Pullback formula and differential operators

Noritomo Kozima (Tokyo Institute of Technology)

In this report, we describe a theory of pullback formula for vector valued Siegel modular forms. The main tools are the Eisenstein series and a differential operator which sends a scalar valued Siegel modular form to the tensor product of two vector valued Siegel modular forms. We investigate $(\mathcal{D}E_k^{p+q})\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}$ precisely, where E_k^{p+q} is the Eisenstein series and $\mathcal{D}$ the differential operator.

In Section 1, we describe pullback formula for scalar valued Siegel modular forms. In Section 2, we define vector valued Siegel modular forms. In Section 3, we describe the differential operator explicitly and "Fundamental Lemmas". In Section 4, we give a general theory from Fundamental Lemmas. In Section 5, we describe results in special cases. In Section 6, we consider a way of computation of $\mathcal{D}E_k^{p+q}$.

1. Introduction

In this section we introduce pullback formula and its applications for scalar valued Siegel modular forms.

Let E_k^n be the holomorphic Eisenstein series of degree n and weight k, i.e.,

$$E_k^n(Z) := \sum_{(C,D)} \det(CZ+D)^{-k}$$

where (C, D) runs over a complete set of representatives of the equivalence class of co-prime symmetric pairs of degree n. The right-hand side converges absolutely and locally uniformly for $k > n+1$. Therefore E_k^n is holomorphic for $k > n+1$.

Next, we define the non-holomorphic Eisenstein series of degree n and weight k by

$$E_k^n(Z, s) := \det(\mathrm{Im}(Z))^s \sum_{(C,D)} \det(CZ+D)^{-k} \left|\det(CZ+D)\right|^{-2s}.$$

Here s is a complex variable. The right-hand side converges for $k + 2\,\mathrm{Re}(s) > n+1$. As is well known, $E_k^n(Z, s)$ has meromorphic continuation on the whole s-plane and satisfies a functional equation. (see Langlands [13], Kalinin [9], Mizumoto [14])

For the holomorphic Eisenstein series, Garrett [7] proved the following formula:

$$E_k^{p+q}\begin{pmatrix} Z^{(p)} & 0 \\ 0 & W^{(q)} \end{pmatrix} = E_k^p(Z)E_k^q(W) + \sum_{r=1}^{\min(p,q)} C_r \sum_{j=1}^{d(r)} D(k, f_{r,j})[f_{r,j}]_r^p(Z)[\theta f_{r,j}]_r^q(W),$$

where C_r is a constant, $d(r)$ is the dimension of the space of cuspforms of degree r and weight k, $\{f_{r,1}, f_{r,2}, \ldots, f_{r,d(r)}\}$ is an orthonormal basis consisting of eigenforms, $[f]_r^p$ denotes the Klingen type Eisenstein series attached to f ([10]), and $(\theta f)(z) := \overline{f(-\overline{z})}$. For an eigenform f and a complex variable s, $D(s, f)$ is defined by

$$D(s, f) := \sum_{T\in\mathbb{T}^{(r)}} \lambda(f, T)\det(T)^{-s},$$

where $\mathbb{T}^{(r)}$ is the set consisting of all elementary divisor forms of degree r and $\lambda(f, T)$ is the eigenvalue on f of the Hecke operator $\left(\Gamma_r \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma_r\right)$. Here Γ_r is the Siegel modular group of degree r. Using this formula, Böcherer [1] studied Fourier coefficients of Klingen type Eisenstein series.

On the other hand, in [4], Böcherer showed that $D(s, f)$ is equal to

$$\zeta(s)^{-1}\prod_{j=1}^{r}\zeta(2s-2j)^{-1}\, L(s-r, f, \underline{\mathrm{St}})$$

where ζ is the Riemann zeta function and $L(*, f, \underline{\mathrm{St}})$ the standard L-function attached to f. Furthermore, in [2], introducing a differential operator $\mathcal{D}_{k,\nu}$ which sends a Siegel modular form of weight k to the product of two Siegel modular forms of weight $k+\nu$, Böcherer showed Garrett's pullback formula for $(\mathcal{D}_{k,\nu}E_k^{2n})\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}$ and proved

$$\frac{L(m, f, \underline{\mathrm{St}})}{\pi^{nk+m(n+1)-n(n+1)/2}(f, f)} \in \mathbb{Q}(f)$$

if the Fourier coefficients of f belong to $\mathbb{Q}(f)$ and m is an integer such that $m > n$ and

$$(1.1) \qquad 1 \le m \le k-n \quad \text{and} \quad m+n \text{ is even.}$$

Here $(\cdot,\cdot)$ denotes a (non-normalized) Petersson inner product and $\mathbb{Q}(f)$ a totally real finite extension of $\mathbb{Q}$. Furthermore Mizumoto [14] proved the same result when (1.1) and

$$n \equiv 3 \pmod 4 \quad \text{or} \quad n = 1 \quad \text{if} \quad m = 1.$$

For the non-holomorphic Eisenstein series, Böcherer [3] showed the following identity:

$$\left(f, E_k^{2n}\left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \overline{s}\right)\right) = (\Gamma\text{-factor}) \cdot L(2s+k-n, f, \underline{\mathrm{St}}) \cdot f(Z),$$

and proved meromorphic continuation and functional equation of standard L-functions.

2. Vector valued Siegel modular forms

Let n be a positive integer. Let (ρ, V_ρ) be a polynomial representation of $GL(n,\mathbb{C})$ on a finite-dimensional complex vector space V_ρ. We fix a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V_ρ such that

$$\langle \rho(g)v, w\rangle = \langle v, \rho({}^t\overline{g})w\rangle \quad \text{for } g \in GL(n,\mathbb{C}),\ v,\ w \in V_\rho.$$

Let $\Gamma_n := Sp(n,\mathbb{R}) \cap M(2n,\mathbb{Z})$ be the Siegel modular group of degree n, and $\mathfrak{H}_n$ the Siegel upper half space of degree n. For $g = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in Sp(n,\mathbb{R})$ and $Z = (z_{\mu\nu})_{1\le \mu,\nu \le n} \in \mathfrak{H}_n$, we put

$$g\langle Z\rangle := (AZ+B)(CZ+D)^{-1}, \quad j(g,Z) := CZ+D,$$

$$\delta(g,Z) := \det(CZ+D), \quad \Delta(g,Z) := (CZ+D)^{-1}C,$$

$$\left(\frac{\partial}{\partial Z}\right) := \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial z_{\mu\nu}}\right)_{1\le\mu,\nu\le n}.$$

Here $\delta_{\mu\nu}$ is the Kronecker's delta. And for a V_ρ-valued function $f\colon \mathfrak{H}_n \to V_\rho$,

$$(f|_\rho g)(Z) := \rho(j(g,Z))^{-1} f(g\langle Z\rangle).$$

We write $|_k$ for $\rho = \det^k$.

A C^∞-function $f\colon \mathfrak{H}_n \to V_\rho$ is called a V_ρ-valued C^∞-modular form of weight ρ if it satisfies $f|_\rho\gamma = f$ for all $\gamma \in \Gamma_n$. The space of all such functions is denoted by $\mathfrak{M}_\rho^\infty$. The space of V_ρ-valued Siegel modular forms of weight ρ is defined by

$$\mathfrak{M}_\rho := \{f \in \mathfrak{M}_\rho^\infty \mid f \text{ is holomorphic on } \mathfrak{H}_n \text{ (and its cusps)}\},$$

and the space of cuspforms by

$$\mathfrak{S}_\rho := \left\{ f \in \mathfrak{M}_\rho \,\middle|\, \lim_{\lambda\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix}\right) = 0 \quad \text{for all } Z \in \mathfrak{H}_{n-1} \right\}.$$

If $\rho = \det^k$, we write $\mathfrak{M}_k^{n\,\infty}$, $\mathfrak{M}_k^n$, and $\mathfrak{S}_k^n$ for $\mathfrak{M}_\rho^\infty$, $\mathfrak{M}_\rho$, and $\mathfrak{S}_\rho$, respectively. For f, $g \in \mathfrak{M}_\rho^\infty$, the (non-normalized) Petersson inner product of f and g is defined by

$$(f,g) := \int_{\Gamma_n\backslash\mathfrak{H}_n} \left\langle \rho(\sqrt{\mathrm{Im}(Z)})f(Z), \rho(\sqrt{\mathrm{Im}(Z)})g(Z)\right\rangle \det(\mathrm{Im}(Z))^{-n-1}\,dZ$$

if the right-hand side is convergent.

3. Differential operators for Siegel modular forms

Let W_j be a j-dimensional vector space

$$W_j := \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_j,$$

where e_1, e_2, ..., e_j are indeterminates. Let $T^l(W_j)$ be the l-th tensor product of W_j for a positive integer l, i.e.,

$$T^l(W_j) := \underbrace{W_j \otimes \cdots \otimes W_j}_{l\text{-times}},$$

and ρ'_j the standard representation of $GL(j, \mathbb{C})$ on $T^l(W_j)$. We put $\rho_j := \det^k \otimes \rho'_j$.

Let ${W_j}^*$ and ${W_j}_*$ be copies of W_j, i.e.,

$$\begin{aligned} {W_j}^* &:= \mathbb{C}{e_1}^* \oplus \mathbb{C}{e_2}^* \oplus \cdots \oplus \mathbb{C}{e_j}^*, \\ {W_j}_* &:= \mathbb{C}e_{1*} \oplus \mathbb{C}e_{2*} \oplus \cdots \oplus \mathbb{C}{e_j}_*, \end{aligned}$$

where ${e_1}^*$, ${e_2}^*$, ..., ${e_j}^*$, e_{1*}, e_{2*}, ..., ${e_j}_*$ are indeterminates. For $w \in W_j$ and $v \in T^l(W_j)$, $w^* \in {W_j}^*$, $w_* \in {W_j}_*$, $v^* \in T^l({W_j}^*)$ and $v_* \in T^l({W_j}_*)$ are defined in the obvious way. And ${\rho'_j}^*$, ${\rho_j}^*$, ${\rho'_j}_*$ and ${\rho_j}_*$ are defined similarly.

On the other hand, we put index sets

$$I^* := \{1^*, 2^*, \ldots, l^*\}, \quad I_* := \{1_*, 2_*, \ldots, l_*\} \quad \text{and} \quad I := I^* \cup I_*$$

and we consider a polynomial ring

$$\mathbb{C}[e_j^{(\alpha)} \mid j \in \mathbb{Z}_{>0},\, \alpha \in I],$$

where $e_j^{(\alpha)}$ is indeterminate for any j and α.

We fix positive integers p and q. For a symmetric matrix S of size $p+q$ and positive integers a, b, we define

$$\begin{aligned} S^{ab} &:= (e_1^{(a^*)}, \ldots, e_p^{(a^*)}, 0, \ldots, 0)\, S\, {}^t(e_1^{(b^*)}, \ldots, e_p^{(b^*)}, 0, \ldots, 0) (= S^{ba}), \\ S^a_b &:= (e_1^{(a^*)}, \ldots, e_p^{(a^*)}, 0, \ldots, 0)\, S\, {}^t(0, \ldots, 0, e_1^{(b_*)}, \ldots, e_q^{(b_*)}), \\ S_{ab} &:= (0, \ldots, 0, e_1^{(a_*)}, \ldots, e_q^{(a_*)})\, S\, {}^t(0, \ldots, 0, e_1^{(b_*)}, \ldots, e_q^{(b_*)}) (= S_{ba}). \end{aligned}$$

Furthermore $S^{a^*b^*}$, $S^{a^*b_*}$ and $S^{a_*b_*}$ denote S^{ab}, S^a_b and S_{ab}, respectively. This notation will be used in Section 6. Now, we consider a product

$$S^{a_1a_2}S^{a_3a_4}\ldots S^{a_{2r-1}a_{2r}}S_{b_1b_2}S_{b_3b_4}\ldots S_{b_{2r-1}b_{2r}}S^{a_{2r+1}}_{b_{2r+1}}S^{a_{2r+2}}_{b_{2r+2}}\ldots S^{a_l}_{b_l}$$

for some $r \geq 0$, where $(a_1, a_2, \ldots, a_l)$ and $(b_1, b_2, \ldots, b_l)$ are permutations of $(1, 2, \ldots, l)$. This product can be expressed as

$$\sum_{\substack{1 \leq i_1, i_2, \ldots, i_l \leq p \\ 1 \leq j_1, j_2, \ldots, j_l \leq q}} (\text{coefficient}) e_{i_1}^{(1^*)} e_{i_2}^{(2^*)} \ldots e_{i_l}^{(l^*)} e_{j_1}^{(1_*)} e_{j_2}^{(2_*)} \ldots e_{j_l}^{(l_*)}.$$

We identify $e_{i_1}^{(1^*)} e_{i_2}^{(2^*)} \ldots e_{i_l}^{(l^*)} e_{j_1}^{(1_*)} e_{j_2}^{(2_*)} \ldots e_{j_l}^{(l_*)}$ with $e^*_{i_1} \otimes e^*_{i_2} \otimes \cdots \otimes e^*_{i_l} \otimes e_{j_1 *} \otimes e_{j_2 *} \otimes \cdots \otimes e_{j_l *}$. Then this product belongs to $T^l(W_p^*) \otimes T^l(W_{q*})$.

A linear combination of these products is called *a homogeneous polynomial of S*.

Examples of homogeneous polynomials:

$l = 1$

$$S_1^1.$$

$l = 2$

$$S_1^1 S_2^2, \quad S_2^1 S_1^2, \quad S^{12} S_{12}.$$

$l = 3$

$$\begin{array}{llllll} S_1^1 S_2^2 S_3^3, & S_1^1 S_3^2 S_2^3, & S_2^1 S_1^2 S_3^3, & S_2^1 S_3^2 S_1^3, & S_3^1 S_1^2 S_2^3, & S_3^1 S_2^2 S_1^3, \\ S^{23} S_{23} S_1^1, & S^{23} S_{31} S_2^1, & S^{23} S_{12} S_3^1, & S^{31} S_{23} S_1^2, & S^{31} S_{31} S_2^2, & \\ S^{31} S_{12} S_3^2, & S^{12} S_{23} S_1^3, & S^{12} S_{31} S_2^3, & S^{12} S_{12} S_3^3. & & \end{array}$$

Let $\mathfrak{P}(S)$ be a homogeneous polynomial of S. We put

$$P(X_1, X_2) = \mathfrak{P}\Big({}^t\!\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \Big),$$

where $X_1 \in M(p, d, \mathbb{C})$ and $X_2 \in M(q, d, \mathbb{C})$. Here d is a positive integer. Then we have
(C1) $P(a_1 X_1, a_2 X_2) = \rho_p'^*(a_1) \otimes \rho_{q*}'(a_2) P(X_1, X_2)$ for any $a_1 \in GL(p, \mathbb{C})$ and $a_2 \in GL(q, \mathbb{C})$,
(C2) $P(X_1 g, X_2 g) = P(X_1, X_2)$ for any $g \in O(d)$.
If
(C3) $P(X_1, X_2)$ is pluri-harmonic for each X_1 and X_2,
and $d := 2k$, then we have the following theorem:

Theorem (Ibukiyama [8, Theorem 1]) *For a C^∞-function $f\colon \mathfrak{H}_{p+q} \to \mathbb{C}$, $g_1 \in Sp(p, \mathbb{R})$ and $g_2 \in Sp(q, \mathbb{R})$, we have*

$$((\mathfrak{P}(\partial) f)|_{\rho_p^*} g_1 |_{\rho_{q*}} g_2)|_{\mathfrak{Z} = \mathfrak{Z}_0} = (\mathfrak{P}(\partial)(f|_k g_1^* g_{2*}))|_{\mathfrak{Z} = \mathfrak{Z}_0},$$

where for $g_1 = \begin{pmatrix} A_1^{(p)} & B_1^{(p)} \\ C_1^{(p)} & D_1^{(p)} \end{pmatrix}$ *and* $g_2 = \begin{pmatrix} A_2^{(q)} & B_2^{(q)} \\ C_2^{(q)} & D_2^{(q)} \end{pmatrix}$, *we define* $g_1^* := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & 1_q & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 1_q \end{pmatrix}$ *and* $g_{2*} := \begin{pmatrix} 1_p & 0 & 0 & 0 \\ 0 & A_2 & 0 & B_2 \\ 0 & 0 & 1_p & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}$, *and we put* $\partial := \left(\dfrac{\partial}{\partial \mathfrak{Z}} \right)$ *and* $\mathfrak{Z}_0 := \begin{pmatrix} Z^{(p)} & 0 \\ 0 & W^{(q)} \end{pmatrix}$.

Applying the above theorem to $f(\mathfrak{Z}) = \delta(g, \mathfrak{Z})^{-k}$ or $f(\mathfrak{Z}) = \delta(g, \mathfrak{Z})^{-k} \left|\delta(g, \mathfrak{Z})\right|^{-2s}$ $\det(\mathrm{Im}(\mathfrak{Z}))^s$, we have the following lemmas:

Fundamental Lemma 1 (holomorphic case) *If $\mathfrak{P}(S)$ is a pluri-harmonic homogeneous polynomial of S, then there exists a homogeneous polynomial $\mathfrak{Q}(X)$ of X such that*

$$\mathfrak{P}(\partial)(\delta(g,\mathfrak{Z})^{-k}) = \delta(g,\mathfrak{Z})^{-k}\mathfrak{Q}(\Delta(g,\mathfrak{Z}))$$

for any $g \in Sp(p+q,\mathbb{R})$ and $\mathfrak{Z} \in \mathfrak{H}_{p+q}$, and $\mathfrak{Q}(X)$ has the following form

$$\mathfrak{Q}(X) = \sum (\text{coefficient}) X^{a_1}_{b_1} X^{a_2}_{b_2} \dots X^{a_l}_{b_l}.$$

Fundamental Lemma 2 (non-holomorphic case) *If $\mathfrak{P}(S)$ is a pluri-harmonic homogeneous polynomial of S, then there exists a homogeneous polynomial $\mathfrak{Q}(X,s)$ of X such that*

$$\begin{aligned}&\big(\mathfrak{P}(\partial)(\delta(g,\mathfrak{Z})^{-k}\,|\delta(g,\mathfrak{Z})|^{-2s}\det(\mathrm{Im}(\mathfrak{Z}))^s)\big)|_{\mathfrak{Z}=\mathfrak{Z}_0}\\ &\quad = \big((\delta(g,\mathfrak{Z})^{-k}\,|\delta(g,\mathfrak{Z})|^{-2s}\det(\mathrm{Im}(\mathfrak{Z}))^s)\mathfrak{Q}(\Delta(g,\mathfrak{Z}) - \frac{1}{2i}(\mathrm{Im}(\mathfrak{Z}))^{-1}, s)\big)|_{\mathfrak{Z}=\mathfrak{Z}_0}\end{aligned}$$

for any $g \in Sp(p+q,\mathbb{R})$ and $\mathfrak{Z} \in \mathfrak{H}_{p+q}$.

Remarks.
(1) In the holomorphic case, we need not restrict $\mathfrak{Z}$ to $\mathfrak{Z}_0$. And $\mathfrak{Q}(\Delta(g,\mathfrak{Z}))$ depends only on the upper-right (or lower-left) block of $\Delta(g,\mathfrak{Z})$.
(2) $\Delta(g,\mathfrak{Z}) - \frac{1}{2i}(\mathrm{Im}(\mathfrak{Z}))^{-1} = -\frac{1}{2i}(j(g,\mathfrak{Z}))^{-1}\,\mathrm{Im}(g\langle\mathfrak{Z}\rangle)^{-1\,t}(j(g,\mathfrak{Z}))^{-1}$.
(3) $\mathfrak{Q}(X)$ and $\mathfrak{Q}(X,s)$ exist independently. We expect $\mathfrak{Q}(X) = \mathfrak{Q}(X,0)$.

Example of pluri-harmonic homogeneous polynomials:

$l = 1$

$$S^1_1.$$

$l = 2$

$$S^1_1 S^2_2 - \frac{1}{d}S^{12}S_{12}, \quad S^1_2 S^2_1 - \frac{1}{d}S^{12}S_{12},$$

$$\begin{aligned}&S^1_1 S^2_2 + S^1_2 S^2_1 - \frac{2}{d}S^{12}S_{12}, && \text{symmetric tensor valued case,}\\ &S^1_1 S^2_2 - S^1_2 S^2_1, && \text{alternating tensor valued case.}\end{aligned}$$

$l = 3$

$$\begin{aligned}S^1_1 S^2_2 S^3_3 &- \frac{1}{(d+2)(d-1)}\{(d+1)(S^{23}S_{23}S^1_1 + S^{31}S_{31}S^2_2 + S^{12}S_{12}S^3_3)\\ &- (S^{23}S_{31}S^1_2 + S^{23}S_{12}S^1_3 + S^{31}S_{23}S^2_1 + S^{31}S_{12}S^2_3 + S^{12}S_{23}S^3_1 + S^{12}S_{31}S^3_2)\}.\end{aligned}$$

4. Applications

We assume that representations of $GL(n,\mathbb{C})$ are irreducible. First we fix a Young diagram $(\lambda_1,\lambda_2,\dots,\lambda_\nu)$ which belongs to $\mathbb{Z}^\nu$ and satisfies $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\nu \geq 0$, $\nu \leq \min(p,q)$ and $\lambda_1+\lambda_2+\cdots+\lambda_\nu = l$. Let

$$c := \sum_{\substack{\sigma\in\mathcal{H}\\ \tau\in\mathcal{V}}} \operatorname{sgn}(\tau)\tau\sigma$$

be the Young symmetrizer of $(\lambda_1,\lambda_2,\dots,\lambda_\nu)$. Here $\mathcal{H}$ is the horizontal permutation group and $\mathcal{V}$ is the vertical permutation group. The Young symmetrizer c belongs to the group algebra $\mathbb{C}[\mathfrak{S}_l]$ where $\mathfrak{S}_l$ is the l-th symmetric group.

As is well known, $\mathbb{C}[\mathfrak{S}_l]$ acts on $T^l(W_j)$ in the obvious way. When $\mathbb{C}[\mathfrak{S}_l]$ acts on $T^l(W_p^*)$ (resp. $T^l(W_{q*})$), we express an element σ of $\mathbb{C}[\mathfrak{S}_l]$ as σ^* (resp. σ_*). We put $V_p^* := c^*(T^l(W_p^*))$ and $V_{q*} := c_*(T^l(W_{q*}))$. Then (ρ_p^*, V_p^*) and (ρ_{q*}, V_{q*}) are irreducible.

From Fundamental Lemma 1, we have the following:

Theorem (Garrett's pullback formula) (Garrett [7], Böcherer [2], Böcherer-Satoh-Yamazaki [5], [12]) *Let k be even and $k > p+q+1$. If $V_p^* \otimes V_{q*}$-valued polynomial $\mathfrak{P}(S)$ is pluri-harmonic homogeneous of S, then*

$$(\mathfrak{P}(\partial)E_k^{p+q})\begin{pmatrix} Z & 0 \\ 0 & W\end{pmatrix} = \sum_{r=\nu}^{\min(p,q)} C_r \sum_{j=1}^{d(r)} D(k, f_{r,j})[f_{r,j}]_r^p(Z)^* \otimes [\theta f_{r,j}]_r^q(W)_*.$$

Here C_r is a constant satisfying

$$2^{r(r+1)-(rk+l)+1} i^{rk+l} \int_{S_r} \langle \rho_{r*}(1_r - \overline{S}S)v_*, \mathfrak{Q}(\begin{pmatrix} * & 1_r \\ 1_r & * \end{pmatrix})\rangle \det(1_r - \overline{S}S)^{-r-1}\, dS = C_r v^*$$

with $S_r := \{S \in M(r,\mathbb{C}) \mid S = {}^tS,\ \ 1_r - \overline{S}S > 0\}$, $d(r)$ is the dimension of $\mathfrak{S}_{\rho_r}$, $\{f_{r,1}, f_{r,2},\dots, f_{r,d(r)}\}$ is an orthonormal basis consisting of eigenforms, Klingen type Eisenstein series $[f]_r^p$ is defined by

$$[f]_r^p(Z) := \sum_{\gamma\in\Gamma_{n,r}\backslash\Gamma_n} f(\operatorname{pr}_r^p(Z))|_{\rho_p}\gamma$$

where $\Gamma_{p,r} := \left\{\begin{pmatrix} A_1^{(r)} & 0 & B_1^{(r)} & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1^{(r)} & 0 & D_1^{(r)} & D_2 \\ 0 & 0 & 0 & D_4 \end{pmatrix} \in \Gamma_p\right\}$ *and* $\operatorname{pr}_r^p \begin{pmatrix} Z_1^{(r)} & Z_2 \\ Z_3 & Z_4 \end{pmatrix} = Z_1$, *and* $(\theta f)(z) := \overline{f(-\overline{z})}$. *And for an eigenform f, $D(f)$ is defined by*

$$D(f) := \sum_{T\in\mathbb{T}^{(r)}} \lambda(f,T)\det(T)^{-k},$$

where $\lambda(f,T)$ is the eigenvalue on f of the Hecke operator $\left(\Gamma_r \begin{pmatrix} T & 0 \\ 0 & T^{-1}\end{pmatrix} \Gamma_r\right)$.

From Fundamental Lemma 2, we have the following:

Proposition (see Böcherer [3], Takayanagi [15, 16], [11]) *Let $n = p = q$. For an eigenform $f \in \mathfrak{S}_{\rho_n}$, then*

$$\Big(f, (\mathfrak{P}(\partial)E_k^{2n})\Big(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, s\Big)\Big)$$
$$= 2^{n(n+1-2s)-(nk+l)+1} i^{nk+l} c(s, \rho_n)\, D(k+2s, f)\, f(Z)^*$$

where $c(s, \rho_n)$ satisfies

$$\int_{S_n} \langle \rho_{n*}(1_n - \overline{S}S)v_*, \mathfrak{Q}(R, \overline{s})\rangle \det(1_n - \overline{S}S)^{s-n-1}\, dS = c(s, \rho_n) v^*$$

with

$$R := -\frac{1}{2i}\begin{pmatrix} S & -2i1_n \\ -2i1_n & 4\overline{S}(1_n - S\overline{S})^{-1} \end{pmatrix}.$$

Conjecture *There exists a non-zero constant c depending only on ρ_n such that*

$$c\big(\frac{s+n-k}{2}, \rho_n\big) = c \cdot \prod_{j=1}^{n} \frac{\Gamma(s+k+\lambda_j - j)}{\Gamma(s+n+k+1-2j)}.$$

Conjecture *If m is a critical point in the sense of Deligne [6] and all Fourier coefficients belong to $\mathbb{Q}(f)$, then*

$$\frac{L(m, f, \underline{\mathrm{St}})}{\pi^{nk+l+m(n+1)-n(n+1)/2}(f,f)} \in \mathbb{Q}(f).$$

5. Special cases

1. Symmetric tensor valued case (Böcherer-Satoh-Yamazaki [5], Takayanagi [15])
   Young symmetrizer

$$c := \sum_{\sigma \in \mathfrak{S}_l} \sigma.$$

Pluri-harmonic homogeneous polynomial

$$\mathfrak{P}(S) := c^* c_* \sum_{\mu=0}^{[l/2]} \frac{(-1)^\mu 2^{l-2\mu}}{\mu!\,(l-2\mu)!\,(k+l-\mu-1)_\mu} S^{12} \ldots S^{2\mu-1,2\mu} S_{12} \ldots S_{2\mu-1,2\mu} S_{2\mu+1}^{2\mu+1} \ldots S_l^l,$$

where the Pochhammer symbol $(a)_\mu := \Gamma(a+\mu)/\Gamma(a)$.

In this case,

$$\mathfrak{Q}(X,s) = c^* c_* \sum_{\mu=0}^{[l/2]} a(l,\mu,k,s) X^{12} \dots X^{2\mu-1,2\mu} X_{12} \dots X_{2\mu-1,2\mu} X_{2\mu+1}^{2\mu+1} \dots X_l^l$$

and

$$\mathfrak{Q}(X) = \mathfrak{Q}(X,0),$$

where

$$a(l,\mu,k,s) := \sum_{h=\mu}^{[l/2]} \binom{h}{\mu} \frac{(-1)^{h-\mu+l}(2k-2+2h)_{l-2h}(-s)_h(k+s)_{l-h}}{h!\,(l-2h)!\,(k-1+h)_{l-h}}.$$

2. Alternating tensor valued case (Takayanagi [16], [11])

Young symmetrizer

$$c := \sum_{\sigma \in \mathfrak{S}_l} \operatorname{sgn}(\sigma)\sigma.$$

Pluri-harmonic homogeneous polynomial

$$\mathfrak{P}(S) := c^* c_* S_1^1 S_2^2 \dots S_l^l.$$

In this case,

$$\mathfrak{Q}(X,s) = c^* c_* \prod_{j=1}^{l} \left(-k-s+\frac{j-1}{2}\right) X_1^1 X_2^2 \dots X_l^l$$

and

$$\mathfrak{Q}(X) = \mathfrak{Q}(X,0).$$

3. Weight $(k+2, \underbrace{k+1,\dots,k+1}_{l-2}, \underbrace{k,\dots,k}_{n-l})$

Young symmetrizer

$$c := \sum_{\substack{\sigma \in \{\mathrm{id.},(12)\} \\ \tau \in \mathfrak{S}_l, \tau(2)=2}} \operatorname{sgn}(\sigma)\sigma.$$

Pluri-harmonic homogeneous polynomial

$$\mathfrak{P}(S) := c^* c_* (S_1^1 S_2^2 - \frac{l}{2(2k-(l-2))} S^{12} S_{12}) S_3^3 \dots S_l^l.$$

In this case,

$$\mathfrak{Q}(X,s) = c^* c_* \prod_{j=1}^{l-1} \left(-k-s+\frac{j-1}{2}\right) \cdot \left\{ (-k-s-\frac{1}{2}+\frac{l}{2(2k-(l-2))}) X_1^1 X_2^2 + \frac{ls}{2(2k-(l-2))} X^{12} X_{12} \right\} X_3^3 \dots X_l^l$$

and

$$\mathfrak{Q}(X) = \mathfrak{Q}(X,0).$$

6. Computation of $\mathfrak{P}(\partial)E_k^{p+q}$

For simplicity, we put

$$\delta := \delta(g, \mathfrak{Z}), \quad \varepsilon := \det(\mathrm{Im}(\mathfrak{Z})), \quad \Delta := \Delta(g, \mathfrak{Z}) \quad \text{and} \quad \mathrm{E} := \frac{1}{2i}(\mathrm{Im}(\mathfrak{Z}))^{-1}.$$

We note that

$$\begin{aligned}
&\partial\delta = \delta\Delta, \quad \partial\varepsilon = \varepsilon\mathrm{E}, \\
&\partial^{\alpha_1\alpha_2}\Delta^{\alpha_3\alpha_4} = -\frac{1}{2}(\Delta^{\alpha_1\alpha_3}\Delta^{\alpha_2\alpha_4} + \Delta^{\alpha_1\alpha_4}\Delta^{\alpha_2\alpha_3}), \\
&\partial^{\alpha_1\alpha_2}\mathrm{E}^{\alpha_3\alpha_4} = -\frac{1}{2}(\mathrm{E}^{\alpha_1\alpha_3}\mathrm{E}^{\alpha_2\alpha_4} + \mathrm{E}^{\alpha_1\alpha_4}\mathrm{E}^{\alpha_2\alpha_3}),
\end{aligned}$$

for α_1, α_2, α_3, $\alpha_4 \in \{1^*, 2^*, \dots, l^*, 1_*, 2_*, \dots, l_*\}$. Using these relations, we obtain that

$$\begin{aligned}
\partial_1^1(\delta^{-k}\,|\delta|^{-2s}\,\varepsilon^s) &= (\delta^{-k}\,|\delta|^{-2s}\,\varepsilon^s)((-k-s)\Delta_1^1 + s\mathrm{E}_1^1), \\
\partial_1^1\partial_2^2(\delta^{-k}\,|\delta|^{-2s}\,\varepsilon^s) &= (\delta^{-k}\,|\delta|^{-2s}\,\varepsilon^s)\big\{((-k-s)\Delta_1^1 + s\mathrm{E}_1^1)((-k-s)\Delta_2^2 + s\mathrm{E}_2^2) \\
&\qquad - \frac{1}{2}((-k-s)(\Delta_2^1\Delta_1^2 + \Delta^{12}\Delta_{12}) + s(\mathrm{E}_2^1\mathrm{E}_1^2 + \mathrm{E}^{12}\mathrm{E}_{12}))\big\}
\end{aligned}$$

More generally, to describe $\mathfrak{P}(\partial)(\delta^{-k}\,|\delta|^{-2s}\,\varepsilon^s)$, we introduce "links" and "chain decomposition".

We fix an index set I. We call a non-ordered pair (α_1, α_2) with α_1, $\alpha_2 \in I$ and $\alpha_1 \neq \alpha_2$ a *link*, and define a set of links $\mathcal{L}(I)$ by

$$\begin{aligned}
\mathcal{L}(I) := \{\{&(\alpha_1, \alpha_2), (\alpha_3, \alpha_4) \dots, (\alpha_{2r-1}, \alpha_{2r})\} \\
&\mid \alpha_1, \alpha_2, \dots, \alpha_{2r} \in I, \quad \alpha_i \neq \alpha_j (i \neq j) \quad \text{for some } r\}.
\end{aligned}$$

For $L = \{(\alpha_1, \alpha_2), (\alpha_3, \alpha_4), \dots, (\alpha_{2r-1}, \alpha_{2r})\} \in \mathcal{L}(I)$, $\overline{L}$ denotes the set $\{\alpha_1, \alpha_2, \dots, \alpha_{2r}\}$. We remark $\#L = r$ and $\#\overline{L} = 2r$.

For L_1, $L_2 \in \mathcal{L}(I)$ with $\overline{L_1} = \overline{L_2}$, L_1 and L_2 are called *chainable* if we can express as

$$L_1 = \{(\alpha_1, \alpha_2), \dots, (\alpha_{2r-1}, \alpha_{2r})\}, \quad L_2 = \{(\beta_1, \beta_2), \dots, (\beta_{2r-1}, \beta_{2r})\}$$

with

$$\beta_i = \begin{cases} \alpha_{i+1}, & \text{for } i = 1, 2, \dots, 2r-1, \\ \alpha_1, & \text{for } i = 2r. \end{cases}$$

If L_1 and L_2 are chainable, so are L_2 and L_1.

For L_1, $L_2 \in \mathcal{L}(I)$ with $\overline{L_1} = \overline{L_2}$, we can express L_1 and L_2 as

$$L_1 = \bigsqcup_{j=1}^{\gamma} \ell_j, \quad L_2 = \bigsqcup_{j=1}^{\gamma} \ell'_j \quad \text{(disjoint union)}$$

such that ℓ_j and ℓ'_j are chainable for each $j = 1, 2, \dots, \gamma$. The decomposition of L_1 is called *the chain decomposition of L_1 with respect to L_2*. The number γ is called *the number of chains* and denoted by $\gamma(L_1, L_2)$.

Let $I := \{1^*, 2^*, \dots, l^*, 1_*, 2_*, \dots, l_*\}$. For $L = \{(\alpha_1, \alpha_2), (\alpha_3, \alpha_4), \dots, (\alpha_{2r-1}, \alpha_{2r})\}$ and a symmetric matrix S of size $p+q$, we put

$$S^L := S^{\alpha_1\alpha_2}S^{\alpha_3\alpha_4}\dots S^{\alpha_{2r-1}\alpha_{2r}}.$$

Then we have the following:

Lemma *For $L_0 \in \mathcal{L}(I)$, we have*

$$\begin{aligned}
&\partial^{L_0}(\delta^{-k}\,|\delta|^{-2s}\,\varepsilon^s) \\
&= (\delta^{-k}\,|\delta|^{-2s}\,\varepsilon^s) \sum_{\substack{L \in \mathcal{L}(I) \\ \overline{L}=\overline{L_0}}} \left(-\frac{1}{2}\right)^{\#L_0 - \gamma(L,L_0)} \prod_{j=1}^{\gamma(L,L_0)} ((-k-s)\Delta^{\ell_j} + s\mathrm{E}^{\ell_j}) \\
&= (\delta^{-k}\,|\delta|^{-2s}\,\varepsilon^s) \cdot \left(-\frac{1}{2}\right)^{\#L_0} \sum_{\substack{L \in \mathcal{L}(I) \\ \overline{L}=\overline{L_0}}} \prod_{j=1}^{\gamma(L,L_0)} ((2k+2s)\Delta^{\ell_j} - 2s\mathrm{E}^{\ell_j})
\end{aligned}$$

where

$$L = \bigsqcup_{j=1}^{\gamma(L,L_0)} \ell_j \quad \text{(chain decomposition with respect to } L_0\text{)}.$$

In particular,

$$\partial^{L_0}(\delta^{-k}) = \delta^{-k} \cdot \left(-\frac{1}{2}\right)^{\#L_0} \sum_{\substack{L \in \mathcal{L}(I) \\ \overline{L}=\overline{L_0}}} (2k)^{\gamma(L,L_0)} \Delta^L.$$

Remarks.
(1) In holomorphic case ($s = 0$), it is enough to calculate the number of chains $\gamma(L, L_0)$ and we need not show chain decomposition explicity.
(2) We put

$$\prod_{j=1}^{\gamma(L,L_0)} ((2k+2s)\Delta^{\ell_j} - 2s\mathrm{E}^{\ell_j}) = (2k+2s)^{\gamma(L,L_0)}(\Delta - \mathrm{E})^L + (\text{remainder terms}).$$

Suppose that $\mathfrak{P}(S)$ is pluri-harmonic. Then from Fundamental Lemma 2, $\mathfrak{P}(\partial)(\delta^{-k}\,|\delta|^{-2s}$ $\varepsilon^s)$ does not depend on remainder terms.

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Noritomo Kozima
Department of Mathematics,
Tokyo Institute of Technology,
Oh-okayama, Meguro-ku, Tokyo,
152-8551, Japan
e-mail: noritomo@math.titech.ac.jp