On p-adic absolute CM-periods

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In the Hakuba conference, Yoshida reported on our collaborative work. Here we tried to reproduce the talk rather faithfully. (§5 is enlarged.) The reader will find a list of notations (will be cited as [N]), which was distributed for the attendants of the talk, after page 13.

§1. Two precursors

Let us first recall two well known precursors of complex and p-adic absolute CM-period symbols. Let K be an imaginary quadratic field of discriminant $-d$. Let h and w be the class number of K and the number of roots of unity contained in K respectively. Let χ be the Dirichlet character which corresponds to the extension $K/\mathbf{Q}$. The Chowla-Selberg formula states

$$\pi p_K(\mathrm{id},\mathrm{id})^2 \sim \prod_{a=1}^{d} \Gamma(\frac{a}{d})^{w\chi(a)/2h}. \tag{1}$$

Here p_K is Shimura's period symbol. For the reader's convenience, its main properties are listed in [N]. The period symbol p_K is a culmination of the theory of complex multiplication of abelian varieties. See Shimura's book [S3]. For an interesting episode on this topic in 1955, see Shimura's article [99e], page 671 in volume IV of his collected papers.

The absolute CM-period symbol gives a (conjectural) generalization of the Chowla-Selberg formula for an arbitrary CM-field.

Let p be a prime number which decomposes completely in K. Let $(p) = \mathfrak{P}\mathfrak{P}^\rho$ be the prime ideal decomposition, where ρ denotes the complex conjugation. We can embed K into $\mathbf{Q}_p$ so that $\mathfrak{P} = \mathcal{O}_K \cap p\mathbf{Z}_p$, where $\mathcal{O}_K$ is the ring of integers of K. Put $\mathfrak{P}^h = (\alpha)$ with $\alpha \in K$. Then the Gross-Koblitz formula [GK] states

$$\log_p(\alpha^\rho/\alpha) = \frac{w}{2}\sum_{a=1}^{d-1} \chi(a)\log_p(\Gamma_p(\frac{a}{d})). \tag{2}$$

Here $\log_p$ is the p-adic logarithmic function and Γ_p is Morita's p-adic gamma function.

In this talk, we will show that Gross-Koblitz formula can be generalized to an arbitrary CM-field in pefect analogy to the absolute CM-period symbol. We have multifold motivations to investigate p-adic analogues of absolute CM-periods. Some of them will be explained in the talk.

§2. Shintani-Kashio formulas

Let F be a totally real algebraic number field of degree n. Let $\mathfrak{a}_1$, $\mathfrak{a}_2$, ..., $\mathfrak{a}_{h_0}$ be integral ideals which represent narrow ideal classes of F. We take a cone decomposition (cf. [N])

$$\mathbf{R}_+^n = \sqcup_{\epsilon \in E_F^+} \epsilon (\sqcup_{j \in J} C_j).$$

For $c \in C_{\mathfrak{f}}$, we put

$$G(c) = \sum_{j \in J} \sum_{z \in R(C_j, c)} \log \frac{\Gamma_{r(j)}(z, A_j)}{\rho_{r(j)}(A_j)},$$

$$W(c) = -\frac{1}{n} \log N(\mathfrak{a}_\mu \mathfrak{f}) \zeta_F(0, c),$$
$$X(c) = G(c) + V(c) + W(c).$$

Here $V(c)$ is a quantity of the form

$$V(c) = \sum_i a_i \log \epsilon_i, \qquad a_i \in F, \quad \epsilon_i \in E_F^+.$$

Let $\zeta_F(s, c)$ be the partial zeta function of the class c. Then Shintani's formula for the derivative of $\zeta_F(s, c)$ at $s = 0$ can be written as

$$\zeta_F'(0, c) = \sum_{\sigma \in J_F} X(c^\sigma). \tag{3}$$

For this formula and a generalization of it for higher derivatives, see Yoshida's book [Y4].

Now let p be a prime number and we fix an embedding $F \subset \mathbf{C}_p$. Let $\mathfrak{p}$ be the prime ideal of F obtained from this embedding. We define [1]

$$G_p(c) = \sum_{j \in J} \sum_{z \in R(C_j, c)} L\Gamma_{p, r(j)}(z, A_j),$$

$$V_p(c) = \sum_i a_i \log_p \epsilon_i,$$

$$W_p(c) = -\log_p(\mathfrak{a}_\mu) \cdot (\zeta_F(0, c) - \zeta_F(0, \underline{\mathfrak{p}}^{-1} c)),$$

[1] $\log_p(\mathfrak{a}_\mu)$ is defined as follows. Take a positive integer e so that $\mathfrak{a}_\mu^e = (\beta)$, $\beta \in F$. We put $\log_p(\mathfrak{a}_\mu) = \log_p \beta / e$. Then $\log_p(\mathfrak{a}_\mu)$ is well defined modulo addition of elements of the form $a \log_p(\epsilon)$, $a \in \mathbf{Q}$, $\epsilon \in E_F^+$.

$$X_p(c) = G_p(c) + V_p(c) + W_p(c).$$

Here $\underline{\mathfrak{p}}$ denotes the class of $\mathfrak{p}$ in $C_{\mathfrak{f}}$ if $\mathfrak{p}$ does not divide $\mathfrak{f}$; if $\mathfrak{p}$ divides $\mathfrak{f}$, the term which contains $\underline{\mathfrak{p}}$ should be ignored. Kashio's formula for the derivative of the p-adic partial zeta function $\zeta_{p,F}(s,c)$ states

$$\zeta'_{p,F}(0,c) = \sum_{\sigma\in J_F} X_p(c^\sigma) - \log_p N(\mathfrak{f})\zeta_{p,F}(0,c). \tag{4}$$

if $\mathfrak{f}$ is divisible by every prime ideal of F lying over p.

§3. Absolute CM-period symbols (complex and p-adic)

Let K be a CM-field which is an abelian extension of F. Let $\widetilde{\mathfrak{f}}\infty_1\infty_2\cdots\infty_n$ be the conductor of K as a class field over F. Put $G = \mathrm{Gal}(K/F)$. For $\tau \in G$, we define

$$g_K(\mathrm{id},\tau) = \pi^{-\mu(\tau)/2}\exp\Big(\frac{1}{|G|}\sum_{\mathfrak{f}|\widetilde{\mathfrak{f}}}\ \sum_{\chi\in(\hat{G}_-)_{\mathfrak{f}}}\frac{\chi(\tau)}{L(0,\chi)}\sum_{c\in C_{\mathfrak{f}}}\chi(c)X(c)\Big).$$

Here $\mu(\tau)$ is 1 if $\tau = \mathrm{id}$, -1 if $\tau = \rho$ and 0 otherwise; $\widehat{G}_-$ denotes the set of odd characters of G; $(\widehat{G}_-)_{\mathfrak{f}}$ denotes the subset of $\widehat{G}_-$ consisting of characters whose conductors have $\mathfrak{f}$ as the finite part; g_K is called the absolute period symbol. Now one of the main conjectures in [Y4] is

Conjecture C.

$$g_K(id,\tau) \sim p_K(id,\tau).$$

The symbol $g_K(\mathrm{id},\tau)$ depends on the choices of $\{C_j\}$ and $\{\mathfrak{a}_\mu\}$. But we can analyze the dependence on these data and can show that the validity of Conjecture C does not depend on the choices of them. The symbol $g_K(\mathrm{id},\tau)$ also depends on the choice of F. In other words, K can be abelian over F_1 and F_2 and $\tau \in \mathrm{Gal}(K/F_1)\cap\mathrm{Gal}(K/F_2)$. To show the consistency of Conjecture C when we change F is more difficult and we obtained only partial results (cf. [Y4]). In the strict sense, $g_K(\mathrm{id},\tau)$ should be written as $g_{K/F}(\mathrm{id},\tau;\{C_j\},\{\mathfrak{a}_\mu\})$.

This conjecture implies that the class invariant $X(c)$ contains information on CM-periods and on units of abelian extension of F. To make this explicit, we remind the reader of the conjecture of Stark-Shintani. Let M be an abelian extension of F of conductor $\mathfrak{m}\infty_2\ldots\infty_n$. For $\sigma \in \mathrm{Gal}(M/F)$, let $\zeta(s,\sigma)$ be the partial zeta function attached to σ. Assume $n \geq 2$. The Stark-Shintani conjecture states that there exists a unit ϵ of M such that

$$\exp(-2\zeta'(0,\sigma)) = \epsilon^\sigma.$$

By Shintani's formula, $\exp(-2\zeta'(0,\sigma))$ can be expressed as a sum of $X(c)$. Here it is important to notice the difference of the infinite parts of the conductors. The inclusion of ∞_1 brings CM-periods. We must refer to [Y4] for detailed explanation of this interesting situation.

In the p-adic case, we put

$$lg_{p,K}(\mathrm{id},\tau) = -\frac{\mu(\tau)}{2h_K}\log_p \alpha_0 + \frac{1}{|G|}\sum_{\mathfrak{f}|\tilde{\mathfrak{f}}}\sum_{\chi\in(\hat{G}_-)_{\mathfrak{f}}}\frac{\chi(\tau)}{L(0,\chi)}\sum_{c\in C_{\mathfrak{f}}}\chi(c)X_p(c).$$

Here $\mathfrak{p}^{h_K} = (\alpha_0)$, $\alpha_0 \in F$. (We remind the reader of that $\mathfrak{p}$ is the prime ideal of F obtained by the fixed embedding $F \subset \mathbf{C}_p$.)

Conjecture P. *Take $\mathfrak{a}_\mu$, $1 \le \mu \le h_0$ so that they are not divisible by $\mathfrak{p}$. Fix an embedding $F \subset K \subset \mathbf{C}_p$ and let $\mathfrak{P}$ be the induced prime ideal of K. Put $\mathfrak{P}^{h_K} = (\alpha)$, $\alpha \in K$. If $\mathfrak{p}$ is completely decomposed in K, then we have*

$$lg_{p,K}(id,\tau) = \frac{1}{2h_K}\log_p(\alpha^{\tau^{-1}\rho}/\alpha^{\tau^{-1}}) + \sum_{i=1}^{n-1} a_i \log_p \epsilon_i$$

with $a_i \in F$, $\epsilon_i \in E_F^+$.

The explanation in the first paragraph after Conjecture C also applies to the symbol $lg_{p,K}(\mathrm{id},\tau)$.

Gross [Gr] generalized the Stark-Shintani conjecture to the p-adic case. It is curious that Conjecture P is closely related to the Gross conjecture and its relation to the p-adic periods is rather indirect. We will explain the latter point in §5. Here let us explain the first point.

To this end, we return to the complex case. We consider the quantity

$$Z_K(\mathrm{id},\tau) = \pi^{-n\mu(\tau)/2}\prod_{\chi\in\hat{G}_-}\exp\left(\frac{\chi(\tau)}{|G|}\frac{L'(0,\chi)}{L(0,\chi)}\right). \tag{5}$$

Then a conjecture of Colmez [Co] and an author [Y2] can be brought to the form

$$Z_K(\mathrm{id},\tau) \sim \prod_{\sigma\in J_F} p_K(\sigma,\tau\sigma). \tag{6}$$

Here we use the same letter σ for (any of) its extension to an isomorphism of K into $\mathbf{C}$. A careful factorization of (6) using (3) suggests Conjecture C. In this respect, we note that Conjecture C is much stronger than (6); we can predict the values of p_K by Conjecture C but it is impossible to do so only by (6) (see, [Y4], p. 74–82).

In analogy to (5), put

$$LZ_{p,K}(\mathrm{id},\tau) = \frac{1}{|G|}\sum_{\chi\in\hat{G}_-}\chi(\tau)\frac{L_p'(0,\omega\chi)}{L(0,\chi)}. \tag{7}$$

Here ω denotes the Teichmüller character and $L_p(s,\omega\chi)$ denotes the p-adic L-function. Let $\mathfrak{p}_1$, ..., $\mathfrak{p}_t$ be all the prime ideals of F lying over p. For simplicity, we assume that p is unramified in K. We put $\sigma_i = (\frac{K/F}{\mathfrak{p}_i})$ using the Artin symbol. The Gross conjecture on $L_p'(0,\omega\chi)$ assumes that one of $\mathfrak{p}_i$ decomposes completely in K. So we may assume that $\mathfrak{p}_1 = \mathfrak{p}$ and that $\mathfrak{P}$ is a prime factor of $\mathfrak{p}$ in K as in our conjecture. Let $K_{\mathfrak{P}}$ be the completion of K at $\mathfrak{P}$. The Gross conjecture is equivalent to

$$\begin{aligned}&LZ_{p,K}(\mathrm{id},\tau)\\ &=\frac{1}{2h_K}\Bigg[\log_p(N_{K_{\mathfrak{P}}/\mathbf{Q}_p}(\alpha^{\tau^{-1}\rho}/\alpha^{\tau^{-1}}))\\ &+\sum_{s=1}^{t-1}\sum_{2\leqq i_1<\cdots<i_s\leqq t}(-1)^s\log_p(N_{K_{\mathfrak{P}}/\mathbf{Q}_p}(\alpha^{\tau^{-1}\sigma_{i_1}^{-1}\cdots\sigma_{i_s}^{-1}\rho}/\alpha^{\tau^{-1}\sigma_{i_1}^{-1}\cdots\sigma_{i_s}^{-1}}))\Bigg].\end{aligned}$$

Now we distinguish three cases.

(I) If $\sigma_i = 1$ for some i, $2 \leqq i \leqq t$, then $L_p'(0,\omega\chi) = 0$ for every $\chi \in \widehat{G}_-$. (This is a part of the Gross conjecture and is proved by Kashio [Kas].) Thus no direct relations are present between the Gross conjecture and our conjecture.

(II) Suppose that $\sigma_i \neq 1$ for all i, $2 \leqq i \leqq t$.

(a) If $\mathfrak{p}$ is unramified and of degree 1 over $\mathbf{Q}$, i.e., $F_{\mathfrak{p}} = \mathbf{Q}_p$ for the completion of F at $\mathfrak{p}$, then the Gross conjecture and our conjecture are essentially equivalent.

(b) If $[F_{\mathfrak{p}} : \mathbf{Q}_p] > 1$, then our conjecture gives an essential refinement of the Gross conjecture. It gives a factorization of $LZ_{p,K}(\mathrm{id},\tau)$; it also implies the Gross conjecture modulo computable numbers of the form $\sum_i a_i \log_p \epsilon_i$, where $a_i \in \widetilde{F}$, ϵ_i is a totally positive unit of $\widetilde{F}$, $\widetilde{F}$ being the normal closure of F over $\mathbf{Q}$.

In the next section, we will give examples.. The case $p = 23$ corresponds to (I). The case $p = 17$ corresponds to (II), (b).

§4. Examples

Let $F = \mathbf{Q}(\sqrt{29})$, $K = \mathbf{Q}(\sqrt{\frac{9+\sqrt{29}}{2}}\,i)$, $\mathfrak{f} = (\frac{9+\sqrt{29}}{2})$. Then K is the maximal ray classfield of F modulo $\mathfrak{f}\infty_1\infty_2$. This is the situation studied in

[Y3], Example 3, [Y4], Chapter III, Example 4.1. The fundamental unit of F is $\epsilon_0 = \frac{5+\sqrt{29}}{2}$ and $\epsilon = \epsilon_0^2 = \frac{27+5\sqrt{29}}{2}$ is the totally positive fundamental unit. We have $h_F = h_0 = 1$. We take $\mathfrak{a}_1 = (1)$ as the representative of the narrow ideal class. We can obtain a cone decomposition of $\mathbf{R}_+^2$ taking $C_1 = C(1, \epsilon)$, $C_2 = C(1)$. We have

$$\mathcal{O}_K = \frac{1}{2}(\sqrt{\frac{9+\sqrt{29}}{2}}\, i + \frac{3+\sqrt{29}}{2})\mathcal{O}_F \oplus \mathcal{O}_F.$$

Let χ be the nontrivial character of $\mathrm{Gal}(K/F)$. We have $L(0,\chi) = 2$. Hence $h_K = 1$.

First we take $p = 23$. Put $\mathfrak{p} = (\frac{11+\sqrt{29}}{2})$, $\mathfrak{p}' = (\frac{11-\sqrt{29}}{2})$. Then both of $\mathfrak{p}$ and $\mathfrak{p}'$ decompose completely in K. The Gross conjecture just tells $L_p'(0,\omega) = L_p'(0,\chi\omega) = 0$, which is a special case of a theorem of Kashio.

Put

$$\alpha = \sqrt{\frac{9+\sqrt{29}}{2}}\, i + 1, \qquad \alpha_0 = \frac{11+\sqrt{29}}{2}.$$

Then we have $\alpha\alpha^\rho = \alpha_0$. We take $\mathfrak{P} = (\alpha)$ and embed K into $\mathbf{Q}_p$ so that the induced prime ideal is $\mathfrak{P}$. By numerical computations, we find

$$lg_{p,K}(\mathrm{id}, \mathrm{id}) = \frac{1}{2}\log_p(\alpha^\rho/\alpha) - \frac{172+\sqrt{29}}{2^2 \cdot 13 \cdot p}\log_p \epsilon. \tag{8}$$

Both sides are elements of $\mathbf{Z}_p$ and the equality holds at least modulo p^{50}. (We don't have a proof of (8).)

Next we take $p = 17$. Then p remains prime in F. For

$$\alpha = \frac{3}{2}(\sqrt{\frac{9+\sqrt{29}}{2}}\, i + \frac{3+\sqrt{29}}{2}) - \sqrt{29},$$

we have $\alpha\alpha^\rho = p$. We verified that

$$\sum_{c \in C_{\mathfrak{f}(p)}} \chi(c) X_p(c) = 2\log_p(\alpha^\rho/\alpha) \tag{9}$$

modulo $p^{50}(\mathcal{O}_K)_p$. In this case, the Gross conjecture states that

$$L_p'(0,\chi\omega) = 2\log_p(N_{K_{\mathfrak{P}}/\mathbf{Q}_p}(\alpha^\rho/\alpha)). \tag{10}$$

We see easily that (9) implies Conjecture P and the Gross conjecture. (Conjecture P implies (9) modulo addition of elements of the form $a\log_p \epsilon$, $a \in F$.) For more examples, see [KY1].

§5. Dictionary

We wish to know the nature of the p-adic absolute period symbol $lg_{p,K}(\mathrm{id},\tau)$. Conjecture P clarifies it when $\mathfrak{p}$ is completely decomposed in K. Though there is some room for improvements, we think that the main feature is revealed in this case by Conjecture P. Now the main problem is to drop the assumption that $\mathfrak{p}$ is completely decomposed in K.

For the formulation of the conjecture in general case, it seems most convenient to consider motives attched to Grössencharacters of type A_0. In fact, in the complex case, a refinement of Conjecture C is formulated with respect to the critical values of the L-functions attached to such Grössencharacters, which eventually turns out to be amenable to numerical tests.

Let K be a CM-field. Let χ be a Grössencharacter of K of conductor $\mathfrak{f}$ such that

$$\chi((\alpha)) = \prod_{\sigma\in J_K} \sigma(\alpha)^{l_\sigma}, \qquad \alpha \equiv 1 \mod{}^{\times}\mathfrak{f}.$$

Here l_σ are integers such that $l_\sigma + l_{\sigma\rho}$ is independent of σ. Put

$$E = \mathbf{Q}(\chi(\mathfrak{a}) \mid \mathfrak{a} \text{ is an integral ideal prime to } \mathfrak{f}),$$

Then E is an algebraic number fields of finite degree.

There exists a motive $M = M(\chi)$ over K with coefficients in E; M is characterized by the property

$$L(M,s) = (L(s,\sigma(\chi)))_{\sigma\in J_E}.$$

We consider four realizations of M; de Rham, p-adic, Betti and cristalline.

First M has the de Rham realization $H_{\mathrm{DR}}(M)$ which is an $E\otimes_{\mathbf{Q}}K$-module of rank 1. For a finite place λ of E, M has the λ-adic realization $H_\lambda(M)$ which is an E_λ-module of rank 1. We have E_λ-linear action of $\mathrm{Gal}(\overline{K}/K)$ on $H_\lambda(M)$. We put

$$H_p(M) = \oplus_{\lambda|p}\, H_\lambda(M).$$

Let $\overline{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. We fix an embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}_p$ and let $\mathfrak{P}$ be the prime ideal of K induced by this embedding. Below we take $K_{\mathfrak{P}}$ as the basic local field.

Using $K \subset \overline{\mathbf{Q}} \subset \mathbf{C}$, we have the Betti realization $H_B(M)$ which is an E-module of rank 1. We have the canonical isomorphism

$$i_p : H_B(M) \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong H_p(M) \tag{11}$$

as $E \otimes_{\mathbf{Q}} \mathbf{Q}_p$-modules of rank 1. The de Rham and p-adic realizations are related in the following way. Let B_{DR} be the discrete valuation field introduced by Fontaine [Fo]. We have

$$I_{\mathrm{DR}} : H_p(M) \otimes_{\mathbf{Q}_p} B_{\mathrm{DR}} \cong H_{\mathrm{DR}}(M) \otimes_K B_{\mathrm{DR}}. \tag{12}$$

Both sides are isomorphic as free $E \otimes_{\mathbf{Q}} B_{\mathrm{DR}}$ -modules of rank 1. The isomorphism is compatible with the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K_{\mathfrak{P}})$ and filtrations.

Suppose that χ is unramified at $\mathfrak{P}$. Then M has the cristalline realization $H_{\mathrm{cris}}(M)$. Let k be the residue field of $K_{\mathfrak{P}}$ and $K_{\mathfrak{P},0}$ be the quotient field of $W(k)$, the ring of Witt vectors over k. We identify $K_{\mathfrak{P},0}$ with the maximal unramified extension of $\mathbf{Q}_p$ contained in $K_{\mathfrak{P}}$. $H_{\mathrm{cris}}(M)$ is a free $E \otimes_{\mathbf{Z}} W(k) \cong E \otimes_{\mathbf{Q}} K_{\mathfrak{P},0}$ -module of rank 1. We have the isomorphism

$$I_{\mathrm{cris}} : H_p(M) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}} \cong H_{\mathrm{cris}}(M) \otimes_{W(k)} B_{\mathrm{cris}}. \tag{13}$$

Both sides are isomorphic as free $E \otimes_{\mathbf{Q}} B_{\mathrm{cris}}$-modules of rank 1. The isomorphism is compatible with the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K_{\mathfrak{P}})$ and Frobenius. Here B_{cris} denotes the subring of B_{DR} introduced by Fontaine.

Let $\varphi \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be a Frobenius and let $\varphi_{\mathfrak{P}}$ be a Frobenius of $\mathfrak{P}$ which lies in $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K_{\mathfrak{P}})$. We may take $\varphi_{\mathfrak{P}} = \varphi^f$ where f denotes the degree of $\mathfrak{P}$ over $\mathbf{Q}$. The action of $\varphi_{\mathfrak{P}}$ on $H_{\mathrm{cris}}(M)$ is given by

$$\varphi_{\mathfrak{P}} \mid H_{\mathrm{cris}}(M) = \chi(\mathfrak{P}) \otimes 1. \tag{14}$$

There is a $K_{\mathfrak{P},0}$-structure on $H_{\mathrm{DR}}(M) \otimes_K K_{\mathfrak{P}}$ which we denote by $H_{\mathrm{DR}}(M/K_{\mathfrak{P},0})$. We have

$$I_0 : H_{\mathrm{cris}}(M) \otimes_{W(k)} K_{\mathfrak{P},0} \cong H_{\mathrm{DR}}(M/K_{\mathfrak{P},0}), \tag{15}$$

$$I_{\mathrm{DR}} = (I_0)_{\mathrm{DR}} \circ (I_{\mathrm{cris}})_{\mathrm{DR}}. \tag{16}$$

Here $(I_{\mathrm{cris}})_{\mathrm{DR}}$ denotes the isomorphism obtained by taking $\otimes_{B_{\mathrm{cris}}} B_{\mathrm{DR}}$ in (13) and $(I_0)_{\mathrm{DR}}$ denotes the isomorphism by taking $\otimes_{K_{\mathfrak{P},0}} B_{\mathrm{DR}}$ in (15).

Hereafter we assume that $K_{\mathfrak{P},0} = K_{\mathfrak{P}}$ for simplicity. This condition is satisfied when $\mathfrak{P}$ is unramified over $\mathbf{Q}$.

We take $0 \neq c_B \in H_{\tau_c,B}(M)$. Then $i_p(c_B)$ is a generator of $H_p(M)$ as an $E \otimes_{\mathbf{Q}} \mathbf{Q}_p$-module. Take $c_{\mathrm{DR}} \in H_{\mathrm{DR}}(M)$ which is a generator of $H_{\mathrm{DR}}(M)$ as an $E \otimes_{\mathbf{Q}} K$-module. Then we define the period $P(\chi)$ by

$$I_{\mathrm{DR}}(i_p(c_B) \otimes 1) = P(\chi)(c_{\mathrm{D}R} \otimes 1). \tag{17}$$

We have $P(\chi) \in (E \otimes_{\mathbf{Q}} B_{\mathrm{DR}})^{\times}$; it is determined up to the multiplication of elements of $(E \otimes_{\mathbf{Q}} K)^{\times}$. Here we regard $E \otimes_{\mathbf{Q}} K$ as a subring of $E \otimes_{\mathbf{Q}} B_{\mathrm{DR}}$. $P(\chi)$ is the p-adic period in the standard sense.

We can choose a generator c_{cris} of $H_{\mathrm{cris}}(M)$ as an $E \otimes_{\mathbf{Z}} W(k)$-module so that

$$I_0(c_{\mathrm{cris}} \otimes 1) = c_{\mathrm{DR}} \otimes 1. \tag{18}$$

We define $\widetilde{P}(\chi) \in (E \otimes_{\mathbf{Q}} B_{\mathrm{cris}})^{\times}$ by

$$I_{\mathrm{cris}}(i_p(c_B) \otimes 1) = \widetilde{P}(\chi)(c_{\mathrm{cris}} \otimes 1). \tag{19}$$

Since $B_{\mathrm{cris}} \subset B_{\mathrm{DR}}$, we can regard $\widetilde{P}(\chi) \in (E \otimes_{\mathbf{Q}} B_{\mathrm{DR}})^{\times}$. Then, by (16) and (19), we have

$$P(\chi) = \widetilde{P}(\chi). \tag{20}$$

Since $H_{\mathrm{cris}}(M, W(k)) \otimes_{W(k)} K_{\mathfrak{P},0}$ is a free $E \otimes_{\mathbf{Q}} K_{\mathfrak{P},0}$-module of rank 1, we can write

$$\varphi^i(c_{\mathrm{cris}} \otimes 1) = Q^{(i)}(c_{\mathrm{cris}} \otimes 1), \qquad 1 \leqq i \in \mathbf{Z} \tag{21}$$

with $Q^{(i)} \in (E \otimes_{\mathbf{Q}} K_{\mathfrak{P},0})^{\times}$. Applying the Frobenius i-times on (19), we obtain

$$\Phi^i_{\mathrm{cris}}(\widetilde{P}(\chi))\varphi^i(c_{\mathrm{cris}} \otimes 1) = \widetilde{P}(\chi)(c_{\mathrm{cris}} \otimes 1).$$

Here Φ_{cris} denotes the action of Frobenius on B_{cris}. Therefore we obtain

$$\Phi^i_{\mathrm{cris}}(\widetilde{P}(\chi))Q^{(i)} = \widetilde{P}(\chi), \qquad 1 \leqq i \in \mathbf{Z}. \tag{22}$$

We put $Q = Q^{(1)}$. In particular, we have

$$\Phi_{\mathrm{cris}}(\widetilde{P}(\chi))Q = \widetilde{P}(\chi). \tag{23}$$

Applying Φ_{cris} on both sides noting that Φ_{cris} acts on $K_{\mathfrak{P},0}$ by the absolute Frobenius φ, we get

$$\Phi^2_{\mathrm{cris}}(\widetilde{P}(\chi))\varphi(Q) = \Phi_{\mathrm{cris}}(\widetilde{P}(\chi)) = \widetilde{P}(\chi)Q^{-1}.$$

Hence we have

$$Q^{(2)} = \varphi(Q)Q.$$

Repeating this process, we get

$$Q^{(i)} = \varphi^{i-1}(Q) \cdots \varphi(Q)Q, \qquad 1 \leqq i \in \mathbf{Z}. \tag{24}$$

We note that

$$Q^{(f)} = \chi(\mathfrak{P}) \otimes 1 \in (E \otimes_{\mathbf{Q}} K_{\mathfrak{P},0})^{\times} \tag{25}$$

which follows from (14).

Using $Q^{(i)}$, we can predict the nature of $lg_{p,K}$ in the general case. Suppose that K is abelian over a totally real number field F as in our construction of $lg_{p,K}$. Take $\tau \in \mathrm{Gal}(K/F)$. We write $lg_{p,K/F}(\mathrm{id},\tau)$ for $lg_{p,K}(\mathrm{id},\tau)$. We have

$$Q^{(i)} \in (E \otimes_{\mathbf{Q}} K_{\mathfrak{P},0})^\times \subset (E \otimes_{\mathbf{Q}} \overline{\mathbf{Q}_p})^\times \cong \prod_{\sigma:E\subset\overline{\mathbf{Q}}} \overline{\mathbf{Q}_p}^\times.$$

Let $Q^{(i)}(\sigma) \in \overline{\mathbf{Q}_p}^\times$ denote the σ-component of $Q^{(i)}$. Let $\mathfrak{p}$ be the prime ideal of F below $\mathfrak{P}$ and let f_0 be the degree of $\mathfrak{p}$ over $\mathbf{Q}$. We take a Hecke character χ of the form

$$\chi((\alpha)) = (\tau(\alpha)/\rho\tau(\alpha))^l, \qquad \alpha \equiv 1 \mod {}^\times\mathfrak{f}$$

with $1 \le l \in \mathbf{Z}$. Let $\widetilde{K}$ be the normal closure of K over $\mathbf{Q}$. We can take l and χ so that $E \subset \widetilde{K}$. Then $E \otimes_{\mathbf{Q}} \widetilde{K} \cong \prod_{\sigma\in J_E} \widetilde{K}$.

Conjecture Q. *We have*

$$lg_{p,K/F}(id,\tau^{-1}) = -\frac{1}{2l}\log_p Q^{(f_0)}(\mathrm{id}) + \sum_i a_i \log_p \epsilon_i + \log_p b$$

with $a_i \in F$, $\epsilon_i \in E_F^+$, $b \in \widetilde{K}$.

Since $\widetilde{P}(\chi)$ is determined up to the multiplication of elements in $(E \otimes_{\mathbf{Q}} K)^\times$, $Q^{(f_0)}$ is determined up to the multiplication of elements of the form $\varphi^{f_0}(c)/c$, $c \in (E \otimes_{\mathbf{Q}} K)^\times$ by (22). Therefore the validity of Conjecture Q does not depend on choices of c_B, c_{cris} and c_{DR}.

We can show, using (25), that Conjecture P is equivalent to the statement that Conjecture Q is true with $b = 1$ if $\mathfrak{p}$ is completely decomposed in K. We also note that the p-adic period can be understood more explicitly in this case.

What is remarkable here is that the nature of the p-adic absolute period symbol $lg_{p,K/F}(\mathrm{id},\tau)$ depends strongly on F in general. (This is not so when $\mathfrak{p}$ is completely decomposed in K.) In the complex case, $p_K(\mathrm{id},\tau)^{2l}$ appears in place of $Q^{(f_0)}(\mathrm{id})^{-1}$. We can show that the dictionary complex $\to$ p-adic is fairly complete.

References

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For a, $b \in \mathbf{C}$, we write $a \sim b$ if $b \neq 0$ and a/b is an algebraic number.

ρ: the complex conjugation

$\mathbf{C}_p$: the completion of an algebraic closure of $\mathbf{Q}_p$

Let F be an algebraic number field.

$\mathcal{O}_F$: the ring of integers of F

E_F, E_F^+: the group of units and the group of totally positive units of F

h_F: the class number

J_F: the set of all isomorphisms of F into $\mathbf{C}$

I_F: the free abelian group generated by J_F

Let F be totally real and put $n = [F : \mathbf{Q}]$.

$\infty_1, \ldots, \infty_n$: infinite places of F

$C_{\mathfrak{f}}$: The ideal class group modulo $\mathfrak{f}\infty_1 \cdots \infty_n$

h_0: the class number of F in the narrow sense

$\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{h_0}$: integral ideals which represent narrow ideal classes

The r-ple gamma function is defined as follows:

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_r)$, $\omega_i > 0$ for $1 \leqq i \leqq r$ and $x > 0$. We define the r-ple zeta function by

$$\zeta_r(s, \omega, x) = \sum_{\Omega = m_1\omega_1 + m_2\omega_2 + \cdots + m_r\omega_r} (x + \Omega)^{-s}.$$

Here $(m_1, m_2, \ldots, m_r)$ extends over all r-tuples of nonnegative integers. We put

$$-\log \rho_r(\omega) = \lim_{x \to +0} \left\{ \frac{\partial}{\partial s} \zeta_r(s, \omega, x) \bigg|_{s=0} + \log x \right\},$$

$$\frac{\partial}{\partial s} \zeta_r(s, \omega, x) \bigg|_{s=0} = \log \frac{\Gamma_r(x, \omega)}{\rho_r(\omega)}.$$

If $r = 1$, $\omega = 1$, we have

$$\Gamma_1(x, 1) = \Gamma(x), \qquad \rho_1(1) = \sqrt{2\pi}.$$

Let F be a totally real number field. Fix an embedding $F \subset \mathbf{C}_p$ and $\mathfrak{p}$ be the induced prime ideal of F. If $\omega_i > 0$, $x > 0$, we have a p-adic r-ple zeta function $\zeta_{p,r}(s, \omega, x)$ satisfying

$$\zeta_{p,r}(-m, \omega, x) = \zeta_r(-m, \omega, x) \qquad \text{for} \quad 0 \leqq m \in \mathbf{Z}, \quad m \equiv 0 \mod p^f - 1.$$

(f is the degree of $\mathfrak{p}$.) We define the p-adic logarithmic r-ple gamma function by

$$\frac{\partial}{\partial s} \zeta_{p,r}(s, \omega, x) \bigg|_{s=0} = L\Gamma_{p,r}(x, \omega).$$

For r linearly independent vectors v_1, v_2, …, $v_r \in \mathbf{R}^n$, put

$$C(v_1, v_2, \ldots, v_r) = \left\{ \sum_{i=1}^{r} t_i v_i \;\middle|\; t_1, t_2, \ldots, t_r > 0 \right\}$$

and call $C(v_1, v_2, \ldots, v_r)$ an r-dimensional open simplicial cone with basis v_1, v_2 , …, v_r. We have a cone decomposition with basis from $\mathcal{O}_F$ (Shintani):

$$\mathbf{R}^n_+ = \sqcup_{\epsilon \in E_F^+} \epsilon \left(\sqcup_{j \in J} C_j \right).$$

Let $r(j)$ be the dimension of C_j and put

$$C_j = C(v_{j1}, v_{j2}, \ldots, v_{jr(j)}), \qquad A_j = (v_{j1}, v_{j2}, \ldots, v_{jr(j)}).$$

For $z \in C_j$, we define the coordinate $x_i(z) \in \mathbf{R}_+$ by

$$z = x_1(z) v_{j1} + x_2(z) v_{j2} + \ldots + x_{r(j)}(z) v_{jr(j)}.$$

Put ${}^t x(z) = (x_1(z), x_2(z), \ldots, x_{r(j)}(z))$. For a fractional ideal $\mathfrak{a}$ of F, we put

$$R(C_j, \mathfrak{a}) = \{ z \in \mathfrak{a} \cap C_j \mid 0 < x_1(z), x_2(z), \ldots, x_{r(j)}(z) \leqq 1,\ {}^t x(z) \in \mathbf{Q}^{r(j)} \}.$$

For $c \in C_{\mathfrak{f}}$, take $\mathfrak{a}_\mu$ so that c and $\mathfrak{a}_\mu \mathfrak{f}$ belong to the same narrow ideal class and define a finite set by

$$R(C_j, c) = \{ z \in R(C_j, (\mathfrak{a}_\mu \mathfrak{f})^{-1}) \mid (z) \mathfrak{a}_\mu \mathfrak{f} = c \ \text{ in } \ C_{\mathfrak{f}} \}.$$

We put

$$G(c) = \sum_{j \in J} \sum_{z \in R(C_j, c)} \log \frac{\Gamma_{r(j)}(z, A_j)}{\rho_{r(j)}(A_j)},$$

$$W(c) = -\frac{1}{n} \log N(\mathfrak{a}_\mu \mathfrak{f}) \zeta_F(0, c),$$

$$X(c) = G(c) + V(c) + W(c).$$

Here $V(c)$ is a quantity of the form

$$V(c) = \sum_i a_i \log \epsilon_i, \qquad a_i \in F, \quad \epsilon_i \in E_F^+.$$

In the p-adic case, we define

$$G_p(c) = \sum_{j \in J} \sum_{z \in R(C_j, c)} L\Gamma_{p, r(j)}(z, A_j),$$

$$V_p(c) = \sum_i a_i \log_p \epsilon_i,$$

$$W_p(c) = -\log_p(\mathfrak{a}_\mu) \cdot (\zeta_F(0, c) - \zeta_F(0, \underline{\mathfrak{p}}^{-1} c)),$$

$$X_p(c) = G_p(c) + V_p(c) + W_p(c).$$

Let K be an abelian extension of F. We assume that K is a CM-field. Let $\widetilde{\mathfrak{f}}$ be the finite part of the conductor of K over F. For $\tau \in G = \mathrm{Gal}(K/F)$, we define

$$g_K(\mathrm{id}, \tau) = \pi^{-\mu(\tau)/2} \exp(\frac{1}{|G|} \sum_{\mathfrak{f} | \tilde{\mathfrak{f}}} \sum_{\chi \in (\hat{G}_-)_{\mathfrak{f}}} \frac{\chi(\tau)}{L(0, \chi)} \sum_{c \in C_{\mathfrak{f}}} \chi(c) X(c)).$$

In the p-adic case, we put

$$lg_{p,K}(\mathrm{id}, \tau) = -\frac{\mu(\tau)}{2h_K} \log_p \alpha_0 + \frac{1}{|G|} \sum_{\mathfrak{f} | \tilde{\mathfrak{f}}} \sum_{\chi \in (\hat{G}_-)_{\mathfrak{f}}} \frac{\chi(\tau)}{L(0, \chi)} \sum_{c \in C_{\mathfrak{f}}} \chi(c) X_p(c).$$

Here $\mathfrak{p}^{h_K} = (\alpha_0)$, $\alpha_0 \in F$.

Let K be a CM-field, Φ be a CM-type of K and $\sigma \in \Phi$. Taking an abelian variety A defined over $\overline{\mathbf{Q}}$ of type (K, Φ), we can define $p_K(\sigma, \Phi) \in \mathbf{C}^\times$ by

$$\int_c \omega_\sigma \sim \pi p_K(\sigma, \Phi) \qquad \text{for every} \quad c \in H_1(A, \mathbf{Z}).$$

Here $\omega_\sigma \neq 0$ is a regular 1-form defined over $\overline{\mathbf{Q}}$ such that $a^* \omega_\sigma = a^\sigma \omega_\sigma$, $a \in K$. There exists a map $p_K : I_K \times I_K \longrightarrow \mathbf{C}^\times$ with the following properties (Shimura).

(1) $p_K(\sigma, \Phi)$ is defined as above if Φ is a CM-type of K and $\sigma \in \Phi$.

(2) $p_K(\sigma_1 + \sigma_2, \tau) \sim p_K(\sigma_1, \tau) p_K(\sigma_2, \tau)$, $p_K(\sigma, \tau_1 + \tau_2) \sim p_K(\sigma, \tau_1) p_K(\sigma, \tau_2)$ for every σ, σ_1, σ_2, τ, τ_1, $\tau_2 \in I_K$.

(3) $p_K(\xi\rho, \eta) \sim p_K(\xi, \eta\rho) \sim p_K(\xi, \eta)^{-1}$ for every ξ, $\eta \in I_K$.

(4) $p_K(\xi, \mathrm{Res}_{L/K}(\zeta)) \sim p_L(\mathrm{Inf}_{L/K}(\xi), \zeta)$ if $\xi \in I_K$, $\zeta \in I_L$ and $K \subset L$, L is a CM-field.

(5) $p_K(\mathrm{Res}_{L/K}(\zeta), \xi) \sim p_L(\zeta, \mathrm{Inf}_{L/K}(\xi))$ if $\xi \in I_K$, $\zeta \in I_L$ and $K \subset L$, L is a CM-field.

(6) $p_{K'}(\gamma\xi, \gamma\eta) \sim p_K(\xi, \eta)$ if γ is an isomorphism of K' onto K.

Conjecture (complex case).

$$g_K(id, \tau) \sim p_K(id, \tau).$$

In the p-adic case, we take an embedding $K \subset \mathbf{C}_p$ and let $\mathfrak{P}$ be the induced prime ideal of K. Put $\mathfrak{P}^{h_K} = (\alpha)$, $\alpha \in K$.

Conjecture (padic case). *If $\mathfrak{p}$ is completely decomposed in K, then we have*

$$lg_{p,K}(id,\tau) = \frac{1}{2h_K}\log_p(\alpha^{\tau^{-1}\rho}/\alpha^{\tau^{-1}}) + \sum_{i=1}^{n-1} a_i \log_p \epsilon_i$$

with $a_i \in F$, $\epsilon_i \in E_F^+$.