Special values of the standard zeta functions of Siegel modular forms

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Abstract In this paper, we consider the relation between the special values of the standard zeta functions and the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbf{Z})$. Furthermore, we give exact values of the standard zeta function for cuspidal Hecke eigenforms with respect to $Sp_2(\mathbf{Z})$ and propose a conjecture concerning the congruence of Saito-Kurokawa lift.

1 Introduction

For a cuspidal Hecke eigenform f of weight k with respect to $Sp_n(\mathbf{Z})$, let $L(f, s, \underline{\mathrm{St}})$ be the standard zeta function of f. Let m be a positive integer such that $m \le k-n$ and $m \equiv n \bmod 2$. Assume that $n \equiv 3 \bmod 4$ or $n = 1$ if $m = 1$. Then the value $\dfrac{L(f, m, \underline{\mathrm{St}})}{< f, f > \pi^{-n(n+1)/2+nk+(n+1)m}}$ belongs to $\mathbf{Q}(f)$ if all the Fourier coefficients of f belong to $\mathbf{Q}(f)$, where $< f, f >$ is the Petersson product and $\mathbf{Q}(f)$ is the field over $\mathbf{Q}$ generated by all Hecke eigenvalues (cf. [B2], [Mi]). In this paper, we consider the denominator of these values and the congruence of Hecke eigenvalues of cusp forms. This type of problem was first considered by Doi and Hida [D-H] in terms of special values of Rankin-Selberg zeta functions in the elliptic modular case. In Section 5, we give a generalization of their result in terms of the special values of standard zeta functions in the Siegel modular form case (cf. Theorems 5.2 and 5.3). The main tool for proving our main results is the pullback formula for Siegel Eisenstein series due to Böcherer [B1], [B2], which we will review in Section 4. This formula has been already used to prove to algebraicity of the special values of the standard zeta functions stated above. However, to complete the proof of our main results, we have to consider the integrality of the Eisenstein series acted by a certain differential operators. We will discuss this integrality in Sections 2 and 3. Furthermore, to formulate our main result reasonably, we have to consider a normalization of the standard zeta values because we have no normalization of Hecke eigenforms in case $n \geq 2$ unlike the elliptic modular case. We discuss this normalization of the standard zeta values in Section 5. Furthermore, in Section 6, we give an explicit formula for this value in terms of Hecke eigenvalues of f and some other elementary quantities in case $n = 2$. This type of formula has been given in elliptic modular cases in [Kat2]. To get our formula in this paper, we need an explicit form of differential operators on the space of Siegel modular forms of degree 4, and an explicit formula for local Siegel series for a half-integral matrix of degree 4. As for the former, the generating function of the differential operators has been given in [I], and by a direct but rather elaborate computation we can get an explicit form of them. As for the latter, we can compute it by using the explicit formula in [Kat1] in principle. However, in this paper, we show a trick which enables us to reduce the computation of local Siegel series of degree 4 to that of degree 2. Finally, in Section 7, we give some numerical examples and propose a conjecture concerning the congruence of Saito-Kurokawa lift.

Notation. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m, n)-matrices with entries in R. In particular put $M_n(R) = M_{nn}(R)$. Here we understand $M_{mn}(R)$ the set of the *empty matrix* if $m = 0$ or $n = 0$. For an (m, n)-matrix X and an (m, m)-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let a be an element of R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset X mod $aM_{mn}(R)$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix A, and R^* denotes the unit group of R. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, for an integral domain R of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices of degree n whose (i, j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. Let R' be a subring of R. Two symmetric matrices A and A' with entries in R are called equivalent over R'

with each other and write $A \underset{R'}{\sim} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2 Fourier coefficients of Siegel-Eisenstein series

For a complex number x put $\mathbf{e}(x) = \exp(2\pi i x)$. For a subring K of $\mathbf{R}$ put

$$GSp_n(K)^+ = \{M \in GL_{2n}(K) \mid J_n[M] = \kappa(M) J_n \text{ with some } \kappa(M) > 0\},$$

and

$$Sp_n(K) = \{M \in GSp_n(K)^+ \mid J_n[M] = J_n\},$$

where $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$. Furthermore, put

$$\Gamma^{(n)} = Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

Let $\mathbf{H}_n$ be Siegel's upper half-space. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(\mathbf{R})^+$ and $Z \in \mathbf{H}_n$ put

$$M < Z >= (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = \det(CZ + D).$$

Furthermore, for a function f on $\mathbf{H}_n$ we define $f|_k M$ as

$$(f|_k M)(Z) = j(M, Z)^{-k} f(M < Z >).$$

A function f on $\mathbf{H}_n$ is called a C^∞-modular form of weight k with respect to $\Gamma^{(n)}$ if it satisfies the following conditions:

(i) f is a C^∞-function on $\mathbf{H}_n$;

(ii) $(f|_k M)(Z) = f(Z)$ for any $M \in \Gamma^{(n)}$;

We call a C^∞-modular form f a holomorphic modular form if

(i) f is holomorphic on $\mathbf{H}_n$;
(ii) if $n = 1$, for any $\alpha > 0, f(z)$ is bounded on the set $\{x + iy \mid y \geq \alpha\}$ for each $\alpha > 0$.

We denote by $\mathfrak{M}_k(\Gamma^{(n)})$ (resp. $\mathfrak{M}_k^{\infty}(\Gamma^{(n)})$ the space of holomorphic (resp. C^{∞}-) modular forms of weight k with respect to $\Gamma^{(n)}$. For a modular form f of weight k with respect to $\Gamma^{(n)}$, let

$$f(Z) = \sum_{A \in \mathcal{H}(\mathbf{Z})_{\geq 0}} a_f(A) \mathbf{e}(\mathrm{tr}(AZ)),$$

be the Fourier expansion of $f(Z)$, where tr denotes the trace of a matrix. We call $f(Z)$ a cusp form if $a_f(A) = 0$ unless A is positive-definite. We denote by $\mathfrak{S}_k(\Gamma^{(n)})$ the submodule of $\mathfrak{M}_k(\Gamma^{(n)})$ consisting of cusp forms. Let dv denote the invariant volume element on $\mathbf{H}_n$ defined by $dv = \det(\mathrm{Im}(Z))^{-n-1} \wedge_{1 \leq j \leq l \leq n} (dx_{jl} \wedge dy_{jl})$. Here for $Z \in \mathbf{H}_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices (x_{jl}) and (y_{jl}). For two C^{∞}-modular forms f and g of weight k with respect to $\Gamma^{(n)}$ we define the Petersson scalar product $< f, g >$ by

$$< f, g >= \int_{\Gamma^{(n)} \backslash \mathbf{H}_n} f(Z) \overline{g(Z)} \det(\mathrm{Im}(Z))^k dv,$$

provided the integral converges.

For a positive integer k we define the Siegel Eisenstein series $E_{n,k}(Z, s)$ of degree n as

$$E_{n,k}(Z, s) = \zeta(1-k) \prod_{i=1}^{[n/2]} \zeta(1-2k+2i) \sum_{M \in \Gamma_{\infty}^{(n)} \backslash \Gamma^{(n)}} j(M, Z)^{-k} (\det(\mathrm{Im}(M < Z >)))^s$$

($Z \in \mathbf{H}_n, s \in \mathbf{C}$), where $\zeta(*)$ is Riemann's zeta function, and $\Gamma_{\infty}^{(n)} = \{\begin{pmatrix} * & * \\ 0_{n,n} & * \end{pmatrix} \in \Gamma^{(n)}\}$. Then $E_{n,k}(Z, s)$ is holomorphic at $s = 0$ as a function of s, and $E_{n,k}(Z, 0)$ is holomorphic as a function of Z unless $k = (n+2)/2 \equiv 2 \bmod 4$, or $k = (n + 3)/2 \equiv 2 \bmod 4$ (cf. [Sh3], [W]). From now on we assume that $E_{n,k}(Z, 0)$ is holomorphic as a function of Z, and write $E_{n,k}(Z) = E_{n,k}(Z, 0)$. To see the Fourier expansion of $E_{n,k}(Z, 0)$, for a half-integral matrix B of degree n over $\mathbf{Z}$, we define the Siegel series $b(B, s)$ by

$$b(B, s) = \sum_{R \in S_n(\mathbf{Q})/S_n(\mathbf{Z})} \mathbf{e}(\mathrm{tr}(BR)) \mu(R)^{-s},$$

where $\mu(R) = [R\mathbf{Z}^n + \mathbf{Z}^n : \mathbf{Z}^n]$. Furthermore we put

$$\Gamma_n(s) = \prod_{j=1} \pi^{j/2}\Gamma(s - (i-1)/2),$$

where $\Gamma(s)$ is Gamma function. For a p-adic number x put $\mathbf{e}_p(x) = \exp(2\pi i\tilde{x})$, where $\tilde{x}$ denotes a rational number such that $\tilde{x} - x \in \mathbf{Z}_p$. To investigate the Siegel series, for a prime number p and a half-integral matrix B of degree n over $\mathbf{Z}_p$ define the local Siegel series $b_p(B, s)$ by

$$b_p(B, s) = \sum_R \mathbf{e}_p(\mathrm{tr}(BR))\mu_p(R)^{-s},$$

where R runs over a complete set of representatives of $S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)$ and $\mu_p(R) = [R\mathbf{Z}_p^n + \mathbf{Z}_p^n : \mathbf{Z}_p^n]$. Then we easily see that for a half-integral matrix B of degree n over $\mathbf{Z}$ we have

$$b(B, s) = \prod_p b_p(B, s).$$

Let m, n be non-negative integers such that $m \geq n \geq 1$. For $A \in \mathcal{H}_m(\mathbf{Z}_p)$ and $B \in S_n(\mathbf{Q}_p)$ define the local density $\alpha_p(A, B)$ and the primitive local density $\beta_p(A, B)$ by

$$\alpha_p(A, B) = \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \#\mathcal{A}_e(A, B),$$

and

$$\beta_p(A, B) = \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \#\mathcal{B}_e(A, B),$$

where

$$\mathcal{A}_e(A, B) = \{X \in M_{mn}(\mathbf{Z}_p)/p^e M_{mn}(\mathbf{Z}_p) \mid A[X] - B \in p^e\mathcal{H}_n(\mathbf{Z}_p)\},$$

and

$$\mathcal{B}_e(A, B) = \{X \in \mathcal{A}_e(A, B) \mid \mathrm{rank}_{\mathbf{Z}/p\mathbf{Z}}(X) = n\}.$$

We define $\chi_p(a)$ for $a \in \mathbf{Q}_p \backslash \{0\}$ as follows;

$$\chi_p(a) = \begin{cases} +1 & \text{if } \mathbf{Q}(\sqrt{a}) = \mathbf{Q} \\ -1 & \text{if } \mathbf{Q}(\sqrt{a})/\mathbf{Q} \text{ is quadratic unramified} \\ 0 & \text{if } \mathbf{Q}(\sqrt{a})/\mathbf{Q} \text{ is quadratic ramified.} \end{cases}$$

For a half-integral matrix B of even degree n define $\xi_p(B)$ by

$$\xi_p(B) = \chi_p((-1)^{n/2} \det B).$$

Let $B \in \mathcal{H}_n(\mathbf{Z})_{>0}$ with n even. Then we can write $(-1)^{n/2} 2^n \det B = \mathfrak{d}_B \mathfrak{f}_B^2$ with $\mathfrak{d}_B$ a fundamental discriminant and $\mathfrak{f}_B \in \mathbf{Z}_{>0}$. Furthermore, let $\chi_B = (\frac{\mathfrak{d}_B}{*})$ be the Kronecker character corresponding to $\mathbf{Q}(\sqrt{(-1)^{n/2} \det B})/\mathbf{Q}$. We note that we have $\chi_B(p) = \xi_p(B)$ for any prime p. Let $H_k = \overbrace{H \perp \ldots \perp H}^{k}$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

For a non-degenerate half-integral matrix B of degree n over $\mathbf{Z}_p$ define a polynomial $\gamma_p(B; X)$ in X by

$$\gamma_p(B;X) = \begin{cases} (1-X)\prod_{i=1}^{n/2}(1-p^{2i}X^2)(1-p^{n/2}\xi_p(B)X)^{-1} & \text{if } n \text{ is even} \\ (1-X)\prod_{i=1}^{(n-1)/2}(1-p^{2i}X^2) & \text{if } n \text{ is odd} \end{cases}.$$

Then the following lemma is well known (e.g. [Ki1], Lemma 1)

Lemma 2.1. *For a non-degenerate half-integral matrix B of degree n over $\mathbf{Z}_p$ there exists a unique polynomial $F_p(B, X)$ in X over $\mathbf{Z}$ with constant term 1 such that*

$$b_p(B, s) = \gamma_p(B; p^{-s}) F_p(B; p^{-s}).$$

Furthermore for any positive integer $k \geq n/2$ and a half integral matrix A of degree $2k$ over $\mathbf{Z}_p$ such that $2A$ is unimodular, we have

$$\alpha_p(A, B) = F_p(B, \xi_p(A)p^{-k}) \gamma_p(B, \xi_p(A)p^{-k})$$

and, in particular,

$$\alpha_p(H_k, B) = F_p(B, p^{-k}) \gamma_p(B, p^{-k}).$$

Remark. For an element $B \in \mathcal{H}_n(\mathbf{Z}_p)$ of rank $m \geq 0$, there exists an element $\tilde{B} \in \mathcal{H}_m(\mathbf{Z}_p) \cap GL_m(\mathbf{Q}_p)$ such that $B \sim \tilde{B} \perp O_{n-m}$. We note that $b_p(\tilde{B}, s)$ does not depend on the choice of $\tilde{B}$ (cf. [Ki1]). Thus we write this as $b_p^{(r)}(B, s)$. Furthemore, $F_p(\tilde{B}, X)$ does not depend on the choice of $\tilde{B}$. Then we put $F_p^{(m)}(B,X)=F_p(\tilde{B},X)$. For an element $B\in\mathcal{H}_n(\mathbf{Z})_{\geq 0}$ of rank $m\geq 0$, there exist an element $\tilde{B}\in\mathcal{H}_m(\mathbf{Z})_{>0}$ such that $B\sim\tilde{B}\perp O_{n-m}$. Then by the above remark $b(\tilde{B},s)$ does not depend on the choice of $\tilde{B}$. Thus we write this as $b^{(m)}(B,s)$. Furthermore, $\det\tilde{B}$ does not depend on the choice of B. Thus we put $\det^{(m)}B=\det\tilde{B}$. Similarly, we write $\chi_B^{(m)}=\chi_{\tilde{B}}$ if m is even.

Now for a semi-positive definite half-integral matrix B of degree $2n$ and of rank m, we put

$$c_{2n,l}(B)=2^{[(m+1)/2]}\prod_p F_p^{(m)}(B,p^{l-m-1})$$

$$\times\begin{cases}\prod_{i=m/2+1}^{n}\zeta(1+2i-2l)L(1+m/2-l,\chi_B^{(m)}) & \text{if } m \text{ is even}\\ (-1)^{(m^2-1)/8}\prod_{i=(m+1)/2}^{n}\zeta(1+2i-2l) & \text{if } m \text{ is odd}\end{cases}.$$

Here we make the convention $F_p^{(m)}(B,p^{l-m-1})=1$ and $L(1+m/2-l,\chi_B^{(m)})=\zeta(1-l)$ if $m=0$. Then we have

Theorem 2.2. *Under the same assumption as above, we have*

$$E_{2n,l}(Z)=\sum_{B\in\mathcal{H}_{2n}(\mathbf{Z})_{\geq 0}}c_{2n,l}(B)\mathbf{e}(\operatorname{tr}(BZ)).$$

Remark. $c_{2n,l}(B)$ is a rational number, and the prime divisor of its denominator is not greater than $(2l-1)!$. This is a weaker version of Böcherer's result [B3].

3 Differential operators

In this section, following [B-S], [I], we introduce some differential operators acting on the space of modular forms. Let $X=(x_{ij})_{1\leq i\leq m,1\leq j\leq d}$ be a matrix of variables, and put $\Delta_{i,j}=\sum_{\nu=1}^{d}\frac{\partial^2}{\partial x_{i\nu}\partial x_{j\nu}}$. A polynomial $P(X)$ in X is called pluriharmonic if $\Delta_{ij}P=0$ for any $1\leq i,j,\leq m$. Take a polynomial mapping $P(X_1,X_2)$ from $M_{n,2l}(\mathbf{C})\times M_{n,2l}(\mathbf{C})$ to $\mathbf{C}$ such that
D-1. $P(X_1,X_2)$ is pluriharmonic for each X_i $(i=1,2)$.
D-2. $P(X_1g,X_2g)=P(X_1,X_2)$ for any $g\in O(2l)$

D-3. $P(a_1X_1, a_2X_2) = (\det a_1)^\nu (\det a_2)^\nu P(X_1, X_2)$ for $a_1, a_2 \in GL(n, \mathbf{C})$.

Assume that $l \geq n$. Then there exists a unique polynomial mapping $Q(W)$ from $S_{2n}(\mathbf{C})$ to $\mathbf{C}$ s.t $P(X_1, X_2) = Q(\begin{pmatrix} X_1{}^tX_1 & X_1{}^tX_2 \\ X_2{}^tX_1 & X_2{}^tX_2 \end{pmatrix})$. We note that $\deg Q = n\nu$. Let $Z = (z_{ij})_{1 \leq i,j \leq 2n}$ be a matrix of variables with $z_{ij} = z_{ji}$, and we write $\frac{\tilde{\partial}}{\partial z_{ij}} = \frac{(1+\delta_{ij})}{2} \frac{\partial}{\partial z_{ij}}$, and $(\frac{\partial}{\partial Z}) = (\frac{\tilde{\partial}}{\partial z_{ij}})_{1 \leq i,j \leq 2n}$. For $f \in C^\infty(\mathbf{H}_{2n})$ we define $\mathcal{D}_Q(f)$ and $\tilde{\mathcal{D}}_Q(f)$ by

$$\mathcal{D}_Q(f) = Q(\frac{\partial}{\partial Z})(f)$$

and

$$\tilde{\mathcal{D}}_Q(f) = \mathcal{D}_Q(f)_{Z_{12}=0},$$

where we write $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix}$ with $Z_1, Z_2 \in \mathbf{H}_n$ and $Z_{12} \in M_n(\mathbf{C})$. We consider the action of the above operators on the Fourier series. Let $A = (a_{ij}) \in \mathcal{H}_{2n}$. Then we have $\mathbf{e}(\mathrm{tr}(AZ)) = \exp(2\pi i(\sum_{\alpha,\beta=1}^{2n} a_{\alpha\beta} z_{\alpha\beta}))$. Then we have

$$\frac{\tilde{\partial}}{\partial_{\alpha\beta}}(\mathbf{e}(\mathrm{tr}(AZ))) = 2\pi i a_{\alpha\beta} \mathbf{e}(\mathrm{tr}(AZ)).$$

Thus we have

$$\mathcal{D}_Q(\mathbf{e}(\mathrm{tr}(AZ))) = (2\pi i)^{n\nu} Q(A) \mathbf{e}(\mathrm{tr}(AZ)).$$

Now let $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix} \in \mathbf{H}_{2n}$ as above, and $f(Z) = \sum_{A \in \mathcal{H}_{2n}(\mathbf{Z})_{\geq 0}} a(A) \mathbf{e}(\mathrm{tr}(AZ))$.

Then we have

$$\tilde{\mathcal{D}}_Q(f)(Z_1, Z_2)$$

$$= (2\pi i)^{n\nu} \sum_{A_1, A_2 \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}} \mathbf{e}(\mathrm{tr}(A_1 Z_1 + A_2 Z_2)) \sum_{R \in M_n(\mathbf{Z})} Q(\begin{pmatrix} A_1 & \frac{1}{2}R \\ \frac{1}{2}{}^tR & A_2 \end{pmatrix}) a(\begin{pmatrix} A_1 & \frac{1}{2}R \\ \frac{1}{2}{}^tR & A_2 \end{pmatrix}).$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GSp_n(\mathbf{R})^+$ put $\gamma^\uparrow = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\gamma^\downarrow = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}$. We define the mapping ι from $Sp_n(\mathbf{R}) \times Sp_n(\mathbf{R})$ to $Sp_{2n}(\mathbf{R})$

by

$$\iota : Sp_n(\mathbf{R}) \times Sp_n(\mathbf{R}) \ni (\gamma_1, \gamma_2) \hookrightarrow \gamma_1^{\uparrow} \gamma_2^{\downarrow} \in Sp_{2n}(\mathbf{R}).$$

Furthermore, for a function $f : \mathbf{H}_n \times \mathbf{H}_n \longrightarrow \mathbf{C}, \gamma_1, \gamma_2 \in Sp_n(\mathbf{R})$ we define

$$f|_l(\gamma_1, \gamma_2)(Z_1, Z_2) = j_{\gamma_1}(Z_1)^{-l} j_{\gamma_2}(Z_2)^{-l} f(\gamma_1(Z_1), \gamma_2(Z_2)).$$

Then we have

Theorem 3.1 ([I])

$$\tilde{\mathcal{D}}_Q(f)|_{l+\nu}(\gamma_1, \gamma_2) = \tilde{\mathcal{D}}(f|_l \iota(\gamma_1, \gamma_2))$$

Now we apply the above theorem to the modular forms. For a subspace $\mathfrak{M}$ of $\mathfrak{M}_l^{\infty}(\Gamma^{(n)})$ let $\mathfrak{M} \otimes \mathfrak{M} = \{\sum_{i,j} a_{ij} f_i(Z_1) f_j(Z_2)$ (finite sum); $f_i, f_j \in \mathfrak{M}, a_{ij} \in \mathbf{C}\}$. Put $C_q(s) = s(s + 1/2) \cdots (s + (q-1)/2)$. We choose Q such that

$$\tilde{\mathcal{D}}_Q(\det Z_{12}^{\nu}) = (-1)^{n\nu} \prod_{\mu=1}^{\nu} (C_n(\mu/2) C_n(l - n + \nu - \mu/2)),$$

and put

$$\overset{\circ}{\mathcal{D}}{}_{n,l}^{\nu} = \tilde{\mathcal{D}}_Q.$$

This coincides with $\overset{\circ}{\mathfrak{D}}{}_{n,l}^{\nu}$ in [B-S]. Then by Theorem 3.1 we easily see

Theorem 3.2. *$\overset{\circ}{\mathcal{D}}{}_{n,l}^{\nu}$ maps $\mathfrak{M}_l^{\infty}(\Gamma^{(2n)})$ to $\mathfrak{M}_{l+\nu}^{\infty}(\Gamma^{(n)}) \otimes \mathfrak{M}_{l+\nu}^{\infty}(\Gamma^{(n)})$. Furthermore $\overset{\circ}{\mathcal{D}}{}_{n,l}^{\nu}$ maps $\mathfrak{M}_l(\Gamma^{(2n)})$ into $\mathfrak{M}_{l+\nu}(\Gamma^{(n)}) \otimes \mathfrak{M}_{l+\nu}(\Gamma^{(n)})$, and in particular if $\nu > 0$, its image is contained in $\mathfrak{S}_{l+\nu}(\Gamma^{(n)}) \otimes \mathfrak{S}_{l+\nu}(\Gamma^{(n)})$.*

4 Pullback formula

$\mathbf{L}_n = \mathbf{L}_{\mathbf{Q}}(GSp_n(\mathbf{Q})^+, \Gamma^{(n)})$ denote the Hecke ring over $\mathbf{Q}$ associated with the Hecke pair $(GSp_n(\mathbf{Q})^+, \Gamma^{(n)})$. For each integer m define an element $T(m)$ of $\mathbf{L}_n$ by

$$T(m) = \sum_{d_1,\ldots,d_n,e_1,\ldots,e_n} \Gamma^{(n)}(d_1 \bot \ldots \bot d_n \bot e_1 \bot \ldots \bot e_n)\Gamma^{(n)},$$

where $d_1, ..., d_n, e_1, ..., e_n$ run over all positive integer satisfying

$$d_i | d_{i+1},\ e_{i+1} | e_i\ (i = 1, ..., n-1), d_n | e_n, d_i e_i = m\ (i = 1, ..., n).$$

Furthermore, for $i = 1, ..., n$ and a prime number p not dividing N, put

$$T_i(p^2) = \Gamma^{(n)}(1_{n-i} \bot p1_i \bot p^2 1_{n-i} \bot p1_i)\Gamma^{(n)}.$$

As is well known, $\mathbf{L}_n$ is generated over $\mathbf{Q}$ by all $T(p)$ and $T_i(p^2)$ $(i = 1, ..., n)$. We denote by $\mathbf{L}'_n$ the subalgebra of $\mathbf{L}_n$ generated by over $\mathbf{Z}$ by all $T(p)$ and $T_i(p^2)$ $(i = 1, ..., n)$. Let $T = \Gamma^{(n)} M \Gamma^{(n)}$ be an element of $\mathbf{L}_n \otimes \mathbf{C}$. Write T as $T = \cup_\gamma \Gamma^{(n)}\gamma$ and for $g \in M_k(\Gamma^{(n)})$ define the Hecke operator $|_k T$ associated to T as

$$f|_k T = \kappa(M)^{kn - n(n+1)/2} \sum_\gamma f|_k \gamma.$$

We call this action the Hecke operator as usual (cf. [A2].) If f is a eigenfunction of a Hecke operator $T \in \mathbf{L}_n \otimes \mathbf{C}$ we denote by $\lambda_f(T)$ its eigenvalue. Furthermore, we denote by $\mathbf{Q}(f)$ the field generated over $\mathbf{Q}$ by eigenvalues of all $T \in \mathbf{L}_n$. As is well known, $\mathbf{Q}(f)$ is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field K let $\mathfrak{O}_K$ denote the ring of integers in K.

Theorem 4.1 *Let $k \geq n+1$. Let $f \in \mathfrak{S}_k(\Gamma^{(n)})$ be a common eigenfunction of all Hecke operators in $\mathbf{L}'_n$. Then $\lambda_f(T)$ belongs to $\mathfrak{O}_{\mathbf{Q}(f)}$ for any $T \in \mathbf{L}'_n$.*

The above theorem is known in case $n = 1, 2$ (cf. [Ku2]), and it seems more or less well known also for general n. But for the reader's convenience, we here give a proof to it. Let R be a subring of $\mathbf{C}$. Let $\mathfrak{S}_k(\Gamma^{(n)})(R)$ be the $\mathbf{Z}$-module consisting of elements of $\mathfrak{S}_k(\Gamma^{(n)})$ whose Fourier coefficients belong to R. It is known that we have $\mathfrak{S}_k(\Gamma^{(n)})(\mathbf{Z}) \otimes \mathbf{C} = \mathfrak{S}_k(\Gamma^{(n)})$ (cf. Shimura [Sh2]). Then Theorem 4.1 follows from the following proposition:

Proposition 4.2. *Let R be a subring of $\mathbf{C}$. Any $T \in \mathbf{L}'_n$ maps $\mathfrak{S}_k(\Gamma^{(n)})(R)$ to itself.*

Proof. The assertion follows from Hafner and Walling [H-W].

Put

$$GSp_n(\mathbf{Q}_p) = \{M \in GL_{2n}(\mathbf{Q}_p); J_n[M] = \kappa(M) J_n \text{ with some } \kappa(M) \in \mathbf{Q}_p^\times\},$$

and let $\mathbf{L}_{np}=\mathbf{L}(GSp_n(\mathbf{Q}_p),GSp_n(\mathbf{Q}_p)\cap GL_{2n}(\mathbf{Z}_p))$ be the Hecke algebra associated with the pair $(GSp_n(\mathbf{Q}_p),GSp_n(\mathbf{Q}_p)\cap GL_{2n}(\mathbf{Z}_p))$. Now assume that f is a common eigenfunction of all Hecke operators, and for each prime number p, let $\alpha_0(p),\alpha_1(p),....,\alpha_n(p)$ be the Satake parameters of $\mathbf{L}_{np}$ determined by f. We then define the standard L-function $L(f,s,\underline{\mathrm{St}})$ by

$$L(f,s,\underline{\mathrm{St}})=\prod_p\prod_{i=1}^n\{(1-p^{-s})(1-\alpha_i(p)p^{-s})(1-\alpha_i(p)^{-1}p^{-s})\}^{-1}.$$

Let $E_{2n,l}(Z)=E_{2n,l}(Z,0)$ be the Eisenstein series in Section 2. We then define $\mathfrak{E}_{2n,l,k}(z_1,z_2)$ as

$$\mathfrak{E}_{2n,l,k}(z_1,z_2)=(-1)^{l/2+1}2^{-n}(2\pi\sqrt{-1})^{l-k}(l-n)\ \overset{\circ}{\mathcal{D}}{}_{n,l}^{k-l}\ (E_{2n,l})(z_1,z_2),$$

where $z_1,z_2\in\mathbf{H}_n$. Let $f(z)=\sum_{A\in\mathcal{H}_n(\mathbf{Z})_{>0}}a(A)\mathbf{e}(\mathrm{tr}(Az))$ be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$. For a positive integer $m\le k-n$ such that $m\equiv n \bmod 2$ put

$$\tilde{\Lambda}(f,m,\underline{\mathrm{St}})=(-1)^{n(n+1)/2+1}2^{-4kn+3n^2+n+(n-1)m+1}$$

$$\times\Gamma(m+1)\prod_{i=1}^n\Gamma(2k-n-i)\frac{L(f,m,\underline{\mathrm{St}})}{<f,f>\pi^{-n(n+1)/2+nk+(n+1)m}},$$

where $\delta=0$ or 1 according as n is even or odd. We note that all the Fourier coefficients of $\tilde{\mathcal{E}}_{2n,l,k}(z_1,z_2)$ are rational and any prime divisor of its denominator is not greater than $(2l-1)!$. Then as a special case of [B3] we have

Theorem 4.3. *Let l,k and n be a positive integers such that $l\le k-n$. Assume that l and n satisfy one of the following conditions:*
C-1. *n and l are even:*
C-2. $n\equiv 3 \bmod 4$ *or* $n=1$, *and l is odd:*
C-3. $n\equiv 1 \bmod 4$, *and l is odd and greater than or equal to 3.*

Let $f\in\mathfrak{S}_k(\Gamma^{(n)})$ be a common eigenfunction of all Hecke operators in $\mathbf{L}_n$. Then we have

$$<f,\mathfrak{E}_{2n,l,k}(*,-\bar{z}))>=L(f,l-n,\underline{\mathrm{St}})f(z),$$

For two semi-positive definite half-integral matrices A_1, A_2 of degree n, put

$$\epsilon_{l,k}(A_1,A_2)=\sum_{A_2-\frac{1}{4}A_1^{-1}[R]\geq 0}\tilde{c}_{2n,l}(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix})Q_{l,k-l}(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}),$$

where $\tilde{c}_{2n,l}(A)=(-1)^{l/2+1}2^{-n}(l-n)c_{2n,l}(A)$ for $A\in\mathcal{H}_{2n}(\mathbf{Z})_{\geq 0}$. Furthermore, for each semi-positive definite half-integral matrix A_1 put

$$\mathcal{F}_{l,k;A_1}(z_2)=\sum_{A_2\in\mathcal{H}_2(\mathbf{Z})_{\geq 0}}\epsilon_{l,k}(A_1,A_2)\mathbf{e}(\mathrm{tr}(A_2z_2)).$$

We note that $\mathcal{F}_{l,k;A_1}(z_2)$ belongs to $\mathfrak{M}_k(\Gamma^{(n)})$, and

$$\mathcal{E}_{2n,l,k}(z_1,z_2)=\sum_{A_1\in\mathcal{H}_n(\mathbf{Z})_{\geq 0}}\mathcal{F}_{l,k;A_1}(z_2)\mathbf{e}(\mathrm{tr}(A_1z_1)).$$

In particular, if $l<k$, $\mathcal{F}_{l,k;A_1}(z_2)$ belongs to $\mathfrak{S}_k(\Gamma^{(n)})$, and

$$\mathcal{E}_{2n,l,k}(z_1,z_2)=\sum_{A_1\in\mathcal{H}_n(\mathbf{Z})_{>0}}\mathcal{F}_{l,k;A_1}(z_2)\mathbf{e}(\mathrm{tr}(A_1z_1)).$$

Take a basis $\{f_i\}_{i=1}^{d_1}$ of $\mathfrak{S}_k(\Gamma^{(n)})$ consisting of primitive forms. Write

$$f_i(z)=\sum_{A\in\mathcal{H}_2(\mathbf{Z})_{>0}}a_i(A)\mathbf{e}(\mathrm{tr}(Az)).$$

Now we compute the value $\Lambda(f,l,\underline{\mathrm{St}})$.

Theorem 4.4. *In addition to the above notation and the assumption, assume that $l\leq k-n-2$. Then for any positive definite half-integral matrix A_1 of degree n we have*

$$\mathcal{F}_{l+n,k;A_1}(z)=\sum_{i=1}^{d_1}\Lambda(f_i,l,\underline{\mathrm{St}})a_i(A_1)\overline{f_i(z)}.$$

Remark 2. Since $\mathcal{F}_{k,k;A_1}(z)$ is not a cusp form, the above formula does not hold for $l=k-n$. However, by modifying the above method, we can get a similar formula for this case.

5 Congruence of modular forms

In this section we consider the congruence between the Hecke eigenvalues of modular forms of the same weight. Let K be an algebraic number filed, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in K. For a prime ideal $\mathfrak{P}$ of $\mathfrak{O}$, we denote by $\mathfrak{O}_{(\mathfrak{P})}$ be the localization of $\mathfrak{O}$ at $\mathfrak{P}$ in K. Then the following lemma can easily be proved.

Lemma 5.1. *Let $f_1, ..., f_d$ be a basis of $\mathfrak{S}_k(\Gamma^{(n)})$ consisting of Hecke eigenforms, and G an element of $\mathfrak{S}_k(\Gamma^{(n)})$. Let K be the composite field of $\mathbf{Q}(f_1), \mathbf{Q}(f_2), ...,$ and $\mathbf{Q}(f_d)$, and $\mathfrak{O} = \mathfrak{O}_K$. Let $\mathfrak{P}$ be a prime ideal of $\mathfrak{O}$. Assume that*

(1) *$c_G(A)$ belongs to $\mathfrak{O}_{(\mathfrak{P})}$ for any A, and $a_{f_1}(A_1) \in \mathfrak{O}^*_{(\mathfrak{P})}$ for some A_1.*

(2) *there exist $c_1, ..., c_d \in K$ such that $\mathrm{ord}_{\mathfrak{P}}(c_1) < 0$ and*

$$G(z) = \sum_{i=1}^{d} c_i f_i(z).$$

Then there exists $i \neq 1$ such that we have

$$\lambda_{f_i}(T) \equiv \lambda_{f_1}(T) \bmod \mathfrak{P}$$

for any $T \in \mathbf{L}'_n$.

Let f be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$ and M be a subspace of $\mathfrak{S}_k(\Gamma^{(n)})$ stable under Hecke operators $T \in \mathbf{L}_n$. A prime ideal $\mathfrak{P}$ of $\mathfrak{O}_{\mathbf{Q}(f)}$ is called a congruence prime of f with respect to M if there exists a Hecke eigenform $g \in \mathfrak{S}_k(\Gamma^{(n)})$ having a different system of Hecke eigenvalues from f such that

$$\lambda_f(T) \equiv \lambda_g(T) \bmod \tilde{\mathfrak{P}}$$

for any $T \in \mathbf{L}'_n$, where $\tilde{\mathfrak{P}}$ is the prime ideal of $\mathfrak{O}_{\mathbf{Q}(f)\mathbf{Q}(g)}$ lying above $\mathfrak{P}$. If $M = \mathfrak{S}_k(\Gamma^{(n)})$, we simply call $\mathfrak{P}$ a congruence prime of f.

Now we consider the relation between the congruence primes and the standard zeta values. To consider this, we have to normalize the standard zeta value $\Lambda(f, l, \underline{\mathrm{St}})$ for a Hecke eigenform f because it is not uniquely determined by the system of Hecke eigenvalues of f. We note that there is no reasonable normalization of cuspidal Hecke eigenform in the higher degree case unlike the elliptic modular case. Thus we define the following quantities:

for a Hecke eigenform f in $\mathfrak{S}_k(\Gamma^{(n)})$, by multiplying a suitable constant c we may assume all $c_f(A)'s$ are elements of $\mathbf{Q}(f)$ with bounded denominator. Let $\mathfrak{J}_f$ be the fractional ideal in $\mathbf{Q}(f)$ generated by all $c_f(A)'s$. Then $\Lambda(f, l, \underline{\mathrm{St}})\mathfrak{J}_f^2$ is a fractional ideal in $\mathbf{Q}(f)$. We note that the value $N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\mathrm{St}}))N(\mathfrak{J}_f)^2$ does not depend on the choice of c, where $N(\mathfrak{J}_f)$ is the norm of the ideal $\mathfrak{J}_f$. Furthermore, for a prime ideal $\mathfrak{P}$ of $\mathfrak{O}$, the order $\mathrm{ord}_{\mathfrak{P}}(\Lambda(f, l, \underline{\mathrm{St}})\mathfrak{J}_f^2)$ does not depend on the choice of c either. In particular, if we assume the multiplicity one property for the Hecke eigenforms, these values are uniquely determined by the system of eigenvalues of f. Then by Theorem 4.4 and Lemma 5.1, we have

Theorem 5.2. *Assume that $\mathfrak{S}_k(\Gamma^{(n)})$ has the multiplicity one property. Let f be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$. Let l be a positive integer satisfying the condition in Theorem 4.4. Let $\mathfrak{P}$ be a prime ideal of $\mathfrak{O}$. Assume that $\mathrm{ord}_{\mathfrak{P}}(\Lambda(f, l, \underline{\mathrm{St}})\mathfrak{J}_f^2) < 0$ and not dividing $(2l-1)!$. Then $\mathfrak{P}$ is a congruence prime of f. In particular, if a rational prime number p divides the denominator of $N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\mathrm{St}}))N(\mathfrak{J}_f)^2$, then p is divided by some congruence prime of f.*

Now a subspace M of $\mathfrak{S}_k(\Gamma^{(n)})$ is called non-splitting if there exists a Hecke eigenform f in M such that $M =< f^\sigma \mid \sigma \in \mathrm{Aut}(\mathbf{C}) >$. Clearly M is stable under the action of $\mathbf{L}_n$. Let $\mathbf{Z}[f]$ be a subring of $\mathfrak{O}_{\mathbf{Q}(f)}$ generated over $\mathbf{Z}$ by all Hecke eigenvalues of f. Then the different $\mathfrak{D}$ of $\mathbf{Z}[f]$ and the discriminant D of $\mathbf{Z}[f]$ does not depend on the choice of f, which will be denoted by $\mathfrak{D}_M$ and D_M, respectively. Then by the above result we easily see the following:

Theorem 5.3. *Let the notation and the assumption be as in Theorem 5.2. Let M be a non-splitting subspace of $\mathfrak{S}_k(\Gamma^{(n)})$. Let f be a Hecke eigenform in M. Let $\mathfrak{D} = \mathfrak{D}_M$ and $D = D_M$. Let $\mathfrak{P}$ be a prime ideal of $\mathfrak{O}$. Assume that $\mathrm{ord}_{\mathfrak{P}}(\Lambda(f, l, \underline{\mathrm{St}})\mathfrak{J}_f^2\mathfrak{D}) < 0$ and not dividing $(2l-1)!$. Then $\mathfrak{P}$ is a congruence prime of f with respect to $M^\perp$. In particular, if a rational prime number p divides the denominator of $N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\mathrm{St}}))N(\mathfrak{J}_f)^2D$, then p is divided by some congruence prime of f with respect to $M^\perp$.*

In Section 7, we will consider the congruence primes of the Saito-Kurokawa lift.

6 Exact standard L-values in case $n = 2$

In this section we obtain a useful formula for computing exact standard L-values in the case of degree 2. The following lemma can easily be proved (e.g. [Ki2]).

Lemma 6.1. *Let $n = n_1 + n_2$ with n_1 even. Let $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z}_p) \cap \frac{1}{2}GL_{n_1}(\mathbf{Z}_p)$ and $A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z}_p) \cap GL_{n_2}(\mathbf{Q}_p)$. Then for any $l \geq n$ we have*

$$\alpha_p(H_l, A_{11} \perp A_{22}) = \alpha_p(H_l, A_{11})\alpha_p(H_{l-n_1} \perp (-A_{11}), A_{22}).$$

Proposition 6.2. *Let n_1 be an even integer. Let $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z}_p) \cap \frac{1}{2}GL_{n_1}(\mathbf{Z}_p)$ and $A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z}_p)$. Let m be the rank of A_{22}. Then we have*

$$F_p^{(n_1+m)}(A_{11} \perp A_{22}, X) = F_p^{(m)}(A_{22}, \xi_p(A_{11})p^{n_1/2}X).$$

Proof. We may assume that A_{22} is non-degenerate. By Lemma 2.1 for any $l \geq n_1 + n_2$ we have

$$\alpha_p(H_l, A_{11} \perp A_{22}) = \gamma_p(A_{11} \perp A_{22}, p^{-l})F_p(A_{11} \perp A_{22}, p^{-l}).$$

By [Kat1, Proposition 2.2], we have

$$\alpha_p(H_l, A_{11} \perp A_{22}) = \beta_p(H_l, A_{11})\alpha_p(H_{l-n_1} \perp (-A_{11}), A_{22}).$$

Again by Lemma 2.1, we have

$$\alpha_p(H_{l-n_1} \perp (-A_{11}), A_{22}) = \gamma_p(A_{22}, \xi_p(A_{11})p^{n_1/2-l})F_p(A_{22}, \xi_p(A_{11})p^{n_1/2-l}).$$

Furthermore we have

$$\beta_p(H_l, A_{11}) = (1 - p^{-l}) \prod_{i=1}^{n_1/2} (1 - p^{2i-2l}) 1 - p^{n_1/2-l}\xi_p(A_{11}))^{-1}$$

(eg. [Ki2]), and by definition we have

$$\gamma_p(A_{11} \perp A_{22}, p^{-l}) = \beta_p(H_l, A_{11})\gamma_p(A_{22}, \xi_p(A_{11})p^{n_1/2-l}).$$

Thus the assertion holds.

Corollary 1. *Let* $A = \begin{pmatrix} A_{11} & A_{12}/2 \\ {}^t A_{12}/2 & A_{22} \end{pmatrix} \in \mathcal{H}_{n_1+n_2}(\mathbf{Z}_p) \cap GL_{n_1+n_2}(\mathbf{Q}_p)$ *with* $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z}_p), A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z}_p),$ *and* $A_{12} \in M_{n_1,n_2}(\mathbf{Z}_p)$. *Let* m *be the rank of* A. *Assume* $2A_{11} \in GL_{n_1}(\mathbf{Z}_p)$. *Then we have*

$$F_p^{(m)}(A, X) = F_p^{(m-n_1)}(A_{22} - \frac{1}{4}A_{11}^{-1}[A_{12}], \xi_p(A_{11})p^{n_1/2}X).$$

Corollary 2. *Let* n_1 *and* n_2 *be positive even integers. Let* $A = \begin{pmatrix} A_{11} & A_{12}/2 \\ {}^t A_{12}/2 & A_{22} \end{pmatrix} \in \mathcal{H}_{n_1+n_2}(\mathbf{Z})_{>0}$ *with* $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z})_{>0}, A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z})_{>0},$ *and* $A_{12} \in M_{n_1,n_2}(\mathbf{Z})$. *Let* p_0 *be a prime number. Let* m *be the rank of* A. *Assume* $2A_{11} \in GL_{n_1}(\mathbf{Z}_p)$ *for any prime number* $p \neq p_0$, *and* $2A_{22} \in GL_{n_2}(\mathbf{Z}_{p_0})$. *Then we have*

$$\prod_p F_p^{(m)}(A, X) = F_{p_0}^{(m-n_2)}(A_{11} - \frac{1}{4}A_{22}^{-1}[{}^tA_{12}], \chi_{A_{22}}(p_0)p_0^{n_2/2}X)$$

$$\times \prod_{p \neq p_0} F_p^{(m-n_1)}(A_{22} - \frac{1}{4}A_{11}^{-1}[A_{12}], \chi_{A_{11}}(p)p^{n_1/2}X).$$

Now for later's computation, we give an explicit form of $F_p^{(1)}(A, X)$ and $F_p^{(2)}(A, X)$ in case $\deg A = 2$.

Proposition 6.3. *Let* $A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \in \mathcal{H}_2(\mathbf{Z})_{\geq 0}$. *Put* $e = e_A =$ $\mathrm{GCD}(a_{11}, a_{12}, a_{22})$

(1) *Assume* rank $A = 1$. *Then we have*

$$F_p^{(1)}(A, X) = \sum_{i=1}^{\mathrm{ord}_p(e_A)} (pX)^i.$$

(2) *Assume* $A > 0$. *Then we have*

$$F_p(A, X) = \sum_{i=0}^{\mathrm{ord}_p(e_A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_A)-i} (p^3X^2)^j$$

$$-\chi_A(p)pX \sum_{i=0}^{\mathrm{ord}_p(e_A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_A)-i-1} (p^3X^2)^j.$$

Now we give an explicit form of differential operator in the case of degree 2 due to Ibukiyama. Let

$$G_l(f_1, f_2, f_3; t)$$
$$= \frac{1}{R(f_1, f_2, f_3; t)^{(2l-5)/2}(\Delta_0(f_1, f_2, f_3; t)^2 - 4f_3t^2)^{1/2}},$$

where

$$\Delta_0(f_1, f_2, f_3; t) = 1 - f_1t + f_2t^2$$

and

$$R(f_1, f_2, f_3; t) = \Delta_0(f_1, f_2, f_3; t) + (\Delta_0(f_1, f_2, f_3; t)^2 - 4f_3t^2)^{1/2}/2.$$

Write

$$G_l(f_1, f_2, f_3; t) = \sum_{m=0}^{\infty} G_{l,m}(f_1, f_2, f_3)t^m,$$

and define a polynomial map $Q_{l,m}(\begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_4 \end{pmatrix})$ from $S_4(\mathbf{C})$ to $\mathbf{C}$ by

$$Q_{l,m}(\begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_4 \end{pmatrix}) = G_{l,m}(\det W_2, \det W_1 \det W_4, \det \begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_4 \end{pmatrix}),$$

where $W_1, W_4 \in S_2(\mathbf{C})$, and $W_2 \in M_n(\mathbf{C})$. Furthermore define a polynomial map $P_{l,m}(X_1, X_2)$ from $M_n(\mathbf{C}) \times M_n(\mathbf{C})$ to $\mathbf{C}$ by

$$P_{l,m}(X_1, X_2) = Q_{l,m}(\begin{pmatrix} X_1{}^tX_1 & X_1{}^tX_2 \\ X_2{}^tX_1 & X_2{}^tX_2 \end{pmatrix}).$$

Then by [I]

Proposition 6.4. *$P_{l,m}(X_1, X_2)$ satisfies the conditions $D-1 \sim D-3$ in Section 3.*

Furthermore, by a direct but rather elaborate calculation we have

Proposition 6.5.

$$G_{l,m}(f_1,f_2,f_3)=\sum_{n=0}^{[m/2]}\begin{pmatrix}2n+l-5/2\\ n\end{pmatrix}f_3^n$$

$$\times\sum_{\nu=0}^{[(m-2n)/2]}(-f_2)^{\nu}\begin{pmatrix}l+m-\nu-5/2\\ m-2n-\nu\end{pmatrix}\begin{pmatrix}m-2n-\nu\\ \nu\end{pmatrix}(2f_1)^{m-2n-2\nu}.$$

We note that $G_{l,m}(f_1,f_2,f_3)\in 2^{-m}\mathbf{Z}[f_1,f_2,f_3]$. Let

$$\mathcal{G}_{l,m}=G_{l,m}(\frac{1}{4}\det(\frac{\partial}{\partial z_{ij}})_{1\le i\le 2,3\le j\le 4},\det(\frac{\tilde{\partial}}{\partial z_{ij}})_{1\le i,j\le 2}\det(\frac{\tilde{\partial}}{\partial z_{ij}})_{3\le i,j\le 4},\det(\frac{\tilde{\partial}}{\partial z_{ij}})_{1\le i,j\le 4}),$$

where $\begin{pmatrix}s\\ m\end{pmatrix}=\frac{\prod_{i=1}^{m}(s-i+1)}{m!}$. We note that

$$(\det(\frac{\partial}{\partial z_{ij}})_{1\le i\le 2,3\le j\le 4})^m((z_{13}z_{24}-z_{14}z_{23})^m)=\Gamma(m+2)\Gamma(m+1).$$

Thus we have

$$\mathcal{G}_{l,m}=d_{l,m}\,\overset{\circ}{\mathcal{D}}{}^m_{n,l},$$

where

$$d_{l,m}=\frac{\Gamma(m+2)\Gamma(m+1)\begin{pmatrix}l+m-5/2\\ m\end{pmatrix}}{2^m\prod_{\mu=1}^m C_2(\mu/2)C_2(l-2+m-\mu/2)}.$$

Furthermore, put

$$\tilde{\Lambda}(f,l,\underline{\mathrm{St}})=\frac{\begin{pmatrix}k-5/2\\ k-l-2\end{pmatrix}}{2^{3k-l-7}}\Gamma(l)\Gamma(k+l-2)\Gamma(k+l-1)\frac{L(f,l,\underline{\mathrm{St}})}{<f,f>(2\pi)^{2k+3l-3}}.$$

We note that

$$\tilde{\Lambda}(f,l,\underline{\mathrm{St}})=d_{l+2,k-l-2}\Lambda(f,l,\underline{\mathrm{St}}),$$

and for a positive definite half-integral matrices A_1 and A_2 of degree 2, let $\epsilon_{l,k}(A_1,A_2)$ be the one in Section 4. Let p_0 be a prime number. Assume that

$2A_1 \in GL_2(\mathbf{Z}_p)$ for any prime number $p \neq p_0$ and $2A_2 \in GL_2(\mathbf{Z}_{p_0})$. Then we have

$$\epsilon_{l,k}(A_1,A_2) = \sum_{R \in M_2(\mathbf{Z})} \tilde{c}_{4,l}(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix})$$

$$\times G_{l,k-l}(\frac{1}{4}\det R, \det A_1 \det A_2, \det \begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix})),$$

where

$$\tilde{c}_{4,l}(A) = (-1)^{l/2+1}2^{-2}(l-2)$$

$$\times F_{p_0}^{(m-2)}(A_1 - \frac{1}{4}A_2^{-1}[{}^tR], \chi_{A_2}(p_0)p_0^{l-m}) \prod_{p \neq p_0} F_p^{(m-2)}(A_2 - \frac{1}{4}A_1^{-1}[R], \chi_{A_1}(p)p^{l-m})$$

$$\times \begin{cases} \prod_{i=m/2+1}^{2} \zeta(1+2i-2l)L(3-l, \chi_A^{(m)}) & \text{if } m = 2,4 \\ \zeta(5-2l) & \text{if } m = 3, \end{cases}$$

for $A = \begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}$. We note that $\epsilon_{l,k}(A_1,A_2)$ is rational number and any prime divisor of its denominator is not greater than $(2l-1)!$. We note that

$$\mathcal{E}_{4,l,k}(z_1,z_2) = \sum_{A_1 \in \mathcal{H}_2(\mathbf{Z})_{>0}} \sum_{A_2 \in \mathcal{H}_2(\mathbf{Z})_{>0}} \epsilon_{l,k}(A_1,A_2)\mathbf{e}(\mathrm{tr}(A_1z_1 + A_2z_2)).$$

We note that $E_{4,l}(Z,0)$ belongs to $\mathfrak{M}_l(\Gamma^{(4)})$ for an even integer $l \geq 4$.

Fix an $A_1 \in \mathcal{H}_2(\mathbf{Z})_{>0}$ and a prime number p. We define $\epsilon_{l,k}(i,A)$ as follows:

$$\epsilon_{l,k}(1,A_1,A) = \epsilon_{l,k}(A_1,A),$$

$$\epsilon_{l,k}(i,A_1,A) = \epsilon_{l,k}(i-1,A_1,pA) + p^{2k-3}\epsilon_{l,k}(i-1,A_1,A/p)$$

$$+p^{k-2} \sum_{D \in GL_2(\mathbf{Z})U_pGL_2(\mathbf{Z})/GL_2(\mathbf{Z})} \epsilon_{l,k}(i-1,A_1,A[D]/p),$$

where $U_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Let $\{f_j\}_{j=1}^d$ be a basis of $\mathfrak{S}_k(\Gamma^{(2)})$ consisting of Hecke eigenforms, and $\lambda_j = \lambda_{f_j}(p)$. Then we have

$$\epsilon_{l+2,k}(i,A_1,A) = \sum_{j=1}^{d} \lambda_j^{i-1}\tilde{\Lambda}(f_j,l,\underline{\mathrm{St}})a_j(A_1)a_j(A)$$

for any $A \in \mathcal{H}_2(\mathbf{Z})_{>0}$. Thus by Theorem 4.5 we have

Proposition 6.6. *In addition to the above assumption, assume that $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$, and $a_1(A_1), a_1(A) \neq 0$. Let $f = f_1, K = \mathbf{Q}(f)$ and $e_i = e_{l+2}(i, A_1, A)$. Put $\tilde{\Lambda}^*(f, l, \underline{\mathrm{St}}) = N_{K/\mathbf{Q}}(\tilde{\Lambda}(f, l, \underline{\mathrm{St}}))N(\mathfrak{I}_f)^2$. Then for any positive even integer $l \leq k-4$ we have*

$$\tilde{\Lambda}^*(f, l, \underline{\mathrm{St}}) = N_{K/\mathbf{Q}}\left(\frac{\begin{vmatrix} e_1 & 1 & \cdots & 1 \\ e_2 & \lambda_2 & \cdots & \lambda_d \\ \vdots & \vdots & \vdots & \vdots \\ e_d & \lambda_2^{d-1} & \cdots & \lambda_d^{d-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_d \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{d-1} & \lambda_2^{d-1} & \cdots & \lambda_d^{d-1} \end{vmatrix}}\right) \frac{N(\mathfrak{I}_f)^2}{N_{K/\mathbf{Q}}(a_1(A_1)a_1(A))}.$$

7 Numerical examples and conjecture

We compute the special values of the standard zeta functions by using Mathematica. Let $J_{k,1}^{\mathrm{cusp}}$ be the space of Jacobi cusp forms of weight k and of index 1 on $\Gamma^{(1)}$, and $\mathcal{V} : J_{k,1}^{\mathrm{cusp}} \longrightarrow \mathfrak{M}_k(\Gamma^{(2)})$ be the injection defined in Theorem 6.2 of Eichler and Zagier [E-Z]. Then $\mathcal{V}(J_{k,1}^{\mathrm{cusp}})$ is the Maass subspace of $\mathfrak{S}_k(\Gamma^{(2)})$, which will be denoted by $\mathfrak{S}_k(\Gamma^{(2)})^*$. Let $\phi_{10,1}(\tau, z)$ and $\phi_{12,1}(\tau, z)$ be the Jacobi cusp forms in $J_{10,1}^{\mathrm{cusp}}$ and $J_{12,1}^{\mathrm{cusp}}$ in Page 40 of [E-Z], respectively. Here $\tau \in \mathbf{H}_1$ and $z \in \mathbf{C}$. Furthermore let $E_{1,k}(\tau)$ be the Eisenstein series of weight k with respect to $\Gamma^{(1)}$ defined in Section 2, and put $E_k(\tau) = \zeta(1-k)^{-1}E_{1,k}(\tau)$. Then it is well known that $E_4^a(\tau)E_6(\tau)^b\phi_{j,1}(\tau, z)$ $(a, b \geq 0, j = 10, 12, 4a + 6b + j = k)$ forms a basis of $J_{k,1}^{\mathrm{cusp}}$. Let $A_0 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}$.

(1) We have dim $\mathfrak{S}_{20}(\Gamma^{(2)}) = 3$, and dim $\mathfrak{S}_{38}(\Gamma^{(1)}) = 2$. Let f_1, f_2 be the basis of $\mathfrak{S}_{38}(\Gamma^{(1)})$ consisting of primitive forms. For $i = 1, 2$ let $\lambda_i = 48(-2025 + \sqrt{D})$ and $48(-2025 - \sqrt{D})$ with $D = 63737521$. Then λ_i is the eigenvalues of the $T(2)$ with respect to f_i. Then they satisfy the equation

$$X^2 + 194400X^2 - 137403408384 = 0,$$

and $\mathbf{Q}(f_i) = \mathbf{Q}(\lambda_i) = K$ with $K = \mathbf{Q}(\sqrt{D})$ (cf. [H-M]). Put $\theta_i = \lambda_i/96$. Then θ_i satisfies the following equation

$$g(X) := X^2 + 2025X - 14909224 = 0.$$

The discriminant of $g(X)$ is D. Thus the discriminant of $\mathbf{Q}(\sqrt{D})$ is D, and the ring of integers in $\mathbf{Q}(\sqrt{D})$ is $\mathbf{Z}(\theta_1)$. Let $h_1(\tau, z) = E_4(\tau)E_6(\tau)\phi_{10,1}(\tau, z)$, and $h_2(\tau, z) = E_4(\tau)^2\phi_{12,1}(\tau, z)$. These form a basis of $J_{20,1}^{\mathrm{cusp}}$. Put $g_i = \mathcal{V}h_i$ for $i = 1, 2$. Then these form a basis of $\mathfrak{S}_{20}(\Gamma^{(2)})^*$ whose A_0-th Fourier coefficient is 1. Furthermore for $i = 1, 2$ put

$$\hat{f}_i = -230g_1 + (-4862 - \theta_i)g_2.$$

Then $\hat{f}_i$ is the Saito-Kurokawa lift of f_i whose A_0-th Fourier coefficient is

$$a_{\hat{f}_i}(A_0) = -5092 - \theta_i.$$

We note that we have $\hat{f}_i = \chi_{20}^{(i)}/2$ for $i = 1, 2$, where $\chi_{20}^{(1)}$ and $\chi_{20}^{(2)}$ are the eigenforms in Kurokawa [Ku1]. Then we have

$$a_{\hat{f}_i}(A_1) = -10(4816 + \theta_i).$$

Furthermore we have

$$\lambda_{\hat{f}_i}(T(2)) = \lambda_i + 3 \cdot 2^{18}.$$

Then

$$N_{K/\mathbf{Q}}(a_{\hat{f}_i}(A_0)) = 2^2 \cdot 3^4 \cdot 5 \cdot 19 \cdot 23,$$

and

$$N_{K_i/\mathbf{Q}}(a_{\hat{f}_i}(A_1)) = -2^5 \cdot 3 \cdot 5^2 \cdot 23 \cdot 2659.$$

By a simple computation we have $N(\mathfrak{I}_{\hat{f}_i}) = 2^5 \cdot 3^2 \cdot 5 \cdot 23$.

Let $\Upsilon 20$ be the cuspidal Hecke eigenform in Skoruppa [Sko]. It is a unique (up to constant) Hecke eigenform in $\mathfrak{S}_{20}(\Gamma^{(2)})$ which is not a Saito-Kurokawa lift. We note that $\Upsilon = \chi_{20}^{(3)}/2$, where $\chi_{20}^{(3)}$ is the Hecke eigenform in [Ku1]. Then $\hat{f}_1, \hat{f}_2$ and $\Upsilon 20$ form a basis of $\mathfrak{S}_{20}(\Gamma^{(2)})$. We have $\mathfrak{I}_\Upsilon = 1$ and $a_\Upsilon(A_0) = 1$ and $a_\Upsilon(A_1) = 2^2$. Furthermore we have $\lambda_\Upsilon(T(2)) = -2^8 \cdot 3^2 \cdot 5 \cdot 73$. Thus by Proposition 6.6, we have the following tables:

l	$\lvert N_{K/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i,l,\underline{\mathrm{St}})\rvert N(\mathfrak{I}_{\hat{f}_i})^2$
2	$3\cdot 5^3\cdot 7\cdot 11\cdot 13^2\cdot 17^2\cdot 29\cdot 31/2^{58}\cdot D$
4	$3^3\cdot 5\cdot 7^2\cdot 11\cdot 13\cdot 17^2\cdot 29\cdot 31\cdot 173\cdot 443/2^{51}\cdot 23\cdot D$
6	$3^2\cdot 5\cdot 11^2\cdot 13^3\cdot 17^2\cdot 29\cdot 31\cdot 227\cdot 1381069/2^{43}\cdot 23^2\cdot D$
8	$3^5\cdot 5^3\cdot 7\cdot 11\cdot 13^2\cdot 17^2\cdot 29\cdot 31\cdot 21347\cdot 58169/2^{34}\cdot 23\cdot D$
10	$3^4\cdot 5^4\cdot 7^2\cdot 13^3\cdot 17^2\cdot 19\cdot 31\cdot 863\cdot 3673\cdot 3426433/2^{29}\cdot 23\cdot D$
12	$3^2\cdot 5\cdot 7^2\cdot 11\cdot 17^2\cdot 29\cdot 37\cdot 293\cdot 691^2\cdot 33721\cdot 96875477/2^{16}\cdot 23\cdot D$
14	$3^4\cdot 5^2\cdot 7^3\cdot 11\cdot 13^2\cdot 17\cdot 29^2\cdot 31\cdot 467196139\cdot 541368271/2^5\cdot D$
16	$2^{13}\cdot 3^{10}\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 29^2\cdot 31^2\cdot 67\cdot 1699\cdot 3617^2\cdot 296551/D$

Here we note that the square of a prime factor of the l-th Bernoulli number appears in the numerator of $N_{K/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i,l,\underline{\mathrm{St}})N(\mathfrak{I}_{\hat{f}_i})^2$.

l	$\lvert\tilde{\Lambda}(\Upsilon 20,l,\underline{\mathrm{St}})\rvert$
2	$3^3\cdot 5\cdot 7\cdot 11\cdot 23\cdot 29\cdot 31/2^{28}$
4	$3^2\cdot 5^2\cdot 13\cdot 23\cdot 29\cdot 31\cdot 113/2^{25}$
6	$3^4\cdot 5\cdot 7\cdot 29\cdot 31\cdot 7549/2^{17}$
8	$2^{15}\cdot 3^3\cdot 5\cdot 7^2\cdot\cdot 11\cdot 29\cdot 31\cdot 37\cdot 4861/2^{16}$
10	$3\cdot 5\cdot 7\cdot 31\cdot 283\cdot 617\cdot 4098371/2^{13}$
12	$3^4\cdot 5^2\cdot 7^2\cdot 11\cdot 29\cdot 31\cdot 337\cdot 91909/2^6$
14	$2^4\cdot 3^4\cdot 7^2\cdot 13\cdot 12893\cdot 2166127$
16	$2^{11}\cdot 3^6\cdot 5^3\cdot 13\cdot\cdot 23\cdot 29\cdot 347162819$

(2) We have dim $\mathfrak{S}_{22}(\Gamma^{(2)})=4$, and dim $\mathfrak{S}_{42}(\Gamma^{(1)})=3$. Let f_1,f_2,f_3 be the basis of $\mathfrak{S}_{42}(\Gamma^{(1)})$ consisting primitive forms. For $i=1,2,3$ let λ_i be the eigenvalues of the $T(2)$ with respect to f_i. Then they satisfy the equation

$$X^3+344688X^2-6374982426624X-520435526440845312=0,$$

and $\mathbf{Q}(f_i)=\mathbf{Q}(\lambda_i)$ (cf. [H-M]). Put $\theta_i=\lambda_i/48$ for $i=1,2,3$. Then θ_i is also an algebraic integer and satisfy the following equation:

$$g(X):=X^3+7181X^2-2766919456X-4705905729536.$$

The discriminant of $g(x)$ is $-2^{10}\cdot 3^4\cdot 5^2\cdot 7^2\cdot 1465869841\cdot 578879197969$. Let $h_1(\tau,z)=E_4(\tau)^3\phi_{10,1}(\tau,z), h_2(\tau,z)=E_6(\tau)^2\phi_{10,1}(\tau,z)$, and $h_3(\tau,z)=E_4(\tau)E_6(\tau)\phi_{12,1}(\tau,z)$. Then These forms a basis of $J_{22,1}^{\mathrm{cusp}}$. Put $g_i=\mathcal{V}h_i$ for $i=$

$1,2,3$. Then these forms a basis of $\mathfrak{S}_{22}(\Gamma^{(2)})^*$ whose A_0-th Fourier coefficient is 1. Furthermore for $i=1,2,3$ put

$$\hat{f}_i = 1155(435776+31\theta_i)g_1 - 220(4760624+79\theta_i)g_2 + (286270336-605563\theta_i+\theta_i^2)g_3.$$

Then $\hat{f}_i$ is the Saito-Kurokawa lift of f_i whose A_0-th Fourier coefficient is

$$a_{\hat{f}_i}(A_0) = -257745664 - 421138\theta_i + \theta_i^2.$$

Then we have

$$a_{\hat{f}_i}(A_1) = 10(395073536 - 642488\theta_i + \theta_i^2)$$

and

$$a_{\hat{f}_i}(A_2) = -352(-3767171584 - 1827333\theta_i + \theta_i^2).$$

Furthermore we have

$$\lambda_{\hat{f}_i}(T(2)) = \lambda_i + 3\cdot 2^{20}.$$

Then

$$N_{K_i/\mathbf{Q}}(a_{\hat{f}_i}(A_0)) = -2^{14}\cdot 3^{13}\cdot 5^4\cdot 7^4\cdot 11^3\cdot 13\cdot 157\cdot 1213$$

and

$$N_{K_i/\mathbf{Q}}(a_{\hat{f}_i}(A_1)) = -2^{24}\cdot 3^8\cdot 5^5\cdot 7^3\cdot 11^3\cdot 13\cdot 157\cdot 1447\cdot 2437.$$

Let $\theta=\theta_1, \hat{f}=\hat{f}_1$, and $K=\mathbf{Q}(f_1)$. Put $\eta_2=\theta(\theta+13)/32, \eta_3=(\theta^2+8)/9+\theta, \eta_5=(1+\theta^2)/5+\theta$, and $\eta_7=(6+5\theta+\theta^2)/7+2\theta+2$. Then by using, "round 2 method", we see that $<1,\theta,\eta_2>$ is 2-maximal in $\mathfrak{O}_K$ (cf. H. Cohen [Co].) We have $[<1,\theta,\eta_2>:<1,\theta,\theta^2>]=2^5$, and therefore the discriminant D of K is not divisible by 2. In the same manner, we see D is not divisible by $3\cdot5\cdot7$, and therefore, we have $D=1465869841\cdot578879197969$ and $\mathfrak{O}_K=<1,\theta,\eta_2,\eta_3,\eta_5,\eta_7>_{\mathbf{Z}}$. We have

$$[\mathfrak{O}_K:\mathbf{Z}[1,\theta,\theta^2]] = 2^5\cdot 3^2\cdot 5\cdot 7.$$

Put $R_1=1155(435776+31\theta_i), R_2=-220(4760624+79\theta_i)$, and $R_3=286270336-605563\theta_i+\theta_i^2$. Then

$$[\mathbf{Z}[1,\theta,\theta^2]:<R_1,R_2,R_3>] = 2^6\cdot 3^3\cdot 5^3\cdot 7^2\cdot 11^3\cdot 13\cdot 157,$$

and

$$[<R_1,R_2,R_3>:<a_{\hat{f}}(A_0),a_{\hat{f}}(A_1),a_{\hat{f}}(A_2)>] = 2^8 3^4.$$

Furthermore we have $< a_{\hat{f}}(A_0), a_{\hat{f}}(A_1), a_{\hat{f}}(A_2) >\subset \mathfrak{I}_{\hat{f}}$. Thus we have Thus $N(\mathfrak{I}_{\hat{f}}) = 2^{19} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^3 \cdot 13 \cdot 157$.

Let $\Upsilon 22$ be the cuspidal Hecke eigenform in Skoruppa [Sko]. It is a unique (up to constant) Hecke eigenform in $\mathfrak{S}_{22}(\Gamma^{(2)})$ which is not a Saito-Kurokawa lift. Then $\hat{f}_1, \hat{f}_2, \hat{f}_3$, and $\Upsilon 22$ form a basis of $\mathfrak{S}_{22}(\Gamma^{(2)})$ and $\mathbf{Q}(\hat{f}_i) = \mathbf{Q}(f_i)$. We have $\mathfrak{I}_\Upsilon = 1$ and $a_\Upsilon(A_0) = 1$ and $a_\Upsilon(A_1) = -2^2 \cdot 3$. Furthermore we have $\lambda_\Upsilon(T(2)) = -2^8 \cdot 3 \cdot 5 \cdot 577$. Thus by Proposition 6.6, we have the following tables:

l	$\lvert N_{K_i/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, l, \underline{\mathrm{St}})\rvert N(\mathfrak{I}_{\hat{f}_i})^2$
2	$3^9 \cdot 5^4 \cdot 7 \cdot 11^3 \cdot 13^5 \cdot 17^2 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37/2^{78} \cdot D$
4	$3^{12} \cdot 5^2 \cdot 7^4 \cdot 11^2 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 29^2 \cdot 31 \cdot 37 \cdot 151 \cdot 1601 \cdot 6551 \cdot 7951/2^{69} \cdot 1423 \cdot D$
6	$3^{12} \cdot 5^9 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 137 \cdot 809$
	$\times 38029874887/2^{57} \cdot 7 \cdot 1423 \cdot D$
8	$3^9 \cdot 5 \cdot 7^5 \cdot 11 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 84521 \cdot 8947751$
	$\times 699588169271/2^{41} \cdot 1423 \cdot D$
10	$3^{10} \cdot 5^9 \cdot 7^3 \cdot 11^4 \cdot 13^4 \cdot 17^2 \cdot 19^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37^2$
	$\times 1423469629 \cdot 27864526583393/2^{28} \cdot 1423 \cdot D$
12	$3^{12} \cdot 5 \cdot 11^2 \cdot 13 \cdot 17 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 37 \cdot 691^3 \cdot 953$
	$\times 243911 \cdot 4251563 \cdot 6617174324030971171/2^{10} \cdot 1423 \cdot D$
14	$2^6 \cdot 3^{12} \cdot 5^5 \cdot 7^8 \cdot 11^3 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37$
	$\times 150197 \cdot 318467 \cdot 1465187 \cdot 13894099 \cdot 63630191/1423 \cdot D$
16	$2^{26} \cdot 3^{19} \cdot 5^5 \cdot 7^2 \cdot 11^3 \cdot 13^3 \cdot 19 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 3617^3$
	$\times 1465869841 \cdot 2775014078857939 \cdot 22683897890722493/1423 \cdot D$
18	$2^{59} \cdot 3^{25} \cdot 5^{15} \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37^2$
	$\times 43867^3 \cdot 365257 \cdot 13553776667/1423 \cdot D$

l	$\lvert\tilde{\Lambda}(\Upsilon 22; l, \underline{\mathrm{St}})\rvert$
2	$3^3 \cdot 5 \cdot 11 \cdot 23 \cdot 29 \cdot 31 \cdot 37/2^{32}$
4	$3^4 \cdot 5 \cdot 11 \cdot 13 \cdot 29 \cdot 31 \cdot 37 \cdot 103 \cdot 157/2^{27} \cdot 1423$
6	$3^6 \cdot 11 \cdot 29 \cdot 31 \cdot 37^2 \cdot 485363/2^{24} \cdot 1423$
8	$3^2 \cdot 29 \cdot 31 \cdot 37 \cdot 149 \cdot 3361493719/2^{18} \cdot 1423$
10	$3^3 \cdot 5 \cdot 11 \cdot 37 \cdot 89 \cdot 1039 \cdot 2741 \cdot 3616027/2^{15} \cdot 1423$
12	$3^4 \cdot 11^2 \cdot 31 \cdot 37 \cdot 421 \cdot 254725279909/2^8 \cdot 1423$
14	$3^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 31 \cdot 37 \cdot 733 \cdot 2131 \cdot 82625047/2 \cdot 1423$
16	$2^5 \cdot 3^7 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 30293340159041/1423$
18	$2^{16} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17 \cdot 31 \cdot 37 \cdot 101 \cdot 439 \cdot 1049 \cdot 49991/1423$

In this case, we have

$$1423 = \mathfrak{P}_i \mathfrak{P}'_i$$

in $\mathfrak{O}_{\mathbf{Q}(f_i)}$, where $\mathfrak{P}_i =< \lambda_i + 967, 1423 >$ and $\mathfrak{P}'_i =< \lambda_i^2 + 778\lambda_i + 660, 1423 >$. We have deg $\mathfrak{P}_i = 1$ and deg $\mathfrak{P}'_i = 2$. Thus from the above table, $\mathfrak{P}_i$ is a congruence prime of $\hat{f}_i$. In fact we have

$$\lambda_{\hat{f}_i}(T(2)) \equiv \lambda_{\Upsilon}(T(2)) \bmod \mathfrak{P}_i.$$

Though we only treat the congruence of the Hecke eigenalues of modular forms in this paper, we can also consider the cogrunece of the Fourier coeff-cients of them in some cases, and in particlar, we can explain the examples of congruences of modular forms in Skoruppa[Sko] in terms of special values of L-functions, which we will discuss in subsequent papers.

Now we consider the congruence prime of the Saito-Kurokawa lift with respect to $(\mathfrak{S}_k(\Gamma^{(2)})^*)^{\perp}$. Let $\hat{f}$ be the Saito-Kurokawa lift of $f \in \mathfrak{S}_{2k-2}(\Gamma^{(1)})$. Then we have

$$L(\hat{f}, s, \underline{\mathrm{St}}) = \zeta(s) \prod_{i=1}^{2} L(f, s+k-i).$$

For positive integers $1 \le m, m' \le 2k-3$, put

$$C(f, m, m') = \frac{L(f, m)L(f, m')}{(2\pi i)^{m+m'} < f, f >},$$

and for $2 \le m \le k-2$,

$$\mathcal{C}(f, m) = C(f, m+k-1, m+k-2).$$

We note that $C(f, m, m') \in \mathbf{Q}(f)$ if $m - m'$ is odd, and therefore $\mathcal{C}(f, m) \in \mathbf{Q}(f)$ (cf. Shimura[Sh1].) On the other hand, put

$$\mathcal{L}(A_f, k) = \prod_{\sigma \in \mathrm{Aut}(\mathbf{C})} L(f^{\sigma}, k)/\Omega_{+}$$

and

$$\mathcal{L}(A_f, k+1) = \prod_{\sigma \in \mathrm{Aut}(\mathbf{C})} L(f^{\sigma}, k+1)/\Omega_{-},$$

where Ω_- and Ω_+ are the periods in Stein [St1]. We note that $\mathcal{L}(A_f, k)\mathcal{L}(A_f, k+1)/N_{\mathbf{Q}(f)/\mathbf{Q}}(\mathcal{C}(f, 2))$ is algebraic, and more precisely we expect that it has no prime factor greater than $(2k-1)!$. We note that 1423 divides both $\mathcal{L}(A_f, k)$ and $N_{\mathbf{Q}(f)/\mathbf{Q}}(\mathcal{C}(f,2))$ for $f = f_i$ in the above example (2). We have

$$\Lambda(\hat{f}, m, \underline{\mathrm{St}}) = \Lambda(\hat{f}, 2, \underline{\mathrm{St}}) \frac{\mathcal{C}(f,m)\zeta(1-m)}{\mathcal{C}(f,2)\zeta(-1)}$$

up to elementary factor. By the above numerical data, we expect that the numerator of $\tilde{\Lambda}(\hat{f}, 2, \underline{\mathrm{St}})$ has relatively small prime factors. Thus a prime factor of the numerator of $\mathcal{C}(f,2)$ greater than $(2k-1)!$ is expected to be a congruence prime of f with respect to $(\mathfrak{S}_k(\Gamma^{(2)})^*)^{\perp}$. However this does not hold in general as will be seen the following example. Hence we have to observe this type of phenomenon more carefully.

(3) We have dim $\mathfrak{S}_{24}(\Gamma^{(2)}) = 5$, and dim $\mathfrak{S}_{46}(\Gamma^{(1)}) = 3$. Let f_1, f_2, f_3 be the basis of $\mathfrak{S}_{46}(\Gamma^{(1)})$ consisting primitive forms. For $i = 1, 2, 3$ let λ_i be the eigenvalues of the $T(2)$ with respect to f_i. Then they satisfy the equation

$$g(x) := X^3 - 3814272X^2 - 44544640241664X + 135250282417024401408 = 0,$$

and $\mathbf{Q}(f_i) = \mathbf{Q}(\lambda_i)$ (cf. [H-M]). Put $\theta_i = \lambda_i/576$ for $i = 1, 2, 3$. Then θ_i is also an algebraic integer and satisfy the following equation:

$$g(X) := X^3 + 6622X^2 - 134261189X + 707735092608 = 0.$$

The discriminant of $g(x)$ is $-2^3 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 227 \cdot 454287770269681529$. Let $h_1(\tau, z) = E_4(\tau)^2 E_6(\tau)\phi_{10,1}(\tau, z), h_2(\tau, z) = E_4(\tau)^3 \phi_{12,1}(\tau, z)$, and $h_3(\tau, z) = E_6(\tau)^2 \phi_{12,1}(\tau, z)$. Then These forms a basis of $J_{24,1}^{\mathrm{cusp}}$. Put $g_i = \mathcal{V} h_i$ for $i = 1, 2, 3$. Then these forms a basis of $\mathfrak{S}_k(\Gamma^{(2)})^*$ whose A_0-th Fourier coefficient is 1. Furthermore for $i = 1, 2, 3$ put

$$R_1(\theta_i) = 379483272 - 890026\theta_i, R_2(\theta_i) = -882(-1783462 + 45\theta_i),$$

$$R_3(\theta_i) = -504343809 + 223340\theta_i + \theta_i^2.$$

and

$$\hat{f}_i = R_1(\theta_i)g_1 + R_2(\theta_i)g_2 + R_3(\theta_i)g_3.$$

Then $\hat{f}_i$ is the Saito-Kurokawa lift of f_i whose A_0-th Fourier coefficient is

$$a_{\hat{f}_i}(A_0) = -16219620 - 98839\theta_i + \theta_i^2.$$

Then we have

$$a_{\hat{f}_i}(A_1) = 2(-650323008 + 843344\theta_i + 5\theta_i^2)$$

and

$$a_{\hat{f}_i}(A_2) = -8(-99687764052 + 37331175\theta_i + 137\theta_i^2).$$

Then

$$N_{K_i/\mathbf{Q}}(a_{\hat{f}_i}(A_0)) = -2^3 \cdot 3^9 \cdot 7^6 \cdot 14149 \cdot 722947657,$$

and

$$N_{K_i/\mathbf{Q}}(a_{\hat{f}_i}(A_1)) = 2^9 \cdot 3^5 \cdot 5^3 \cdot 7^5 \cdot 1553 \cdot 3559 \cdot 722947657.$$

Let $\theta = \theta_1, \hat{f} = \hat{f}_1$, and $K = \mathbf{Q}(f_1)$. Put $\eta_2 = \theta(\theta + 13)/32, \eta_3 = (\theta^2 + 8)/9 + \theta, \eta_5 = (1+\theta^2)/5+\theta$, and $\eta_7 = (6+5\theta+\theta^2)/7+2\theta+2$. Then by using, "round 2 method", we see that $< 1, \theta, \eta_2 >$ is 2-maximal in $\mathfrak{O}_K$ (cf. H. Cohen [Co].) We have $[< 1, \theta, \eta_2 >:< 1, \theta, \theta^2 >] = 2^5$, and therefore the discriminant D of K is not divisible by 2. In the same manner, we see D is not divisible by $3 \cdot 5 \cdot 7$, and therefore, we have $D = 2 \cdot 227 \cdot 454287770269681529$ and $\mathfrak{O}_K =< 1, \theta, \eta_2, \eta_3, \eta_5, \eta_7 >_{\mathbf{Z}}$. We have

$$[\mathfrak{O}_K : \mathbf{Z}[1, \theta, \theta^2]] = 2 \cdot 3 \cdot 5 \cdot 7.$$

Put $R_j = R_j(\theta)$ for $j = 1, 2, 3$. Then

$$[\mathbf{Z}[1, \theta, \theta^2] :< R_1, R_2, R_3 >] = 2^5 \cdot 3^2 \cdot 7^4 \cdot 722947657$$

and

$$[< R_1, R_2, R_3 >:< a_{\hat{f}}(A_0), a_{\hat{f}}(A_1), a_{\hat{f}}(A_2) >] = 2^8 3^4.$$

Furthermore we have $< a_{\hat{f}}(A_0), a_{\hat{f}}(A_1), a_{\hat{f}}(A_2) >\subset \mathfrak{J}_{\hat{f}}$. Thus we have Thus $N(\mathfrak{J}_{\hat{f}}) = 2^{14} \cdot 3^7 \cdot 5 \cdot 7^5 \cdot 722947657$. Let $\Upsilon 24a$ and $\Upsilon 24b$ be the cuspidal Hecke eigenform in Skoruppa [Sko]. They are Hecke eigenforms and form a basis of $(\mathfrak{S}_k(\Gamma^{(2)})^*)^{\perp}$. Then $\hat{f}_1, \hat{f}_2, \hat{f}_3$, and $\Upsilon 24a, \Upsilon 24b$ form a basis of $\mathfrak{S}_{24}(\Gamma^{(2)})$ and $\mathbf{Q}(\hat{f}_i) = \mathbf{Q}(f_i)$. Then we have

$$\tilde{\Lambda}(\hat{f}, 2, \underline{\mathrm{St}})$$

$$= -3^{13} \cdot 5^6 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 17^3 \cdot 19^2 \cdot 29^3 \cdot 31^2 \cdot 37 \cdot 41 \cdot 1213/2^{91} \cdot D.$$

We note that 1213 divides $\mathcal{C}(f, 2)$ but it is not a congruence prime of $\hat{f}$ with respect to $(\mathfrak{S}_k(\Gamma^{(2)})^*)^{\perp}$. Indeed it does not divide either $N_{K/\mathbf{Q}}(\lambda_{\hat{f}_i}(T(2)) -$

$\lambda_{\Upsilon 24a}(T(2))$ or $N_{K/\mathbf{Q}}(\lambda_{\hat{f}_i}(T(2)) - \lambda_{\Upsilon 24b}(T(2))$. On the other hand, from the numerical table in W. Stein [St2], we see that

$$\mathcal{L}(A_f, k) = 157 \cdot 83,$$

and

$$\mathcal{L}(A_f, k+1) = 1213.$$

Thus for $4 \leq l \leq 20$, the prime numbers 157 and 83 divide the denominator of $\tilde{\Lambda}(\hat{f}, 2, \underline{\mathrm{St}})$ and therefore, by Theorem 5.3, they are congruence primes of $\hat{f}$ with respect to $(\mathfrak{S}_{24}(\Gamma^{(2)})^*)^{\perp}$. From the above consideration, we would propose the following conjecture:

Conjecture. *Let $\mathfrak{P}$ be a prime ideal of $\mathbf{Q}(f)$ not dividing $(2k-1)!$. Then $\mathfrak{P}$ is a congruence divisor of $\hat{f}$ with respect to $(\mathfrak{S}_k(\Gamma^{(2)})^*)^{\perp}$ if and only if $\mathfrak{P}$ divides the numerator of $\mathcal{L}(A_f, k)$.*

This is an analogue of the Doi-Hida-Ishii conjecture concerning the congruence primes of the Doi-Naganuma lifting [D-H-I]. We also note that this type of conjecure has been proposed by Harder [Ha] in the case of vector valued Siegel modular forms. We can formulate this type of conjecture for the congruence primes of the Ikeda lifting, which we will discuss in a subsequent paper.

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