Holomorphic components of nonholomorphic differential operators (after restriction)

The purpose of the talk is to describe some relations between Maaß-Shimura type differential operators and holomorphic differential operators. This kind of connection may be of interest, because the Maaß-Shimura operators can (ultimately) be explained by Lie-theory. our main tool is the theory of nearly holomorphic functions due to Shimura.

§1 Examples (Here we describe two simple examples)

Example 1: The Maaß-Shimura differential operator $\delta_k^{(n)}$ on $\mathbb{H}_n$ (changing weights from k to $k+2$ can be defined by

$$\delta_k^{(n)} = det(Y)^{-k+\frac{n-1}{2}} det(\partial_{ij}) det(Y)^{k-\frac{n-1}{2}}.$$

For n=2 it has the explicit form

$$\delta_k^{(2)} = \frac{k(k-\frac{1}{2})}{det(2iY)} + (k-\frac{1}{2}) \cdot \frac{2i(y_1\partial_{11} + 2y_{12}\partial_{12} + y_4\partial_{22})}{det(2iY)} + det(\partial_{ij})$$

The degree 1 operators are of the form $\delta_k^{(1)} = \frac{k}{2iy} + \frac{\partial}{\partial z}$; we therefore get a decomposition

$$\left(\delta_k^{(2)}\right)_{|z_{12}=0} = \frac{k-\frac{1}{2}}{k} \cdot \underbrace{\left(\delta_k^{(1)} \otimes \delta_k^{(1)}\right)}_{\text{Maaß diff.operator}} + \underbrace{\left(\frac{1}{2k}\partial_{11}\partial_{22} - \partial_{12}^2\right)_{z_{12}=0}}_{\text{holomorphic diff. operator}} \qquad (k \neq 0)$$

We can view the holomorphic part $\mathcal{D}$ as a kind of holomorphic projection, because for (holomorphic) cusp forms F on $\mathbb{H}_2$ of weight k and f,g on $\mathbb{H}$ of weight k+2 we have

$$(\int\int \delta_k^{(2)} F)(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}) \overline{f(z_1)g(z_2)} y_1^k y_2^k dz_1 dz_2 =$$

$$(\int\int \mathcal{D}(F)(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}) \overline{f(z_1)g(z_2)} y_1^k y_2^k dz_1 dz_2$$

It is a nice extra feature that the equality of the integrals above holds true already after *one* integration!

The cases $k = \frac{1}{2}$ and $k = 0$ deserve special attention:
For $k = \frac{1}{2}$ only the holomorphic part appears.
For $k = 0$ the decomposition is of different nature

$$\left(\delta_0^{(2)}\right)_{|z_{12}=0} = \underbrace{-\frac{1}{2}\left(\frac{\partial_{11}}{2iy_{22}} + \frac{\partial_{22}}{2iy_{11}}\right) - \partial_{12}^2}_{\text{strange new (?) operator}} + \underbrace{\partial_{11} \cdot \partial_{22}}_{\delta_0^{(1)} \otimes \delta_0^{(1)}}$$

Example 2: This is a (minor) digression from the main topic
Let f and g be elliptic modular forms of weight k and l respectively. Then by an easy calculation

$$f \cdot \delta_l^{(1)}(g) =$$

$$\frac{l}{k+l}\delta_{k+l}^{(1)}(f \cdot g) + \frac{1}{k+l}\underbrace{\left(k \cdot f\left(\frac{\partial}{\partial z}g\right) - l\left(\frac{\partial}{\partial z}f\right)g\right)}_{\text{a Rankin Cohen bracket}}$$

Again the cases $l = 0$ and $k + l$ need special attention:
For $l = 0$ both sides of the equation above are holomorphic anyway.
For $k + l = 0$ no reasonable decomposition seems available; if we symmetrize the situation, then it becomes better:

$$k \cdot f \cdot (\delta_l^{(1)}g) - l \cdot g \cdot \delta_k^{(1)}(f) = \underbrace{kf(\frac{\partial}{\partial z}g) - lg(\frac{\partial}{\partial z}f)}_{\text{Rankin Cohen bracket}}$$

As before we can view the Rankin-Cohen-bracket as a kind of holomorphic projection, because

$$\int f \cdot (z)\delta_l^{(1)}(g)(z) \cdot \overline{h(z)}y^{k+l}dxdy = \int [f,g] \cdot \overline{h(z)}y^{k+l}dxdy\,;$$

this is true for all holomorphic cusp forms h of weight $k + l + 2$ - at least if f, g are holomorphic cusp forms of weights k and l.

Concerning this integral - and the one from example 1 - one does not need to work with cusp forms; it would be sufficient to work in a space

$$L_{hol}(\Gamma\backslash\mathbb{H}; y^r dxdy)$$

of holomorphic functions which are Γ-invariant for a subgroup Γ of $SL(2,\mathbb{Z})$ with respect to the $|_{r+2}$-action and square-integrable w.r.t. $y^r dxdy$. The group Γ does not need to be of finite index !

Conclusion: Both these examples show that it is possible to get various types of holomorphic differential operators from nonholomorphic differential operators of Maaß-Shimura type. If the weights are not too special, we can hope for reasonable results.

§2 Maaß-Shimura operators and restrictions

Concerning restrictions of functions on $\mathbb{H}_n$ to products of (diagonally embedded) lower-dimensional upper half spaces, we will tacitly use the same notations as in [3].

We recall that a nearly holomorphic function on $\mathbb{H}_n$ is a polynomial in the entries of Y^{-1} with holomorphic functions as coefficients. In particular, $\mathcal{N}_n^\nu$ denotes the complex vector space of nearly holomorphic functions of degree $\leq \nu$. The following (completely elementary!) remark is crucial:

Remark: *The restriction of a nearly holomorphic function F on $\mathbb{H}_n$ to $\mathbb{H}_p \times \mathbb{H}_q$ with $p+q=n$ is again nearly holomorphic (in both sets of variables)*

Let us start from a differential operator D on $\mathbb{H}_n$ with the following properties

1. It is a polynomial in the (holomorphic) derivatives and the entries of Y^{-1}
2. It changes the automorphy factor (k,ρ) to $(k‘,\rho‘)$

Let F be a holomorphic function F on $\mathbb{H}_n$ and apply such a differential operator D to it and restrict it to $\mathbb{H}_p \times \mathbb{H}_q$; we call this operator D^o. Upon restriction to $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ the representation $det^{k'} \otimes \rho'$ decomposes as a direct sum of tensor products. Let $(det^{k_1'} \otimes \rho_1) \otimes (det^{k_2'} \otimes \rho_2)$ be such a summand occuring in the restriction of ρ. Let $D^0_{\rho_1,\rho_2}$ be the restriction to this representation. From the remark above, $D^0_{\rho_1,\rho_2}(F) \in \mathcal{N}_p^{\nu_p} \otimes \mathcal{N}_q^{\nu_q}$ for suitable ν_p, ν_q. When we apply the structure theorem of Shimura to $D^0_{\rho_1,\rho_2}(F)$, this means that there are appropriate Maaß-Shimura operators D_p^i and D_q^j acting on functions on $\mathbb{H}_p$ and $\mathbb{H}_q$ and moreover holomorphic functions f_{ij} on $\mathbb{H}_p \times \mathbb{H}_q$ such that

$$D^o_{\rho_1,\rho_2}(F) = \sum_{i,j}(D^i_p \otimes D^j_q)(f_{ij}).$$

An inspection of the proof of Shimura shows that (in this situation) the functions f_{ij} are polynomials in the (holomorphic!) derivatives of F, evaluated at $w = 0$. These polynomials (as well as the operators D^i_p and D^j_q) do not depend on the individual F at all. We therefore get

Theorem: (first version)
For a given differential operator D as above on $\mathbb{H}_n$ there are Maaß-Shimura operators D^i_p and D^j_q and holomorphic differential operators $\mathcal{D}_{ij}$, which are polynomials in the holomorphic partial derivatives (evaluated at $w = 0$) such that

$$D^o_{\rho_1,\rho_2} = \sum_{ij} \left(D^i_p \otimes D^j_q\right) \circ \mathcal{D}_{ij}$$

These equations are true at least if some inequality between the degree and the weights is satisfied ($k'_i \geq C_i$ where the constant C_i depends on ν_i and on ρ_i‘). If we denote the trivial operator (=identity operator) by $D^0_p \otimes D^0_q$, then we can consider the operator $\mathcal{D}_{00}$ as a holomorphic component of the Maaß-Shimura operator D^0. In the sequel, we write $\mathcal{D}'_{00}$ for this to distinguish it (at least formally) from the one obtained below.

There is a second version of this statement, which is more focussed on the "holomorphic component". It is more explicit with respect to the range of applicability :

For D and F as above, there are nearly holomorphic functions G_0, G_1, G_2, G_3 on $\mathbb{H}_p \times \mathbb{H}_q$ such that

$$D^0_{\rho_1,\rho_2}(F) = G_0 + \mathcal{L}_p(G_1) + \mathcal{L}_q(G_2) + (\mathcal{L}_p \otimes \mathcal{L}_q)(G_3)$$

where the $\mathcal{L}_i$ are kind of Laplacians on $\mathbb{H}_i$ (whose image is orthogonal to holomorphic functions under suitable conditions, e.g. orthogonal to cusp forms). In the scalar-valued case ($\rho_1 = 1$, $\rho_2 = 1$) this is true at least under the conditions

$$k'_1 \notin [p + \frac{3 - t_1}{2}, \, p + t_1]$$

$$k_2 \notin [q + \frac{3-t_2}{2},\, q+t_2]$$

where t_1, t_2 denotes the degree of $D^0(F)_{\rho_1,\rho_2}$ as nearly holomorphic on $\mathbb{H}_p$ and $\mathbb{H}_q$ respectively (for the vector-valued case there is a less explicit condition) Again we can reformulate this as

Theorem (second version)
Under the same conditions as above,

$$D_{\rho_1,\rho_2} = \mathcal{D}''_{00} + \mathcal{L}_p(\mathcal{P}_1) + \mathcal{L}_q(\mathcal{P}_2) + \mathcal{L}_p \otimes \mathcal{L}_q(\mathcal{P}_3)$$

with some differential operators $\mathcal{P}_i$ sending functions on $\mathbb{H}_n$ to functions on $\mathbb{H}_p \times \mathbb{H}_q$; again $\mathcal{D}''_{00}$ is a polynomial in the holomorphic partial derivatives, evaluated at $w = 0$.

Remarks

- These general statements explain the example 1 from the beginning.
- There should also be a Lie-theoretic explanation for the Theorem above, for this see the contribution of Schulze-Pillot [10].
- Of course, in some sense the $\mathcal{D}_{ij}$ for $(i,j) \neq (0,0)$ are less interesting (and more difficult to describe) than $\mathcal{D}_{0,0}$. In [2] there is a completely explicite description of the equality above for the case D^0 changes the automorphy factor from $(det_n^k \otimes sym^l)$ to $(det_p^k \otimes sym^{l+2\nu}) \otimes (det_q^k \otimes sym^{l+2\nu})$.

- There are cases with $\mathcal{D}_{0.0} = 0$:
  Take e.g. n=3 and $p = 2$, $q = 1$, D=Maass operator on $\mathbb{H}_3$. An explicit calculation for the decomposition of the theorem shows that $\mathcal{D}'_{00} = 0$. This can also be predicted by abstract reasoning of representation theory.
- Clearly, we can get the same kind of statement for any kind of embedding
  $$\mathbb{H}_{n_1} \times \ldots \mathbb{H}_{n_r} \hookrightarrow \mathbb{H}_n \qquad (n = n_1 + \ldots n_r)$$
- Ususally, the differential operators D come up as families, parametrized by the (initial) weight k. Then more explicit statements are possible, e.g. concerning the dependence of coefficicents on k.

- If we take the explict form of D for granted, we can get an explicit form of $\mathcal{D}''_{00}$ by "holomorphic projection" of D^0; it is enough to do this for a suitable class of "test functions", see the example below.

Example: We take again our "standard case" $n_1 = n_2 = n$ and a Maaß -type operator D on $\mathbb{H}_{2n}$ changing the weight from k to $k + 2\nu$. We work in the space

$$L^2_{hol}(A_n \backslash \mathbb{H}_n\,;\ det(Y)^{k+2\nu} d^* Z),$$

where A_n denotes the group of (integral) translations (i.e. $A_n = \begin{pmatrix} 1_n & * \\ 0_n & 1_n \end{pmatrix}$).
As a test function we choose

$$F_{\mathcal{T}}(Z) := e^{2\pi i tr(\mathcal{T} Z)}$$

with $\mathcal{T}$ symmetric, semiintegral, positiv definit. We define two polynomials arising fom D^0 and $\mathcal{D}_{00}$ by

$$\begin{aligned} D^0(F_{\mathcal{T}}) &= Q(\mathcal{T}, y_1^{-1}, y_4^{-1}) e^{2\pi i tr(T_1 z_1 + T_4 z_4)} \\ \mathcal{D}_{00}(F_{\mathcal{T}}) &= P(\mathcal{T}) e^{2\pi i tr(T_1 z_1 + T_4 z_4)} \end{aligned}$$

Clearly $\mathcal{P}$ determines $\mathcal{D}_{00}$.
Both $D^0(F_{\mathcal{T}})$ and $\mathcal{D}_{00}(F_{\mathcal{T}})$ define elements of the L^2-space from above (w.r.t. z_1 and z_4); their scalar products against $(z_1, z_4) \longmapsto e^{2\pi i tr(T_1 z_1 + T_4 z_4)}$ coincide. This gives rise (after the two trivial integrations over $\Re(z_i)$ mod 1) to an identity

$$\int\!\!\int Q(\mathcal{T}, y_1^{-1}, y_4^{-1})\,(det(y_1) det(y_4))^{k+2\nu-\frac{n+1}{2}}\ e^{-4\pi tr(T_1 y_1 + T_4 y_4)} d^* y_1 d^* y_4 =$$

$$P((\mathcal{T}) \int\!\!\int (det(y_1) det(y_4))^{k+2\nu-\frac{n+1}{2}}\ e^{-4\pi tr(T_1 y_1 + T_4 y_4)} d^* y_1 d^* y_4$$

The integrations go over the space of positive definite matrices.
This can be considered to be an analytic method to get an explicit formula for $\mathcal{P}$ and then for $\mathcal{D}_{00}$. On both sides of the equality above, the integrals can be expressed in terms of gamma functions and polynomials in the entries of $\mathcal{T}$, divided by powers of $det(t_1)$ and $det(t_4)$.
This method (*analytic construction*) should only be applied, if algebraic or combinatorial formulas are not (yet) available; it can however give us some idea about the (combinatorial) nature of the polynomials in question.

Some questions about the two versions:
The following questions should be answered within the framework of the theory of nearly holomorphic functions (*not* depending on the very special way, in which our nearly holomorphic functions arise from applying D to a holomorphic function F)

Q1 Are the f_{ij} and the G_l uniquely determined by F ?

Q2 Are the $(D_p^i \otimes D_q^j)(f_{ij})$ orthogonal to cusp forms (for $i, j) \neq (0, 0)$) ?

Q3 Is $\mathcal{D}'_{00}(F) = \mathcal{D}''_{00}(F)$?

The function G_0 is indeed always uniquely determined by F and therefore also the differential operators $\mathcal{D}''_{00}$, see [4] In the special situation of our theorem, there is another approach to the question **Q3**: For simplicity, we consider only the scalar-valued case ($\rho = 1$). We take the test function $F_{\mathcal{T}}$ as before; by using the differential operators $\mathcal{D}'_{00}$ and $\mathcal{D}''_{00}$ we obtain two polynomials $\mathcal{P}'(\mathcal{T})$ and $\mathcal{P}''(\mathcal{T})$; these polynomials have to be proportional by Ibukiyama [8], at least if $2k \geq n$, therefore the differential operators $\mathcal{D}'_{00}$ and $\mathcal{D}''_{00}$ have to be proportional. To see that the constant is actually equal to 1 (again for $2k \geq n$), one has to look at the terms of highest degree in the derivatives ∂w_{ij}, evaluated in $w = 0$.

On the doubling method
Here we use the differential operators $\mathcal{D}^0_{\alpha,\nu}$ from [8, 1] for the embedding $\mathbb{H}_n \times \mathbb{H}_n \hookrightarrow \mathbb{H}_{2n}$; they change the weight from $\alpha = k + s$ to $\alpha + \nu$ (for arbitrary $\alpha \in \mathbb{C}$.
For a (holomorphic) function F on $\mathbb{H}_{2n}$ we can look at $F \longmapsto \mathcal{D}^0_{k,\nu}(F)$ but also at the "disturbed version"

$$F \longmapsto \mathcal{K}^0_{\alpha,\nu}(F) := det(y_1)^s det(y_4)^s \mathcal{D}^0_{\alpha,\nu}(F \cdot \det(Y)^{-s})$$

which changes the weight in the same way (from k to $k+\nu$). By the uniqueness of the Ibukiyama type differential operators (for large k) it is clear that the holomorphic component G_0 of $\mathcal{K}^0_{\alpha,\nu}(F)$ must be a multiple of $\mathcal{D}^0_{k,\nu}(F)$. The constant was determinend explicitly in [5]. This allows to compare the two integrals

$$\int \mathcal{D}^0_{k,\nu}(F)(z_1, z_4)\overline{f(z_1)g(z_4)} det(y_1) det(y_4) dz_1 dz_4$$

and

$$\int\int \mathcal{K}^0_{\alpha,\nu}(F)(z_1,z_4)\overline{f(z_1)g(z_4)}det(y_1)det(y_4)dz_1dz_4$$

for two (holomorphic cusp forms f and g. In particular in the situation of the doubling method, this allows us to substitute the integral involving $\mathcal{D}^0_{k,\nu}(E^{2n}_k(Z,s)$ by an integral, which is easier to unfold, namely an integral involving $\mathcal{D}^0_{k+s,\nu}(G^{2n}(Z)_{k+s,s})$, see also [3]. Here

$$\begin{aligned} G^n_{k+s,s}(Z) &= \sum_{C,D} det(CZ+D)^{-k-s}\overline{det(CZ+D)^{-s}} \\ E^n_k(Z,s) &= det(Y)^s\cdot G^n_{k+s,s}(Z) \end{aligned}$$

§3 A criterion for nonvanishing of the holomorphic component (often quite useful!)

Tacitly we assume that we can apply a version of the Theorem above (weights are high compared to degree as polynomials in the entries of Y^{-1}). We formulate this more generally for any embedding

$$\mathbb{H}_{n_1}\times\dots\mathbb{H}_{n_r}\hookrightarrow\mathbb{H}_n \qquad (n=n_1+\dots n_r)$$

We denote by w_{ij} those variables of $Z\in\mathbb{H}_n$ which lie "outside " the block diagonal given by the embedding above.

Criterion: *Suppose that the differential operator D^0 contains a nonzero summand of the form*

$$\left(monom\ in\ the\ \frac{\partial}{\partial w_{ij}}\right)_{|w=0}$$

(i.e. a summand which does not involve entries of Y^{-1} nor derivatives from the block diagonals). Then, with the obvious generalization of the notations above

$$D^0_{\rho'_1,\dots,\rho'_r}=\mathcal{D}_{0,\dots,0}+\dots$$

with $\mathcal{D}_{0,\dots 0}\neq 0$.

Example: We consider the Maaß-operator D for $\mathbb{H}_{2n}$, changing the weight from det^k to det^{k+2l} and consider the case $r=2, n_1=n_2=n$. Then $D^0(F)$

is nearly holomorphic of degree l and it clearly contains a summand of the form $det\left(\frac{\partial}{\partial w_{ij}}\right)^{2l}$. Therefore we see that (at least as long as $k+2l > n+l$) there are holomorphic differential operators raising weights from k to $k+2l$ for the embedding $\mathbb{H}_n \times \mathbb{H}_n \hookrightarrow \mathbb{H}_{2n}$ which come from Maaß-type operators on $\mathbb{H}_{2n}$. Ibukiyama [8] , using invariant theory in his construction, relied somewhat on the condition $2k \geq 2n$.
Remark: The example from above can be generalized to the case of an arbitrary embedding $n = n_1 + \dots n_r$ and weight change from k to $(k+2l, \dots, k+2l)$; again we can produce a nontrivial holomorphic differential operator for this case provided that the initial k is large enough and the "blocks" are not too big: The reasoning as before works, if $det_n(Z)$ has summands which involve only the entries of w.

§4 Rankin-Cohen brackets from the point of view of Maaß differential operators.

Here we follow the strategy described in [1] (where the case of the Jacobi group was considered).
From the point of view of representation theory, this has been studied by Harris [6],[7].
We use iterates of Maaß-Shimura operators, which we denote by

$$\delta_{k,r}^{(n)} := \delta_{k+2r-2}^{(n)} \circ \cdots \circ \delta_k^{(n)}$$

We start from two holomorphic functions F and G on $\mathbb{H}_n$ and consider the product

$$H(F,G) := \delta_{k,\nu_1}^{(n)}(F) \cdot \delta_{l,\nu_2}^{(n)}(G).$$

This changes the weight from k and l to $k+l+2\nu$ with $\nu = \nu_1 + \nu_2$. Then H is nearly holomorphic of degree $n(\nu)$. If $k+l+\nu$ is large , then there are (possibly vector valued) holomorphic functions H_i and Shimura-type differential operators D_i such that

$$H(F,G) = H_o(Z) + \sum_i D^i(H_i)$$

Again by inspection of Shimura's proof, the $H_i = H_i(F, G)$ can be written as

$$H_i = \mathcal{R}_i(F, G) \qquad (0 \leq i)$$

where the $\mathcal{R}_i$ are bilinear (holomorphic) differential operators not depending on F and G. The operator $\mathcal{R}_0$ can be viewed as holomorphic component of the operator

$$H : (F, G) \longmapsto \delta(F) \cdot \delta(G);$$

this operator can be written as

$$H = \mathcal{R}_0 + \sum_i D^i \circ \mathcal{R}_i.$$

Whenever this decomposition holds (i.e. essentially for $k+l$ large), we would like to see that $\mathcal{R}_0 \neq 0$. To see such nonvanishing, the best thing is to use the analytic construction mentioned earlier. This method was used extensively in the case of Jacobi forms [1].

§5 Final remarks

In this note we completely ignored the viewpoints of Lie-theory and representation theory. Our basic tools were Maaß-Shimura type operators, which come from Lie-theory, therefore Lie theory has something to say about this topic (see [10])! One can study the theorem in §2 from the viewpoint of branching rules for holomorphic discrete series; our section 4 on Rankin-Cohen brackets is related to properties of tensor products for holomorphic discrete series.

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