Nearly holomorphic modular forms and differential operators

Atsuo YAMAUCHI

2004 Oct. 25th and 26th in Hakuba

1 The case of elliptic modular forms

This lecture is a review of [3], [7], [8] and [9] in case of Siegel modular forms.

As is well known, the elliptic Eisenstein series of weight 2 is non-holomorphic. In [1], Hecke showed that

$$E_2(z;\Gamma) = \left[\sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (P\cap\Gamma)\backslash\Gamma} (cz+d)\,|cz+d|^{-2s} \right]_{s=0}$$

on $z \in \mathfrak{H} = \{z \in \mathbb{C}\,|\mathrm{Im}(z) > 0\}$ is as follows.

$$E_2(z;\Gamma) = \frac{c_0}{\mathrm{Im}(z)} + \sum_{n=0}^{\infty} a_n \cdot \exp(2\pi\sqrt{-1}nz/N),$$

where $c_0, a_n \in \mathbb{C}$ and a positive integer N. Using this, we can consider a non-holomorphic modular form f as

$$f(z) = \sum_{i=0}^{p} \frac{f_i(z)}{\mathrm{Im}(z)^i} \tag{1.1}$$

with holomorphic functions f_i on $\mathfrak{H}$.

We can define differential operators D_k and E (with $k \in \mathbb{Z}$) as

$$\begin{aligned} D_k f &= 2\sqrt{-1} \cdot \frac{\partial f}{\partial z} + \frac{kf}{\mathrm{Im}(z)}, \\ Ef &= 2\sqrt{-1} \cdot \mathrm{Im}(z)^2 \frac{\partial f}{\partial \overline{z}}. \end{aligned}$$

Then for any C^∞-function f on $\mathfrak{H}$ and any $\alpha \in \mathrm{SL}(2, \mathbb{R})$, we have

$$\begin{aligned} (D_k f)|_{k+2}\alpha &= D_k(f|_k \alpha), \\ (Ef)|_{k-2}\alpha &= E(f|_k \alpha). \end{aligned}$$

For f as in (1.1), we obtain

$$\begin{aligned} D_k f &= \sum_{i=0}^{p} \left(\frac{(k-i) f_i}{\mathrm{Im}(z)^{i+1}} + \frac{2\sqrt{-1}}{\mathrm{Im}(z)^i} \cdot \frac{\partial f_i}{\partial z} \right), \\ Ef &= \sum_{i=1}^{p} \frac{i f_i}{\mathrm{Im}(z)^{i-1}}. \end{aligned}$$

It can be proved that $f \in C^\infty(\mathfrak{H}, \mathbb{C})$ can be written as (1.1) if and only if $E^{p+1} f = 0$.

Now we can define the space of nearly holomorphic (elliptic) modular forms as

$$\mathcal{N}_k^p(\Gamma) = \left\{ f \in C^\infty(\mathfrak{H}, \mathbb{C}) \,\middle|\, \begin{array}{l} f|_k \gamma = f \text{ for any } \gamma \in \Gamma, \\ E^{p+1} f = 0, \\ f \text{ is finite at every cusp,} \end{array} \right\}$$

for any congruence subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$. Then any $f \in \mathcal{N}_k^p(\Gamma)$ is written as (1.1).

2 Nearly holomorphic Siegel modular forms

For a positive integer m, put

$$\begin{aligned} \mathrm{Sp}(m, \mathbb{R}) &= \left\{ \gamma \in \mathrm{GL}(2m, \mathbb{R}) \,\middle|\, {}^t\gamma \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix} \right\}, \\ \mathfrak{H}_m &= \left\{ Z \in \mathbb{C}_m^m \,|\, {}^t Z = Z \text{ and } \mathrm{Im}(Z) \text{ is positive definite} \right\}. \end{aligned}$$

Then $\mathfrak{H}_m$ is an $\frac{m(m+1)}{2}$-dimensional complex manifold. As is well known, $\mathrm{Sp}(m,\mathbb{R})$ acts on $\mathfrak{H}_m$ by

$$\alpha(Z) = (A_\alpha Z + B_\alpha)(C_\alpha Z + D_\alpha)^{-1},$$

where $\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{pmatrix} \in \mathrm{Sp}(m,\mathbb{R})$ with $A_\alpha, B_\alpha, C_\alpha, D_\alpha \in \mathbb{R}_m^m$, and $Z \in \mathfrak{H}_m$. Set

$$\begin{aligned} \mu(\alpha, Z) &= C_\alpha Z + D_\alpha, \\ \eta(Z) &= 2\sqrt{-1}(\overline{Z} - Z) = 2 \cdot \mathrm{Im}(Z). \end{aligned}$$

For any rational representation (ρ, X) of $\mathrm{GL}(m,\mathbb{C})$ (*i.e.* $\rho : \mathrm{GL}(m,\mathbb{C}) \to \mathrm{GL}_{\mathbb{C}}(X)$), any $\alpha \in \mathrm{Sp}(m,\mathbb{R})$, and a function $f : \mathfrak{H}_m \to X$, we define another X-valued function $f|_\rho\alpha$ on $\mathfrak{H}_m$ by

$$(f|_\rho \alpha)(Z) = \rho(\mu(\alpha, Z))^{-1} f(\alpha(Z)).$$

In case $\rho(a) = \det(a)^k$, $f|_\rho\alpha$ is written as $f|_k\alpha$.

Put

$$T = T_m = \left\{ u \in \mathbb{C}_m^m \,\middle|\, {}^t u = u \right\}.$$

For $f \in C^\infty(\mathfrak{H}_m, X)$, define $\mathrm{Hom}(T, X)$-valued functions $Df, \overline{D}f, D_\rho f$ and Ef by

$$\begin{aligned} (Df)(u) &= \sum_{i=1}^m \sum_{j=1}^m \left(\frac{1}{2}(1+\delta_{ij}) \frac{\partial f}{\partial z_{ij}} \right) \cdot u_{ij}, \\ (\overline{D}f)(u) &= \sum_{i=1}^m \sum_{j=1}^m \left(\frac{1}{2}(1+\delta_{ij}) \frac{\partial f}{\partial \overline{z_{ij}}} \right) \cdot u_{ij}, \\ (Ef)(u) &= (\overline{D}f)(\eta u {}^t\eta), \\ D_\rho f &= \rho(\eta)^{-1} D(\rho(\eta) f), \end{aligned}$$

where $u = (u_{ij})_{1 \le i,j \le m} \in T$.

Let us define representations $\rho \otimes \tau$ and $\rho \otimes \pi$ of $\mathrm{GL}(m,\mathbb{C})$ on $\mathrm{Hom}(T, X)$ by

$$\begin{aligned} \left\{ (\rho \otimes \tau)(a) h \right\}(u) &= \rho(a) h({}^t a u a) \qquad \text{for any } u \in T, \\ \left\{ (\rho \otimes \pi)(a) h \right\}(u) &= \rho(a) h(a^{-1} u {}^t a^{-1}) \qquad \text{for any } u \in T, \end{aligned}$$

for $h \in \mathrm{Hom}(T, X)$.

Then by a formal computation, we obtain

$$\begin{aligned} D_\rho(f|_\rho \alpha) &= (D_\rho f)|_{\rho\otimes\tau} \alpha, \\ E(f|_\rho \alpha) &= (Ef)|_{\rho\otimes\pi} \alpha, \end{aligned}$$

for any $\alpha \in \mathrm{Sp}(m, \mathbb{R})$ and any $f \in C^\infty(\mathfrak{H}_m, X)$.

The k-th iterates of D_ρ and E take values in

$$\mathrm{Hom}(T, \mathrm{Hom}(T, \ldots, \mathrm{Hom}(T, X) \cdots)).$$

This vector space can be identified with the space of all multilinear maps of T^k into X, which we denote by $\mathcal{M}\updownarrow_k(T, X)$. We define representations $\rho \otimes \tau^k$ and $\rho \otimes \pi^k$ of $\mathrm{GL}(m, \mathbb{C})$ on $\mathcal{M}\updownarrow_k(T, X)$ by

$$\begin{aligned} \left\{(\rho \otimes \tau^k)(a)h\right\}(u_1, \ldots, u_k) &= \rho(a)h({}^t a u_1 a, \ldots, {}^t a u_k a), \\ \left\{(\rho \otimes \pi^k)(a)h\right\}(u_1, \ldots, u_k) &= \rho(a)h(a^{-1} u_1 {}^t a^{-1}, \ldots, a^{-1} u_k {}^t a^{-1}), \end{aligned}$$

for $h \in \mathcal{M}\updownarrow_k(T, X)$ and $(u_1, \ldots, u_k) \in T^k$. Then clearly

$$\begin{aligned} (D_{\rho\otimes\tau^{k-1}} \cdots D_{\rho\otimes\tau} D_\rho)(f|_\rho \alpha) &= (D_{\rho\otimes\tau^{k-1}} \cdots D_{\rho\otimes\tau} D_\rho f)|_{\rho\otimes\tau^k} \alpha, \\ E^k(f|_\rho \alpha) &= (E^k f)|_{\rho\otimes\pi^k} \alpha, \end{aligned}$$

for any $\alpha \in \mathrm{Sp}(m, \mathbb{R})$.

We can view that τ (resp. π) is a representation of $\mathrm{GL}(m, \mathbb{C})$ on T by $\tau(a)u = {}^t a u a$ (resp. $\pi(a) = a^{-1} u {}^t a^{-1}$), and $\rho \otimes \tau^k$ (resp. $\rho \otimes \pi^k$) can be identified with $\rho \otimes (\tau^{\otimes k})$ (resp. $\rho \otimes (\pi^{\otimes k})$). Then we identify the space $\mathcal{M}\updownarrow_k(T, X)$ with $X \otimes T^{\otimes k}$ or $\mathrm{Hom}(T^{\otimes k}, X)$.

But actually, the images of $D_{\rho\otimes\tau^{k-1}} \cdots D_{\rho\otimes\tau} D_\rho$ and E^k take values in

$$\mathcal{S}_k(T, X) = \left\{ h \in \mathcal{M}\updownarrow_k(T, X) \;\middle|\; \begin{array}{l} h(u_{\varepsilon(1)}, \ldots, h_{\varepsilon(k)}) = h(u_1, \ldots, u_k) \\ \text{for any } \varepsilon \in \mathfrak{S}_k \end{array} \right\},$$

which is identified with $X \otimes \mathrm{Sym}^k T$ or $\mathrm{Hom}(\mathrm{Sym}^k T, X)$.

Put

$$r(Z) = (r_{ij}(Z))_{1 \le i,j \le m} = (Z - \overline{Z})^{-1}.$$

Then we have the following lemma.

Lemma 2.1. *Assume that $f \in C^\infty(\mathfrak{H}_m, X)$ and $E^{p+1} f = 0$ for some non-negative integer p. Then f can be written as a polynomial of $\{r_{ij}\}_{1 \le i,j \le m}$ of degree at most p, with coefficients in holomorphic functions on $\mathfrak{H}_m$.*

Note that, if f is written as a polynomial of $\{r_{ij}\}_{1 \le i,j \le m}$ of degree p with coefficients in holomorphic functions on $\mathfrak{H}_m$, $D_\rho f$ (resp. Ef) is written as that of degree $p + 1$ (resp. $p - 1$).

For a non-negative integer p and a congruence subgroup Γ of $\mathrm{Sp}(m,\mathbb{Z})$, we denote by $\mathcal{N}_\rho^p(\Gamma)$, the space of all $f \in C^\infty(\mathfrak{H}_m, X)$ satisfying the following properties (1)–(3).

(1) $f|_\rho \gamma = f$ for any $\gamma \in \Gamma$.

(2) $E^{p+1} f = 0$.

(3) f is finite at every cusp if $m = 1$.

We denote by $\mathcal{N}_\rho^p$ the union of $\mathcal{N}_\rho^p(\Gamma)$ for all congruence subgroups Γ of $\mathrm{Sp}(m,\mathbb{Z})$. If m is odd, the Eisenstein series of weight $(m+3)/2$ is contained in $\mathcal{N}_{(m+3)/2}^m$. (See, Proposition 4.1 of [8].)

Consider a subspace Y of $\mathrm{Sym}^k T$ which is stable under the action of $\mathrm{GL}(m,\mathbb{C})$ by $\mathrm{Sym}^k\tau$ (resp. $\mathrm{Sym}^k\pi$) and denote by $D_\rho^Y f$ (resp. $E^Y f$), the restriction of $(D_{\rho\otimes\tau^{k-1}} \circ \cdots \circ D_\rho) f$ (resp. $E^k f$) to Y for any $f \in C^\infty(\mathfrak{H}_m, X)$. In this case $D_\rho^Y f$ and $E^Y f$ are $\mathrm{Hom}(Y, X)$-valued.

Since $\det^2$ is a subrepresentation of $\mathrm{Sym}^m\tau$, there exists a one-dimensional subspace Y of $\mathrm{Sym}^m T$ such that

$$(\mathrm{Sym}^m\tau)(a) v_Y = \det(a)^2 v_Y,$$

for any $a \in \mathrm{GL}(m,\mathbb{C})$, where $0 \neq v_Y \in Y$. Hence we can define the differential operator $\Delta_\rho : C^\infty(\mathfrak{H}_m, X) \to C^\infty(\mathfrak{H}_m, X)$ by

$$(\Delta_\rho f) = (D_{\rho\otimes\tau^{k-1}} \circ \cdots \circ D_\rho \circ f)(v_Y).$$

Then we have

$$\det(\mu(\alpha, Z))^{-2} (\Delta_\rho f)|_\rho \alpha = \Delta_\rho (f|_\rho \alpha),$$

for any $\alpha \in \mathrm{Sp}(m,\mathbb{R})$. If $\rho(a) = \det(a)^k$, then Δ_ρ is essentially same as Maass operator M_k, which is concretely written in [2]. (Strictly, $\Delta_\rho = \mathrm{const} \times \det(\mathrm{Im}(Z))^{-1} M_k$.)

3 Holomorphic projection

Given a rational representation (ρ, X) of $\mathrm{GL}(m,\mathbb{C})$, we define a contraction map $\theta_X : \mathcal{M}\updownarrow_k(T, \mathcal{M}\updownarrow_k(T, X)) \to X$ by

$$\theta_X \varphi = \sum \varphi(c_1, \ldots, c_k; c_1, \ldots, c_k),$$

where $c_1, \ldots, c_k$ run independently over the standard basis of T. Then we have

$$\begin{aligned} \theta_X \circ (\rho \otimes \tau^k \otimes \pi^k)(a) &= \theta_X \circ (\rho \otimes \pi^k \otimes \tau^k)(a) \\ &= \rho(a) \circ \theta_X, \end{aligned}$$

for any $a \in \mathrm{GL}(m, \mathbb{C})$.

Theorem 3.1. *(Proposition 3.4 of [7], Proposition 3.3 of [8])*
Let $\rho(a) = \det(a)^\nu \rho_0(a)$ for $a \in \mathrm{GL}(m, \mathbb{C})$ with $\nu \in \mathbb{Z}$ and a rational representation (ρ_0, X) of $\mathrm{GL}(m, \mathbb{C})$. Let $f \in \mathcal{N}_\rho^p(\Gamma)$ with a congruence subgroup Γ of $\mathrm{Sp}(m, \mathbb{Z})$. If ν is larger than an integer $N(\rho_0, p)$ depending omly on ρ_0 and p, then

$$f = \sum_{l=0}^{p} (\theta_X \circ D_{\rho\otimes\pi^l\otimes\tau^{l-1}} \circ \cdots \circ D_{\rho\otimes\pi^l\otimes\tau} \circ D_{\rho\otimes\pi^l})(g_l), \tag{3.1}$$

with holomorphic modular forms g_l with respect to Γ and the representations $\rho \otimes \pi^l$.

For example, elliptic Eisenstein series of weight 2 can not be written as (3.1). But any nearly holomorphic (scalar-valued) elliptic modular form can be made from holomorphic ones and E_2 (non-holomorphic Eisenstein series of weight 2) by using differential operators D_k. It is because nearly holomorphic elliptic modular form of weight k is contained in $\mathcal{N}_k^{k/2}$ or $\mathcal{N}_k^{(k-1)/2}$ according to the parity of k. In case of $f \in \mathcal{N}_k^p(\Gamma) \quad (k > 2p)$, it can be written as

$$f = \sum_{l=0}^{p} D_{k-2} \circ \cdots \circ D_{k-2l} \circ g_l$$

with holomorphic modular forms g_l of weights $k - 2l$ with respect to Γ. On the other hand, $f \in \mathcal{N}_{2p}^p(\Gamma)$ can be written as

$$f = c \cdot D_{2p-2} \circ \cdots \circ D_2 \circ E_2 + \sum_{l=0}^{p-1} D_{2p-2} \circ \cdots \circ D_{2p-2l} \circ g_l$$

with a constant c, (non-holomorphic) Eisenstein series $E_2 \in \mathcal{N}_2^1(\Gamma)$, and holomorphic modular forms g_l of weights $2p - 2l$ with respect to Γ.

Define an operator $L_\rho : C^\infty(\mathfrak{H}_m, X) \to C^\infty(\mathfrak{H}_m, X)$ by

$$L_\rho = -\theta_X \circ D_{\rho\otimes\pi} \circ E.$$

Then clearly we have

$$(L_\rho f)|_\rho \alpha = L_\rho(f|_\rho \alpha),$$

for any $\alpha \in \mathrm{Sp}(m, \mathbb{R})$. Hence we have $L_\rho(\mathcal{N}_\rho^p(\Gamma)) \subset \mathcal{N}_\rho^p(\Gamma)$. Moreover, the image of L_ρ is orthogonal to any cusp forms with respect to ρ.

In case $\rho(a) = \det(a)^k$, we write L_ρ by L_k. Using this L_k, we can make an orthogonal projection of nearly holomorphic modular forms to holomorphic ones as follows.

Theorem 3.2. *(Theorem 3.3 of [9], Proposition 15.3 of [10])*
Let p, k be non-negative integers such that

$$k > m + p \quad \text{or} \quad k < m + \frac{3-p}{2}.$$

For any irreducible subspace Y of $\mathrm{Sym}^i T$ with respect to $\mathrm{Sym}^i \tau$, put

$$\alpha_Y = i(k - (m+1) + (1-i)c_Y),$$

where $-1/2 \leq c_Y \leq 1$ denotes a rational number depending only on Y (not on k). Put $A_i = \prod_Y (1 - \alpha_Y^{-1} L_k)$ for $0 < i \leq p$, where Y runs over all the irreducible subspaces of $\mathrm{Sym}^i T$ (with respect to $\mathrm{Sym}^i \tau$), and $\mathfrak{A} = \prod_{i=1}^p A_i$. Take any $f \in \mathcal{N}_k^p$. Then $\mathfrak{A} f$ is a holomorphic modular form of weight k and $f = \mathfrak{A} f + L_k t$ with $t \in \mathcal{N}_k^p$.

Note that $< f, g > = < \mathfrak{A} f, g >$ for any holomorphic cusp form g of weight k.

This theorem essentially uses the fact that the space $\mathrm{Sym}^i T$ is a direct sum of irreducible (rational) representations of $\mathrm{GL}(m, \mathbb{C})$, and each irreducible constituent has **multiplicity one**. (See, §12 of [10] and [6].) The author doesn't know how to generalize this theorem to the case of vector-valued modular forms. In fact, let ρ be a rational representation of $\mathrm{GL}(m, \mathbb{C})$ on a $\mathbb{C}$-vector space X. Then the irreducible decomposition of the space $X \otimes \mathrm{Sym}^i T$ does not satisfy the property of multiplicity one in general.

Professor Böecherer conjectures that both holomorphic projections of Theorems 3.1 and 3.2 are equal. It means that $\mathfrak{A} f$ in Theorem 3.2 coincides with g_0 in Theorem 3.1 for any scalar valued modular form f (of sufficiently high weight.)

References

[1] E. Hecke, Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik, Abh. Math. Sem. Hamburg 5(1927), 199-224.

[2] H. Maass, Siegel's Modular Forms and Dirichlet Series, Lecture Notes in Math. **216**(1971), Springer-Verlag.

[3] G. Shimura, Arithmetic of differential operators on symmetric domains, Duke Math. J. **48**(1981), 813-843.

[4] ____, Confluent hypergeometric functions on tube domains, Math. Annalen **260**(1982), 269-302.

[5] ____, On Eisenstein series, Duke Math. J. **50**(1983), 417-476.

[6] ____, On differential operators attached to certain representations of classical groups, Invent. Math. **77**(1984), 463-488.

[7] ____, On a class of nearly holomorphic automorphic forms, Ann. of Math. **123**(1986), 347-406.

[8] ____, Nearly holomorphic functions on hermitian symmetric spaces, Math. Annalen **278**(1987), 1-28.

[9] ____, Differential operators, holomorphic projection, and singular forms. Duke Math. J. **76**(1994), 141-173.

[10] ____, Arithmeticity in the Theory of Automorphic Forms, Mathematical Surveys and Monographs vol.**82**(2000), AMS.