', \mathtt{d}_4' \circ \mathtt{d}_4') &= \frac{2^{61} \cdot 3^{16} \cdot 5^{12} \cdot 7^5 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23}{131 \cdot 593 \cdot 691^3 \cdot 3617^2 \cdot 43867}, \\ (\mathtt{d}_5', \mathtt{d}_2' \circ \mathtt{d}_4') &= -2^{54} \cdot 3^{12} \cdot 5^{10} \cdot 7^2 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 691^{-1} \cdot 3617^{-2} \cdot 43867^{-1}. \end{aligned}$$

$$\begin{aligned}
\langle \Delta, \Delta \rangle =& 0.000001035362056804320922347816812225164593224907\cdots \\
\langle \phi_{20}, \phi_{20} \rangle =& 0.000008265541531659703164230062760258225715343908\cdots \\
\frac{\langle \Delta, \Delta \rangle}{\langle h, h \rangle} =& 0.098872279065281741186752369945336382997115288715\cdots \\
\tilde{\Lambda}(9, \Delta, \mathrm{Ad}) =& 0.139584317666868979132086560789461824236408711579\cdots \\
\doteqdot & 2^{19} \cdot 3^2 \cdot 5^{-1} \cdot 7^{-1} \langle \Delta, \Delta \rangle, \\
\Lambda(18, \phi_{20})\Lambda(19, \phi_{20}) =& 2^{23} \cdot 3^4 \cdot 7^2 \cdot 17 \cdot 283^{-1} \cdot 617^{-1} \langle \phi_{20}, \phi_{20} \rangle, \\
\Lambda(11, \mathrm{Ad}(\Delta) \boxtimes \phi_{20}) =& 0.000000033447080614408498864020192110373963031495\cdots \\
\doteqdot & 2^{24} \cdot 3^2 \cdot 5^2 \langle \Delta, \Delta \rangle^2 \langle \phi_{20}, \phi_{20} \rangle \langle h, h \rangle^{-1}.
\end{aligned}$$

We can now calculate the Petersson norm $\langle F_4', F_4' \rangle$. By Proposition 6.1 and Lemma 5.1, we have

$$\begin{aligned}
\langle F_4', F_4' \rangle =& 2^{-29} \frac{(\mathtt{d}_4', \mathtt{d}_4')^2}{(\mathtt{d}_1', \mathtt{d}_4' \circ \mathtt{d}_4')} |\mathcal{A}_{3,9}|^{-1} \tilde{\Lambda}(9, \Delta, \mathrm{Ad}) \Lambda(18, \phi_{20}) \Lambda(19, \phi_{20}) \\
\doteqdot & 2^{-6} \cdot 3^{-5} \langle \phi_{20}, \phi_{20} \rangle \langle \Delta, \Delta \rangle.
\end{aligned}$$

Here, $\mathcal{A}_{3,9} = \zeta(-11)\zeta(-21)\zeta(-19)\zeta(-17)$. By Lemma 5.1, we have

$$\begin{aligned}
\frac{\langle F_5' |_{\mathfrak{h}_1 \times \mathfrak{h}_3}, F_2' \times F_4' \rangle^2}{\langle F_2', F_2' \rangle \langle F_4', F_4' \rangle} &= \langle F_2', F_2' \rangle \langle F_4', F_4' \rangle \left(\frac{(\mathtt{d}_5', \mathtt{d}_2' \circ \mathtt{d}_4')}{(\mathtt{d}_2', \mathtt{d}_2')\ (\mathtt{d}_4', \mathtt{d}_4')} \right)^2 \\
&\doteqdot 2^8 \cdot 3 \cdot 5^2 \langle \Delta, \Delta \rangle^2 \langle \phi_{20}, \phi_{20} \rangle.
\end{aligned}$$

On the other hand, we have

$$\Lambda(11, \mathrm{st}(g) \boxtimes f) \tilde{\Lambda}(1, f, \mathrm{Ad}) \tilde{\xi}(2) \doteqdot 2^{42} \cdot 3 \cdot 5^2 \langle \Delta, \Delta \rangle^3 \langle \phi_{20}, \phi_{20} \rangle^2 \langle h, h \rangle^{-1}$$

Hence the equation (C) holds approximately in this case with $\alpha = 34$. Other examples are shown in Table C.

We give another example $n = k = 6$, $r = 0$, $g = 1$, $f = \Delta$, and $F = G = F_{24}$. Then by computer calculation,

$$\Lambda(12, \mathrm{st}(g) \boxtimes f) \prod_{i=1}^{6} \tilde{\Lambda}(2i-1, f, \mathrm{Ad}) \tilde{\xi}(2i) \doteqdot \frac{2^{73} \langle \Delta, \Delta \rangle^6 \Lambda(12, \Delta)}{3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23}.$$

On the other hand, using Böcherer's result [3], one can show

$$\frac{\langle f, f \rangle}{\langle h, h \rangle} \langle F, F \rangle = \frac{\langle \Delta, \Delta \rangle^6 \Lambda(12, \Delta)}{2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23}.$$

Therefore it seems (C) holds in this case as well. Notice that the assumption $k > n$ is not satisfied in this case and that $\Lambda(12, \Delta)$ is not a critical value in the sense of Deligne [8].

• Tabe A: Standard L-functions

$$
\begin{aligned}
L(s,F_3,\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11}L(s+i,\phi_{22}),\\
L(s,F_4,\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{9\le i\le 10}L(s+i,\phi_{20}),\\
L(s,F_5,\mathrm{st}) =&\zeta(s)\prod_{8\le i\le 11}L(s+i,\phi_{20}),\\
L(s,F_6,\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11}L(s+i,\phi_{22})\prod_{8\le i\le 9}L(s+i,\phi_{18}),\\
L(s,F_7,\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{9\le i\le 10}L(s+i,\phi_{20})\prod_{7\le i\le 8}L(s+i,\phi_{16}),\\
L(s,F_8,\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{7\le i\le 10}L(s+i,\phi_{18}),\\
L(s,F_9,\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11}L(s+i,\phi_{22})\prod_{6\le i\le 9}L(s+i,\phi_{16}),\\
L(s,F_{11},\mathrm{st}) =&\zeta(s)\prod_{6\le i\le 11}L(s+i,\phi_{18}),\\
L(s,F_{12},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{5\le i\le 10}L(s+i,\phi_{16}),\\
L(s,F_{14},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{9\le i\le 10}L(s+i,\phi_{20})\prod_{7\le i\le 8}L(s+i,\phi_{16})\prod_{5\le i\le 6}L(s+i,\Delta),\\
L(s,F_{16},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{7\le i\le 10}L(s+i,\phi_{18})\prod_{5\le i\le 6}L(s+i,\Delta),\\
L(s,F_{13},\mathrm{st}) =&\zeta(s)\prod_{4\le i\le 11}L(s+i,\phi_{16}),\\
L(s,F_{17},\mathrm{st}) =&\zeta(s)\prod_{8\le i\le 11}L(s+i,\phi_{20})\prod_{4\le i\le 7}L(s+i,\Delta),\\
L(s,F_{18},\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11}L(s+i,\phi_{22})\prod_{8\le i\le 9}L(s+i,\phi_{18})\prod_{4\le i\le 7}L(s+i,\Delta),\\
L(s,F_{20},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{9\le i\le 10}L(s+i,\phi_{20})\prod_{3\le i\le 8}L(s+i,\Delta),\\
L(s,F_{22},\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11}L(s+i,\phi_{22})\prod_{2\le i\le 9}L(s+i,\Delta),\\
L(s,F_{23},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{i=1}^{10}L(s+i,\Delta),\\
L(s,F_{24},\mathrm{st}) =&\zeta(s)\prod_{i=0}^{11}L(s+i,\Delta).
\end{aligned}
$$

• Table B: Liftings

type	form	degree	g	f	F	r	n	k
Duke-Imamoglu	F_3	2		ϕ_{22}				
Miyawaki	F_4	3	Δ	ϕ_{20}	F_5	1	1	10
Duke-Imamoglu	F_5	4		ϕ_{20}				
Miyawaki	F_6	4	F_3	ϕ_{18}	F_{11}	2	1	9
Miyawaki	F_7	5	F_4	ϕ_{16}	F_{13}	3	1	8
Miyawaki	F_8	5	Δ	ϕ_{18}	F_{11}	1	2	9
Miyawaki	F_9	6	F_3	ϕ_{16}	F_{13}	2	2	8
Duke-Imamoglu	F_{11}	6		ϕ_{18}				
Miyawaki	F_{12}	7	Δ	ϕ_{16}	F_{13}	1	3	8
Miyawaki	F_{14}	7	F_7	Δ	F_{24}	5	1	6
Miyawaki	F_{16}	7	F_8	Δ	F_{24}	5	1	6
Duke-Imamoglu	F_{13}	8		ϕ_{16}				
Miyawaki	F_{17}	8	F_5	Δ	F_{24}	4	2	6
Miyawaki	F_{18}	8	F_6	Δ	F_{24}	4	2	6
Miyawaki	F_{20}	9	F_4	Δ	F_{24}	3	3	6
Miyawaki	F_{22}	10	F_3	Δ	F_{24}	2	4	6
Miyawaki	F_{23}	11	Δ	Δ	F_{24}	1	5	6
Duke-Imamoglu	F_{24}	12		Δ				

• Table C: The autor has checked that the equation (C) holds up to at least 30 decimals in the following cases:

G	g	f	F	r	n	k	α
Δ	Δ	ϕ_{22}	F_3	1	0	11	12
F_3	F_3	ϕ_{20}	F_5	2	0	10	35
F_4	F_4	ϕ_{18}	F_{11}	3	0	9	56
F_5	F_5	ϕ_{16}	F_{13}	4	0	8	75
F_6	F_6	ϕ_{16}	F_{13}	4	0	8	75
F_9	F_9	Δ	F_{24}	6	0	6	107
F_{11}	F_{11}	Δ	F_{24}	6	0	6	107
F_3	1	ϕ_{22}	F_3	0	1	11	12
F_4	Δ	ϕ_{20}	F_5	1	1	10	34
F_6	F_3	ϕ_{18}	F_{11}	2	1	9	55
F_7	F_4	ϕ_{16}	F_{13}	3	1	8	74
F_{14}	F_7	Δ	F_{24}	5	1	6	106
F_{16}	F_8	Δ	F_{24}	5	1	6	106
F_5	1	ϕ_{20}	F_5	0	2	10	33
F_8	Δ	ϕ_{18}	F_{11}	1	2	9	53
F_9	F_3	ϕ_{16}	F_{13}	2	2	8	72
F_{17}	F_5	Δ	F_{24}	4	2	6	104
F_{18}	F_6	Δ	F_{24}	4	2	6	104
F_{11}	1	ϕ_{18}	F_{11}	0	3	9	50
F_{12}	Δ	ϕ_{16}	F_{13}	1	3	8	68
F_{20}	F_4	Δ	F_{24}	3	3	6	100
F_{13}	1	ϕ_{16}	F_{13}	0	4	8	63
F_{22}	F_3	Δ	F_{24}	2	4	6	94
F_{23}	Δ	Δ	F_{24}	1	5	6	86

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