Holomorphic differential operators with iteration

Basically this is a report on a class of differential operators introduced in [1]; we add some more recent (unpublished) aspects of these operators

There are only a few cases (i.e. a few weights) for which there exist (automorphy preserving) holomorphic differential operators on $\mathbb{H}_n$, e.g. for n=1 $\frac{d}{dz}$ changes the weight from 0 to two .
To improve our chance to construct such holomorphic differential operators, we can make the group smaller for which we ask for the property of preserving automorphy, e.g. any holomorphic differential operator preserves periodicity, i.e. automorphy for the subgroup $\begin{pmatrix} 1_n & * \\ 0_n & 1_n \end{pmatrix}$.
Here we will concentrate on the case of the natural (diagonal) embedding

$$\iota_n : Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R}) \hookrightarrow Sp(2n,\mathbb{R})$$

which was already dealt with in [1]. For $g \in Sp(n,\mathbb{R})$ we will often write $g^{\uparrow}$ instead of $\iota_n(g, 1_{2n})$. We will decompose $Z \in \mathbb{H}_{2n}$ into block matrices of size n, for which we use simultaneously the notations $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ and

$$Z = \begin{pmatrix} \tau & w \\ w^t & z \end{pmatrix} \qquad (\tau, z \in \mathbb{H}_n,\ w \in \mathbb{C}^{(n,n)})$$

The restriction $w = 0_n$ gives us an embedding (also called ι_n)

$$\iota_n : \mathbb{H}_n \times \mathbb{H}_n \hookrightarrow \mathbb{H}_{2n}$$

Our basic differential operators on $\mathbb{H}_{2n}$ are

$$\partial_{ij} = \begin{cases} \frac{\partial}{\partial_{ii}} & \\ \frac{1}{2}\frac{\partial}{\partial_{ij}} & \text{for } i \neq j \end{cases} ;$$

the matrix of these operators (and its decomposition into block matrices) will be denoted by

$$\partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix}.$$

We need the calculus of minors of the matrix ∂; for details on this we refer to [7, chap.III, §6]. For a matrix of size n we denote by $A^{[p]}$ the $\binom{n}{p} \times \binom{n}{p}$- matrix of its p-minors $(0 \leq p \leq n)$ and by Ad^pA its adjoint. Then we define (for $p+q \leq n$) the $\binom{n}{p+q} \times \binom{n}{p+q}$ matrix $A^{[p]} \sqcap A^{[q]}$ as in [7]. In the special case $p+q=n$ this just means

$$A^{[p]} \sqcap B^{[q]} = tr\left(A^{[p]} \cdot Ad^{[p]}B\right)$$

We are looking for differential operators $\mathcal{D} = \mathcal{D}_{k,\nu}$ on $\mathbb{H}_{2n}$ satisfying

$$\begin{aligned} \mathcal{D}_{k,\nu}(F \mid_k g^{\uparrow}) &= \mathcal{D}_{k,\nu}(F) \mid_{k+\nu} g^{\uparrow} \\ \mathcal{D}_{k,\nu}(F \mid_k g^{\downarrow}) &= \mathcal{D}_{k,\nu}(F) \mid_{k+\nu} g^{\downarrow} \end{aligned}$$

for all $g \in Sp(n,\mathbb{R})$. Here ν can be any integer (not necessarily positive). Note that $\mathcal{D}_{k,\nu}(F)$ is again a function on $\mathbb{H}_{2n}$, therefore such operators can be iterated !
When we restrict $\mathcal{D}(F)$ to the subdomain defined by $w = 0_n$, then we get another type of differential operators $\mathcal{D}^o$ satisfying - for $g \in Sp(n, \mathbb{R}$ -

$$\begin{aligned} \mathcal{D}^o(F \mid_k g^{\uparrow}) &= \mathcal{D}^o(F) \mid_{k+\nu}^{(1)} g \\ \mathcal{D}^o(F \mid_k g^{\downarrow}) &= \mathcal{D}^o(F) \mid_{k+\nu}^{(4)} g \end{aligned}$$

Here the upper index (1) (or (4)) indicates that the slash operator has to be applied to the variable z_1 (or z_4) ; note that $\mathcal{D}^o(F)$ is a function on $\mathbb{H}_n \times \mathbb{H}_n$. This class of differential operators and their connection with pluriharmonic polynomials was carefully investigated by Ibukiyama [9].
Our operators have stronger properties than those of Ibukiyama, in particular, they can be iterated. But we have to pay a price for this: We have to allow our operators to have nonconstant coefficients, more precisely the coefficients of $\mathcal{D}$ will be allowed to be polynomials in the entries w_{ij} of w.

Before going on, we have to emphasize a property of the variable w, which will be crucial for all what follows:
Remark 1: *For $l \in \mathbb{Z}$ we define the function*

$$\varphi_l : \begin{cases} \mathbb{H}_{2n} & \longrightarrow & \mathbb{C} \\ Z & \longmapsto & det(w)^l \end{cases}$$

Then φ_l is a "symmetric function of weight $-l$ " for $Sp(n,\mathbb{R})^\uparrow$ and $Sp(n,\mathbb{R})^\downarrow$, i.e. for all $g \in Sp(n,\mathbb{R})$ we have

$$\begin{aligned} \varphi_l \mid_{-l} g^\uparrow &= \varphi_l \\ \varphi_l \mid_{-l} g^\downarrow &= \varphi_l \\ \varphi_l(Z[V]) &= \varphi_l(Z) \end{aligned}$$

with $Z[V] = \begin{pmatrix} \tau & w' \\ w & z \end{pmatrix}$.

It is an easy excercise to see that (up to a constant) $det(w)^l$ is the only holomorphic function on $\mathbb{H}_{2n}$ with such a property (the function does not depend on the real parts of z and τ...)

An inspection of the action of $GL(n,\mathbb{R}) \hookrightarrow Sp(n,\mathbb{R})$ shows that $\mathcal{D}$ must be of the form $\mathcal{Q}(\partial_1, \partial_2, \partial_4, w)$ with a polynomial[1]

$$\mathcal{Q} = \mathcal{Q}(X, R, Y, w),$$

where X and Y are symmetric matrices and $R, w \in M_n(\mathbb{C})$. This polynomial has to satisfies

$$\begin{aligned} \mathcal{Q}(A^tXA, A^tR, Y, A^{-1}w) &= det(A)^\nu \mathcal{Q}(X, R, Y, w) \\ \mathcal{Q}(X, RA, A^tYA, wA^{-t}) &= det(A)^\nu \mathcal{Q}(X, R, Y, w) \end{aligned}$$

for all $A \in GL(n,\mathbb{C})$. As for the degrees of such polynomials, it is clear that

$$r_1 - r_2 = n\nu$$

where

- r_1 = degree of $\mathcal{P}$ as a polynomial in the entries of X,R,Y (for $\mathcal{D}$ this means the degree in the derivatives ∂_{ij}
- r_2 = degree of $\mathcal{D}$ as a polynomial in the w_{ij}

For obvious reasons, we will call ν the weight of the polynomial $\mathcal{Q}$.

It should be remarked that

$$det(w)^\mu \cdot \mathcal{D}$$

is again such an operator, changing the weight by $\nu - \mu$, in particular, multiplication by $det(w)$ changes the weight from k to k-1.

There are several delicate problems here

[1]Some care is necessary about this notation, because w and ∂_2 do not commute ! We tacitly assume that the variables w_{ij} are on the left side of the partial derviatives involved in ∂_2

- Construct such operators $\mathcal{D}$ explicitly (in a natural way ?)
- Give a characterization of the space of all such polynomials $\mathcal{Q}$ (dimension ?); as for this problem, see also topic 5 of the appendix.
- Describe their iteration
- In particular, construct such operators raising the weight with the additional property that $\mathcal{D}^o \neq 0$

Sometimes in the iteration a lot of combinatorial problems may arise, but I also want to point out that in important applications (integral representations of L - functions using the doubling method, see e.g.[4] or topic 1 of the appendix) the explicit combinatorics of iterated operators does not really matter !

We shall present three different methods to construct such operators (mainly for $\nu = 1$):

<u>**Method I:**</u> (our original method from 1985 [1])
Here we consider $\nu = 1$. We start from the assumption, that $\partial_2^{[n]} = det(\frac{\partial}{\partial w_{ij}})$ is the "main part " of the differential operator (constant term w.r.t. w) and look for "correction terms". For $n = 1$ the operator is then quite simple and can in fact be found implicitly in the book by Eichler-Zagier, namely

$$\mathcal{D}_k = (-k + \frac{1}{2})\partial_2 + w \cdot (\partial_1 \partial_4 - \partial_2 \partial_2)$$

The case of arbitrary degree n is given in my paper. It is a quite complicated but completely explicit formula. Here is a decription of the method of proof in [1].
To construct such a differential operator, we first consider a somewhat simpler situation: We look only at at functions of type

$$F(Z) = g(z_4, w)e^{tr(Tz_1)}$$

which we may call functions of (nondegenerate) Jacobi type (with T a symmetric matrix of size n with maximal rank);
in a **first step** we only request our operator to preserve automorphy for the

group $Sp(n, \mathbf{R})^{\downarrow}$, i.e. essentially for $M = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}^{\downarrow}$.
Then we can see that

$$\mathcal{D} := \sum_{p+q=n} \begin{pmatrix} n \\ q \end{pmatrix} \tilde{C}(-k + \frac{n}{2}) \Delta(p,q)$$

has the requested properties. Here

$$\begin{aligned} \Delta(p,q) &= T^{[p]} w^{[p]} \left(\partial_4 - \partial_3 T^{-1} \partial_2\right)^{[q]} \sqcap \partial_2^{[p]} \\ \tilde{C}(s) &:= \prod_{p \neq q} C_p(s) \qquad\qquad (C_p(s) = s \cdot \ldots (s + \frac{p-1}{2}) \end{aligned}$$

This is the really hard work here: To see that this linear combination does it!
An inspection of the proof shows that the solution is unique up to scalars, if we only look at linear combinations of the $\Delta(p,q)$.
This expression in terms of the $\Delta(p,q)$ is actually quite useful for computations related to the doubling method, see below.
In a **second step** we get an expression not involving T^{-1}; we also substitute expressions involving the entries of T by the corresponding entries of ∂_1:

$$\begin{aligned} \Delta(p,q) &= \sum_{\alpha+\beta=n} (-1)^{\beta} \delta(p,\alpha,\beta) \\ \delta(p,\alpha,\beta) &= z_2^{[\alpha]} \cdot \partial_4^{[\alpha]} \sqcap \left((1_n^{[p]} \sqcap z_2^{[\beta]} \partial_3^{[\beta]} Ad^{[p+\beta]} \partial_1) \partial_2^{[p+\beta]}\right) \end{aligned}$$

In a **third step** we have to convince ourselves that the expression obtained is symmetric with respect to

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \longmapsto \begin{pmatrix} z_4 & z_3 \\ z_2 & z_4 \end{pmatrix}.$$

This shows that $\mathcal{D} = \mathcal{D}_{k,1}$ preserves automorphy for $Sp(n, \mathbf{R})^{\uparrow} \times Sp(n, \mathbf{R})^{\downarrow}$.

Method II and Method III can be viewed as attempts to understand these operators in a more conceptual way.

Method II: (NOT useful, but quite instructive)

There is a holomorphic differential operator on $\mathbb{H}_{2n}$ which has all the requested properties, unfortunately only for weight $k = \frac{2n-1}{2}$:

$$\partial^{[2n]}(F \mid_{\frac{2n-1}{2}} g) = \partial^{[2n]}(F) \mid_{\frac{2n+3}{2}} g,$$

valid for all g in the big group $Sp(2n, \mathbb{R})$. So if we want to get an operator changing the weight from k to k+1, we should consider

$$det(w)^{-k+\frac{2n+1}{2}} \partial^{[2n]}(F \cdot det(w)^{k-\frac{2n-1}{2}})$$

clearly the operator has the requested automorphy properties, but there is a serious disadvantage, namely the coefficients of this differential operator are not polynomials in the entries of w, but in $det(w)^{-1} \cdot \mathbb{C}[w_{ij}]$, in fact the operator is of the form

$$det(w)\partial^{[2n]} + \cdots + c\,\det(w)^{-1}$$

The last term is not a problem by itself (we can just remove it), but there are other more complicated terms with det(w) in the denominator, which are somewhat mysterious. If we multiply with det(w), we get a polynomial of weight 0, i.e. we do not change the weight k at all .
As for the special case $n = 1$, we almost get back the previous operator, namely

$$w(\partial_1\partial_4 - \partial_2^2) + (-k + \frac{1}{2})\,\partial_2 + \frac{1}{w}.$$

Is there any way in general to remove the "fractional part" ???

Method III "The Rankin-Cohen bracket method"
Here we must quote some results from Ibukiyama-Eholzer [6]: We need more than just the existence of bilinear differential operators acting on pairs F and G of holomorphic functions on $\mathbb{H}_{2n}$. In our notation, those Rankin brackets are of the form

$$\mathcal{R}^{2n}_{k_1,k_2}(F, G) = \sum_{\alpha+\beta=2n} \binom{2n}{\alpha} (-1)^\alpha s_{2n-\alpha}(k_2 - \frac{\alpha}{2}) s_{2n-\beta}(k_1 - \frac{\beta}{2}) \partial^{[\alpha]} F \sqcap \partial^{[\beta]} G$$

with

$$s_i(x) = x(x - \frac{1}{2}) \ldots (x - \frac{i-1}{2})$$

They satisfy (for all $g \in Sp(2n,\mathbb{R})$)

$$\mathcal{R}^{2n}_{k_1,k_2}(F\mid_{k_1} g, G\mid_{k_2} g) = \left(\mathcal{R}^{2n}_{k_1,k_2}(F,G)\right)\mid_{k_1+k_2+2} g$$

In [6] those operators are only considered under the assumption

$$k_i \geq n,$$

but the formulas hold true for any weights k_i (the coefficients of the differential operators being polynomials in the k_i). In particular, we now define from this a differential operator $\mathcal{R}^{2n}_k$ mapping holomorphic functions on $\mathbb{H}_{2n}$ to functions on the same space:

$$\mathcal{R}^{2n}_k(F) := \mathcal{R}^{2n}_{k,-1}(F, det(w))$$

Clearly this operator is a polynomial in the entries of w and it satisfies our desired transformation properties, raising the weight k to $k+1$. It remains to show that this construction is nonzero:
Certainly the coefficients

$$s_{2n-\alpha}(-1-\frac{\alpha}{2})\cdot s_{2n-\beta}(k-\frac{\beta}{2})$$

are nonzero as long as k is large enough. As a polynomial in the w_{ij}, the term of highest degree is

$$s_{2n}(k-n)det(w)\cdot \partial^{[2n]}.$$

The term free of w should be a multiple of $\partial_2^{[n]}$ as is clear from the relation to harmonic polynomials of weight one. To show that the constant involved is actually different from zero, we look at $\mathcal{R}^{2n}_k(F)$ for $F = det(w)$. Clearly $\partial^{[\alpha]}(detw)\sqcap\partial^{[\beta]}(detw) = 0$ for any α, β with $\alpha+\beta = 2n$ unless $\alpha = \beta = n$ It is therefore sufficient to show that $\partial^{[n]}(detw)\sqcap\partial^{[n]}(detw)$ is different from zero. This follows from

$$\begin{aligned}(-1)^n\left(\frac{1}{2}\right)^{2n}(n+1)!\,n! &= \partial^{[2n]}det(w)^2\\ &= \sum_{\alpha+\beta=2n}\binom{2n}{\alpha}\partial^{[\alpha]}det(w)\sqcap\partial^{[\beta]}det(w)\\ &= \frac{(2n)!}{n!n!}\left(\partial^{[n]}detw\right)\sqcap\left(\partial^{[n]}detw\right)\end{aligned}$$

This means that $\mathcal{R}_k^{2n}(det(w)$ is a constant different from zero, hence the coefficient of $\partial^{[n]}$ is nonzero (for all but finitely many k).
Finally we mention that one generalize this construction to write down directly (i.e.without iteration) operators which raise the weight by $\nu \in \mathbb{N}$, namely Rankin-Cohen operators

$$\mathcal{R}_{k,-\nu,\nu}^{2n}(F, det(w)^\nu)$$

Their combinatorics however is known only in part ([6]).

Conclusion If we allow nonconstant coefficients (polynomials in the w_{ij}), we can define holomorphic differential operators acting on $\mathbf{H}_{2n}$ preserving automorphy for $Sp(n,\mathbb{R})^\uparrow \times Sp(n,\mathbb{R})^\downarrow$. If the "constant term" in these operators (and its iterates) is different from zero, we get back the polynomials considered by Ibukiyama.
Remarks:

- Our constructions should work for all symmetric domains of tube type (a description in terms of Jordan algebras should be possible).
- Method III should work also for other kinds of subgroups and suitable auxiliary functions analogous to det(w) (see also topic 3 in the appendix)
- To extend method III to vector-valued cases (including the possibility of iteration), we need a Rankin-Cohen bracket starting also from possibly vector-valued cases...

Appendix: Some additional topics

Topic 1: Use of this operator in doubling method (e.g.[4])

When we apply the doubling method not with a Siegel Eisenstein series E^{2n} series itself, but with $\mathcal{D}^{2n,\nu}_{k-\nu}E^{2n}_{k-\nu}$, then by unfolding the integral, we end up with computing

$$\left(\mathcal{D}^{2n,\nu}_{k-\nu} det(z+w+w^t+\tau)^{-k+\nu}\right)_{w=0}.$$

It is clear that the result is a multiple of

$$det(\tau+z)^{-k}$$

but we also need the constant. The point is now that this constant can easily be computed using the following properties of the $\Delta(p,q)$

- For all functions F on $\mathbb{H}_{2n}$, which only depend on $z+w+w'+\tau$ we have
$$\Delta(p,q)(F)=0 \qquad \text{for all } q>0$$
- For $h_k(Z)=det(z+w+w'+\tau)^{-k}$ we have
$$\partial^{[n]_w}h_k=c_n(-k)h_{k+1}$$
with $c_n(s)=s\cdot(s+\frac{1}{2})\dots(s+\frac{n-1}{2})$.

This means that we can determine the constant in question directly without any combinatorial problems !
Such combinatorial problems came up in other works on the doubling method, related to vector-valued modular forms, see [2].
The equation above also shows that the "constant term" of any iterate $\mathcal{D}$ is different from zero except for the cases where $c_n(-k)c_n(-k-1)...c_n(-k-\nu+1)$ is zero.

Topic 2: We have not been able to check whether the differential operators obtained by method 1 and method 3 are equal. To do it directly (if possible) will involve some combinatorial work. Another possibility would be a uniqueness statement for these differential operators (?).

Topic 3: More general operators: Triple case

Just for simplicity we consider the special case $SL(2,\mathbb{R})^3 \hookrightarrow Sp(3,\mathbb{R})$; the more general case $\mathbb{H}_n^3 \hookrightarrow \mathbb{H}_{3n}$ should work along the same lines. We decompose an element of $\mathbb{H}_3$ as

$$Z = \begin{pmatrix} \tau_1 & w_1 & w_2 \\ w_1 & \tau_2 & w_3 \\ w_2 & w_3 & \tau_3 \end{pmatrix}$$

If we want to follow the same lines as before, we should first study a function φ_l on $\mathbb{H}_3$ defined by

$$\varphi_l(Z) := (w_1 w_2 w_3)^l$$

For elements in $SL(2,\mathbb{R})^3$ the "weight k action" on a function F on $\mathbb{H}_3$ is then changed into a weigkt k-2l action on $F \cdot \varphi_l$.
Now we try the method II:
Then the weight k-action on F becomes a weight 3 action on

$$\partial^{[3]} \left(F \cdot (w_1 w_2 w_3)^{\frac{k-1}{2}} \right).$$

After multiplikation with $(w_1 w_2 w_3)^{\frac{-k+3}{2}}$ we end up with

$$w_1 w_2 w_3 \partial^{[3]} F + ... + const \times F.$$

We cannot get a reasonable holomorphic operator raising the weight properly: If we remove the constant term F and divide by $w_1 w_2 w_3$ to raise the weight, we end up with meromorph coefficients... (e.g. $(w_1 w_2)^{-1}$ will occur.
Method III is more promising:
This time we look at Rankin brackets changing the weight by $2\nu = 4$

$$\mathcal{R}^3_{k_1,k_2,2}(F,G) = \sum_{i,j=0}^{3} C_{ij} (P_i \cdot P_j)(\partial_Z, \partial_{Z'})(F,G)_{Z=Z'}$$

using the notation of [6, section 5.2] and we put

$$(\mathcal{D}_k)(F) := \mathcal{R}^3_{k,-2,2}(F, w_1 w_2 w_3)$$

This differential operator changes the weight k action on F to a weight k+2 action on $\mathcal{D}_k(F)$. Again we should show that this operator has a nonzero constant coefficient: For k large enough we only have to look at the contribution of $i = j = 3$. Similarly as before we study the special function $F(Z) := w_1 w_2 w_3$; this come down to look at

$$\partial_{w_1}\partial_{w_2}\partial_{w_3}(w_1 w_2 w_3),$$

(or its square) which is nonzero.

Topic 4: Arbitrary (complex) weights)

On several occasions (see also [3, 5]), we have to extend the validity of constructions of differential operators. It is based on the following principle, which we formulate in the setting of vector-valued automorphy factors: First we fix a branch of $det(cz+d)^s$ on $\mathbb{H}_n$. Assume that we already have (say, for infinitely many $k \in \mathbb{N}$) an explicit differential operator D_k which changes the automorphy factor from $det^k \otimes \rho$ to $\det^{k+\nu} \otimes \rho$‘ for a possibly fixed $M \in Sp(n, \mathbb{R})$. The operator D_k should be a polynomial in the complex derivatives with coefficients, which are polynomials in k (depending only on ρ, ρ‘$, \nu$). Then we can define formally an operator D_s for $s \in \mathbb{C}$. We claim that

$$D_s(F \mid_{det^s \otimes \rho} M) = D_s(F) \mid_{det^{s+\nu} \otimes \rho`} M$$

Proof: Consider a test function

$$F_{T,v} := v \cdot exp(tr(TZ)) \qquad (v \in V_\rho,\ T = T^t \in \mathbb{C}^{(n,n)})$$

Then (after choosing a basis of $V_{\rho'}$, (the components of) both sides of the equation above are (after multiplying both sides with an appropriate power of $det(cz+d)$ of the form

$$exptr(T \cdot M < Z >) \times \text{ polynomial in z and T}$$

The coefficients of these polynomials depend polynomially on s and are equal for infinitely many k, therefore they must coincide.

One can use this principle to extend a known construction of holomorphic differential operators beyond the original range of definition.

Topic 5: **Characterization of the polynomial $\mathcal{Q}$ in terms of Gauß-transform**

Here k, ν and so on have the same meaning as in first part of this paper.

We use the same kind arguments as in [1, 8, 9].
First we recall from [8, chap.II] the notion of *Gauß-transform* of a polynomial function $P : \mathbb{C}^r \longrightarrow \mathbb{C}$, defined by

$$\widehat{P}(y) := \int_{\mathbb{R}^r} P(x+y) exp(-\pi x^t \cdot x) dx \qquad (y \in \mathbb{C}^r)$$

It is related to the Laplacian Δ on $\mathbb{C}^r$ by

$$\widehat{P} = \sum_{j \geq 0} \frac{1}{j!} \left(\frac{\Delta}{4\pi} \right)^j P \qquad \text{(a finite sum!)}$$

We define a polynomial

$$\mathcal{P} : \left\{ \begin{array}{ccc} \mathbb{C}^{(m,n)} \times \mathbb{C}^{(m,n)} \times \mathbb{C}^{(n,n)} & \longrightarrow & \mathbb{C} \\ (\mathfrak{x}, \mathfrak{y}, w) & \longmapsto & \mathcal{Q}(\pi i \mathfrak{x}^t \cdot \mathfrak{x} \,,\, \pi i \mathfrak{x}^t \cdot \mathfrak{y} \,,\, \pi i \mathfrak{y}^t \cdot \mathfrak{y}, w) \end{array} \right.$$

with $m = 2k$.
For $w = 0$ this is a pluriharmonic polynomial in both variables $\mathfrak{x}$ and $\mathfrak{y}$; it occurs in [1, 9]. We try to find a property of $\mathcal{P}$, which can be viewed as a substitute for being pluriharmonic. For a fixed $\mathfrak{x}_o \in \mathbb{C}^{(m,n)}$ we study the function (with $Z \in \mathbb{H}_{2n}$)

$$\mathfrak{y} \longmapsto g(Z, \mathfrak{y}) := \sum_{G \in \mathbb{Z}^{(m,n)}} exp(\pi i tr(Z[(\mathfrak{x}_0, G + \mathfrak{y})^t)$$

The reciprocity law for theta constants of general type gives

$$g(Z, \mathfrak{y}) = i^{\frac{mn}{2}} \left(\sum_{G \in \mathbb{Z}^{(m,n)}} exp \pi i tr(Z[(\mathfrak{x}_0, G)^t] + 2G^t \mathfrak{y}) \right) |_k \, I^{\downarrow}$$

with $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in Sp(n, \mathbb{R})$

Application of the differential operator $\mathcal{D}$ yields

$$\begin{aligned}\mathcal{D}g(Z,\mathfrak{y}) &= \sum_G \mathcal{P}(\mathfrak{x}_0, G+\mathfrak{y}, w) exp\pi itr(Z[(\mathfrak{x}_0, G+\mathfrak{y})^t] \\ &= i^{\frac{mn}{2}} det(z_4)^{-\frac{m}{2}-\nu} \times \\ &\quad \sum_G \mathcal{P}(\mathfrak{x}_0, G, w\cdot z_4^{-1}) exp(I^{\downarrow} < Z > [(\mathfrak{x}_0, G,)^t] + 2G^t \cdot \mathfrak{y})\end{aligned}$$

This is a periodic holomorphic function of $\mathfrak{y}$; its Fourier expansion is of the form

$$\mathcal{D}g(Z,\mathfrak{y}) = \sum_G A(G) exp(2\pi itr(G^t \cdot \mathfrak{y})$$

We can compute the Fourier coefficients $A(G)$ in two ways: The formula above gives immediately a first formula:

$$A(G) = i^{\frac{mn}{2}} det(z_4)^{-\frac{m}{2}-\nu} \mathcal{P}(\mathfrak{x}_0, G, w\cdot z_4^{-1}) exp(I^{\downarrow} < Z > [(\mathfrak{x}_0, G,)^t])$$

The second formula is obtained by starting from the definition of such a Fourier coefficient

$$\begin{aligned}A(G) &= \int_{\mathfrak{r}\ mod\ 1} exp(-2\pi itr(G^t\mathfrak{y})\mathcal{D}g(z,\mathfrak{y})d\mathfrak{r} \qquad (\mathfrak{y} = \mathfrak{r} + i\mathfrak{s}) \\ &= \int_{\mathbb{R}^{(m,n)}} \mathcal{P}(\mathfrak{x}_0, \mathfrak{y}, w) exp\pi itr(Z[(\mathfrak{x}_0, \mathfrak{y})^t] - 2G^t\mathfrak{y})d\mathfrak{r} \qquad (\text{all } \mathfrak{s}) \\ &= exp(\pi tr(z_1[\mathfrak{x}_0^t]) exp(-\pi itr(z_4^{-1}[G^t - (\mathfrak{x}_0^t z_2)] \times \\ &\quad \int \mathcal{P}(\mathfrak{x}_0, \mathfrak{y}, w) exp(\pi z_4[(\mathfrak{y} - Gz_4^{-1} + \mathfrak{x}_0 z_2 z_4^{-1})^t]) d\mathfrak{r}\end{aligned}$$

We consider the special points

$$z_4 = i\cdot 1_n\,, \quad \mathfrak{x}_0 \in \mathbb{R}^{(m,n)}\,, \quad z_2 = w \in \mathbb{R}^{(n,n)}\,, \quad i\mathfrak{s} = -iG + i\mathfrak{x}_0 z_2$$

Then

$$A(G) = exp(\pi tr(z_1[\mathfrak{x}_0^t]) exp(\pi tr(1_n[G^t - (\mathfrak{x}_0^t z_2)] \int \mathcal{P}(\mathfrak{x}_0, \mathfrak{r} - iG + i\mathfrak{x}_0 z_2, w) exp(-\pi 1_n[\mathfrak{r}^t]) d\mathfrak{r}$$

Comparison yields (first for $\mathfrak{x}_0 \in \mathbb{R}^{(m,n)}$, $w \in \mathbb{R}^{(n,n)}$, $G \in \mathbb{Z}^{(m,n)}$

$$i^{-n\nu} \mathcal{P}(\mathfrak{x}_0, G, -iw) = \widehat{\mathcal{P}}(\mathfrak{x}_0, -iG + i\mathfrak{x}_0 \cdot w, w),$$

where $\widehat{\mathcal{P}}$ denotes the Gauß-transform of the polynomial (viewed as polynomial of the variable $\mathfrak{y}$).
Finally we get for all $\mathfrak{x}_0, \mathfrak{y} \in \mathbb{C}^{(m,n)}$, $w \in \mathbb{C}^{(n,n)}$

$$\mathcal{P}(\mathfrak{x}_0, \mathfrak{y}, w) = \widehat{\mathcal{P}}(\mathfrak{x}_0, \mathfrak{y} + i\mathfrak{x}_0 \cdot w, w)$$

and the same kind of law w.r.t. $\mathfrak{x}$ (with $\mathfrak{y}_0$ fixed). One should therefore study the class of all polynomial functions $P : \mathbb{C}^{(m,n)} \times \mathbb{C}^{(m,n)} \times \mathbb{C}^{(n,n)} \longrightarrow \mathbb{C}$ with the following properties

(1) symmetry:

$$P(\mathfrak{y}, \mathfrak{x}, w^t) = P(\mathfrak{x}, \mathfrak{y}, w)$$

(2) det^ν action of $GL(n, \mathbb{C})$:

$$P(\mathfrak{x}_0, \mathfrak{y} \cdot A, w \cdot A^{-t}) = det(A)^\nu P(\mathfrak{x}_0, \mathfrak{y}, w) \qquad (A \in GL(n, \mathbb{C}))$$

(3) Substitute for harmonicity:

$$\mathcal{P}(\mathfrak{x}_0, \mathfrak{y}, w) = \widehat{\mathcal{P}}(\mathfrak{x}_0, \mathfrak{y} + i\mathfrak{x}_0 \cdot w, w)$$

It would be very interesting to understand this class of polynomial functions. Of course in (2) and (3) there is a also a version, where we can fix $\mathfrak{y}_0$; this is already taken care of by the symmetry property (1).

Example: $n = 1, \nu = 1$ with

$$\mathcal{P} = (\pi i)^2 w\{(\sum_j x_j^2)(\sum_j y_j^2) - (\sum_j x_j x_j)^2\} + (-k + \frac{1}{2})\pi i(\sum_j x_j y_j)$$

Here (as above, we take the Laplacian w.r.t. $\mathfrak{y}$)

$$\Delta\mathcal{P} = (\pi i)^2 w(2m-2)(\sum_j x_j^2)$$

Indeed,

$$\begin{aligned}
\widehat{P}(\mathfrak{x}, \mathfrak{y} + i\mathfrak{x}w, w) &= P(\mathfrak{x}, \mathfrak{y} + i\mathfrak{x}w, w) + (\pi i)^2 \frac{(2m-2)}{4\pi} w(\sum x_j^2) \\
&= (\pi i)^2 w(\pi i)^2\{(\sum_j x_j^2)(\sum_j y_j^2) - (\sum_j x_j x_j)^2\} + \pi i(-k + \frac{1}{2})(\sum_j x_j y_j) + \\
&\quad \pi i(iw)(-k + \frac{1}{2})(\sum x_j x_j) - \pi w(k - \frac{1}{2})(\sum_j x_j^2) \\
&= \mathcal{P}(\mathfrak{x}, \mathfrak{y}, w)
\end{aligned}$$

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Siegfried Böcherer
Kunzenhof 4B
79117 Freiburg
boecherer@t-online.de