Differential Operators on Siegel Modular Forms and Related Topics

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1 Introduction

In the workshop, I talked on the following contents.
(1) Differential operators compatible with the group action and the restriction of the domain and their relation with invariant pluri-harmonic polynomials. This includes Rankin-Cohen type operators and Eichler-Zagier-Böcherer type operators.
(2) Explicit description of the operators in several cases, partly joint work with Zagier, Eholzer, Kuzumaki.
(3) Application to explicit determination of the structure of rings or modules of scalar-valued or vector valued Siegel modular forms with integral or half-integral weight, partly joint with Aoki.
(4) Related theory of special functions, holonomic systems or system of regular singular differential equations, joint work partly with Don Zagier and partly with Takako Kuzumaki.
(5) Differential operators from vector valued Jacobi forms to a product of scalar valued Jacobi forms, a joint work with Ryoji Kyomura. (This part was talked by Kyomura.)

My study began when Don Zagier asked me in 1990 at Kyushu University if we can find a differential operator $\mathbb{D}$ acting on Siegel modular forms F of weight k of degree three such that the restriction of $\mathbb{D}(F)$ to the diagonal of the Siegel upper half space is in the tensor product of modular forms of one variable of weight $k+a_1$, $k+a_2$, $k+a_3$ with $a_i=\nu_j+\nu_k$, ν_i, ν_j, $\nu_k \in \mathbb{Z}_{\geq 0}$. He himself had a solution when $k=2$ and was looking for a solution for general k. His aim was to use this to the arithmetic intersection theory. This aim was not pursued by us at all after that. But anyway, I could answer that question and gave a generalization in 1990, including the case of Rankin-Cohen type operators(The generalization was published in 1999 though the joint paper for the former part is still in preparation.) For a long time I was thinking that these theories are just theoretical exercises and formal calculations, even after I studied jointly with Eholzer on explicit shapes of Rankin-Cohen operators in 1996. But later around in 2000, I found that these operators are really useful in various stages, for example for actual constructions of unknown Siegel modular forms. By using this kind of construction, we could construct scalar valued or vector valued Siegel modular forms of integral or half-integral weight, which are difficult to be constructed by any other known methods. These constructions were used to give an experimental evidence for my conjecture on Shimura correspondence between Siegel modular forms of degree two.

Since 1990, I continued the study of these operators unexpectedly for a long time and found that this is a more fruitful and enjoyable area than the first appearance. There are still a lot of mysterys to be solved, and many things are related. I believe this is a growing interesting area with wide scope.

In this short note I will skip any details, but would like to write some essences of various flavors of the theory.

2 Problem Setting

We prepare complex domains $D_1 \subset \mathbb{C}^N$ and $D_2 \subset D_1$. We assume that a group G is acting on D_1 biholomorphically and that D_2 is invariant by G. We also take two finite dimensional vector spaces V_1 and V_2 over $\mathbb{C}$. Let $J_i(g, Z_i)$ be a $GL(V_i)$-valued function on $G \times D_i$ which is holomorphic with respect to Z_i. We assume that J_i are automorphy factors. Namely, for any V_i-valued holomorphic functions $F(Z)$ on D_i, and for an element $g \in G$, we define an operation by $(F|_{J_i}[g])(Z) = J_i(g, Z)^{-1}F(gZ)$ and we assume that this gives an group action of G. If $V_1 = V_2$ and J_2 is obtained by the restriction of J_1 to $D_2 \times G$, then the group action is compatible with the restriction. So, for example, if F is a modular form on D_1, we get a modular form on D_2 by this restriction. We would like to generalize this procedure. For simplicity, we assume that $V_1 = \mathbb{C}$ and we write $V_2 = V$.

Problem Find a linear V-valued holomorphic differential operator $\mathbb{D}$ with constant coefficients such that for any holomorphic function F on D_1 we have

$$Res_{D_2}(\mathbb{D}(F|_{J_1}[g])) = (Res_{D_2}(\mathbb{D}(F)))|_{J_2}[g]$$

where Res_{D_2} means the restriction of functions on D_1 to the functions on D_2.

Of course the existence or the uniqueness of such operator is not assured at all in general. Besides, in case we know the existence theoretically, it is not always easy to get an explicit operator.

We treat here two typical examples related with Siegel modular forms. But before giving our general formulation, we give a prototype of our operators.

2.1 Prototype

For any natural number n, we denote by H_n the Siegel upper half space of degree n and by $Sp(n,\mathbb{R})$ the symplectic group of size $2n$. Before I started this study, there were essentially two kinds of known operators. One is the restriction from Siegel modular forms of degree two to the diagonals given in Eichler-Zagier [4] and the other is a so-called Rankin-Cohen operator. (There was also a theory by Böcherer in slightly different formulation. cf. [2].)

Case(I):
Let F be a Siegel modular form of $Sp(2,\mathbb{Z})$ of weight k. We denote by $G_{2k}(u,v)$ the polynomial of two variables defined by the generating function as follows.

$$\frac{1}{(1-2ut+vt^2)^{k-1}} = \sum_{\nu=1}^{\infty} G_{2k}^{\nu}(u,v)t^{\nu}.$$

Then $G_{2k}^{\nu}(x,1)$ is the usual Gegenbauer polynomial of degree ν. For $Z \in H_2$, we put $Z = \begin{pmatrix} \tau & z \\ z & w \end{pmatrix}$ and put

$$\mathbb{D} = G_{2k}^{\nu}\left(\frac{1}{2}\frac{\partial}{\partial z}, \frac{\partial^2}{\partial\tau\partial w}\right).$$

Then the restriction of $\mathbb{D}(F)$ to the diagonals becomes a tensor product of modular forms of H_1 of weight $k+\nu$. Namely we have

$$Res_{H_1}(\mathbb{D}(F))\left(\frac{a\tau+b}{c\tau+d}, \frac{aw+b}{cw+d}\right) = (c\tau+d)^{\nu}(cw+d)^{\nu}Res_{H_1}(\mathbb{D}(F))(\tau,z).$$

Case(II):
Let f or g be a modular form on H_1 of $SL_2(\mathbb{Z}) = Sp(1,\mathbb{Z})$ of weight k or l, respectively. Then

$$\{f,g\}_{\nu} = \sum_{\mu=0}^{\nu}(-1)^{\mu}\binom{l+\nu-1}{\mu}\binom{k+\nu-1}{\nu-\mu}\frac{\partial^{\mu} f}{\partial\tau^{\mu}}\frac{\partial^{\nu-\mu} g}{\partial\tau^{\nu-\mu}}.$$

is a modular form of $SL_2(\mathbb{Z})$ of weight $k+l+2\nu$. This is called a Rankin-Cohen operator. We note that this operator is better understood if we put

$$\mathbb{D}_\nu = \sum_{\mu=0}^{\nu} (-1)^\mu \binom{l+\nu-1}{\mu} \binom{k+\nu-1}{\nu-\mu} \frac{\partial^\nu}{\partial\tau^\mu \partial w^{\nu-\mu}}.$$

and take the restriction of the function $\mathbb{D}_\nu(f(\tau)g(w))$ on $H_1 \times H_1$ to the diagonal $(\tau,\tau) \cong H_1$,

Both cases will be generalized as we explain below.

2.2 General case

Corresponding to the prototype, we explain two cases separately.
Case (I):Restriction of $D_1 = H_n$ to the diagonals $D_2 = H_{n_1} \times \cdots \times H_{n_r}$ with $n = n_1 + \cdots + n_r$.
Here D_2 is embedded naturally into D_1 by the mapping

$$H_{n_1} \times \cdots \times H_{n_r} \ni (Z_1, \dots, Z_r) \to \begin{pmatrix} Z_1 & 0 & \dots & 0 \\ 0 & Z_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & Z_r \end{pmatrix} \in H_n$$

We put $G = Sp(n_1, \mathbb{R}) \times \cdots \times Sp(n_r, \mathbb{R})$. The action of G on D_2 is the usual one (i.e. the component wise action). The group G acts on D_1 through the diagonal embedding of G into $Sp(n, \mathbb{R})$.

$$G \ni (g_1, \dots, g_r) \to \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & g_r \end{pmatrix} \in Sp(n, \mathbb{R}).$$

We denote by (ρ_i, V_i) a rational irreducible representation of GL_{n_i} $(1 \leq i \leq r)$. We put $J(g_i, Z_i) = \rho_i(CZ_i + D)$ for $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in Sp(n_i, \mathbb{R})$. We fix a natural number k. For any function F on H_n and $g \in Sp(n, \mathbb{R})$, we write

$$(F|_k[g])(Z) = \det(CZ + D)^{-k} F(gZ)$$

and for any $V_1 \otimes \cdots \otimes V_r$ valued function on $F(Z_1, \dots, Z_r)$ on D_2 and $g = (g_1, \dots, g_r) \in G$, we write

$$F|_{\rho_1,\rho_2,\dots,\rho_r}[g] = \rho_1(CZ_i + D)^{-1} \otimes \cdots \otimes \rho_r(CZ_i + D)^{-1} F(gZ).$$

We would like to get a $V_1 \otimes \cdots \otimes V_r$ valued linear holomorphic differential operator $\mathbb{D}$ with constant coefficients acting on holomorphic functions on H_n which satisfies the following condition.

Condition 1. For any holomorphic function F on D_1 and $g \in G$, we have

$$Res_{D_2}(\mathbb{D}(F|_k[g])) = Res_{D_2}(\mathbb{D}(F))|_{\det^k \rho_1,\ldots,\det^k \rho_r}[g].$$

This condition means that the differential operator $\mathbb{D}$ gives an intertwining operator between representation of $Sp(n,\mathbb{R})$ and its restriction to G. We characterize such $\mathbb{D}$ in a more concrete way by using pluri-harmonic polynomials with an invariance property since it gives a way to calculate an explicit shape of the operator when we need. Also this interpretation enables us to treat everything inside the representation theory of compact orthogonal group and give a link to a theory of special functions or system of differential operators.

We first explain necessary notion. Let d and n be natural numbers. For any function P on the set $M_{n,d}$ of $n \times d$ real matrices and any natural numbers i, j with $1 \leq i, j \leq n$, we define differential operators $\Delta_{ij} = \Delta_{ij}(X)$ by

$$\Delta_{ij} = \Delta_{ij}(X) = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial x_{i\nu} x_{j\nu}}.$$

Any polynomial $P(X)$ on $M_{n,d}$ is called pluri-harmonic if $\Delta_{ij}P = 0$ for all i, j with $1 \leq i, j \leq n$. This is equivalent to say that $P(AX)$ is harmonic in the usual sense for any $A \in GL_n$ as a function of nd variables

For any $Z = (z_{kl}) \in H_n$, we put $\frac{\partial}{\partial Z} = \left(\frac{1+\delta_{lk}}{2}\frac{\partial}{\partial z_{kl}}\right)$. We denote by R an $n \times n$ symmetric matrix with variable components, and we denote by $Q(R)$ a homogeneous (vector valued) polyonomial of components of R, namely polynomial of $n(n+1)/2$ variables. Then any $V_1 \otimes \cdots \otimes V_r$ valued operator $\mathbb{D}$ with constant coefficients of fixed rank is expressed as $\mathbb{D} = Q(\frac{\partial}{\partial Z})$ by some $V_1 \otimes \cdots \otimes V_r$-valued polynomial Q.

Theorem 1 *We assume that $d \geq n$. An operator $\mathbb{D} = Q(\frac{\partial}{\partial Z})$ satisfies the above condition 1 if and only if Q satisfies the following two conditions.*
(1) $Q({}^tARA) = (\rho_1(A_1) \otimes \cdots \otimes \rho_r(A_r))Q(R)$, where A is any matrix of the form

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_r \end{pmatrix}$$

$(A_i \in GL_{n_i},\ A \in GL_n)$.
(2) We write a matrix with variable component by

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{pmatrix} \in M_{n,d}$$

where $X_l \in M_{n_l,d}$. Then the polynomial $P(X) = Q(X^tX)$ is pluri-harmonic with respect to each $X_l \in M_{n_l,d}$ $(1 \leq l \leq r)$, that is, $\Delta_{ij}(X_l)P = 0$ for each l and each i, j with $1 \leq i, j \leq n_l$.

Sometimes it is better to formulate this theorem by using the polynomials P instead of Q. This can be done (under the condition $d \geq n$) by imposing the following conditions on vector valued polynomials $P(X)$ on $M_{n,d}$.
(1) $P(AX) = \rho_1(A_1) \otimes \cdots \otimes \rho_r(A_r)P(X)$.
(2) $P(X)$ is pluri-harmonic for each X_i.
(3) $P(Xh) = P(X)$ for any element $h \in O(d)$.
Then by the fundamental theorem of invariant theory (see Weyl's book on classical groups), there exists a polynomial $Q(R)$ such that $Q(X^tX) = P(X)$ as in the theorem.

Case (II): Rankin-Cohen type operators

We take $D_1 = H_n^r$ and D_2 is the diagonals of D_1, namely $H_n \cong D_2 = \{(Z, Z, \ldots, Z)\} \subset H_n^r$. We put $G = Sp(n, \mathbb{R})$ and embed it to $Sp(n, \mathbb{R})^r$ diagonally. Then we can define an action of G on H_n^r and on H_n. This time we take a rational irreducible representation (ρ, V) of GL_n and put $J(g, Z) = \rho(CZ + D)$ for any $Z \in H_n$ and $g \in G$. For any V-valued function F on H_n, we write

$$(F|_\rho[g])(Z) = \rho(CZ + D)^{-1}F(gZ).$$

For a function F on D_1 and natural numbers $k_1, \ldots, k_r$, we write also

$$(F|_{k_1,\ldots,k_r}[g])(Z_1, \ldots, Z_r) = \prod_{i=1}^{r} \det(CZ_i + D)^{-k_i}F(gZ_i).$$

We would like to get a V-valued linear holomorphic differential operator $\mathbb{D}$ on holomorphic functions on D_1 which satisfies the following condition. We fix natural numbers $k_1, \ldots, k_r$.

Condition 2. For any function F on $D_1 = H_n^r$, we have

$$Res_{D_2}(\mathbb{D}(F|_{k_1,\dots,k_r}[g]) = (Res_{D_2}(\mathbb{D}(F)))|_{\det^{k_1+\cdots+k_r}\otimes\rho}.$$

Again this means that $\mathbb{D}$ gives an intertwining operator from the tensor product of holomorphic discrete series representations of $Sp(n,\mathbb{R})$ to an irreducible component.

If we apply this operator to functions $F_1(Z_1)\times\cdots\times F_r(Z_r)$ where F_i is a function on H_n, then it is more close to the usual Rankin-Cohen operator. Namely we can construct from r numbers of Siegel modular forms F_i of weight k_i a Siegel modular form of weight $\det^{k_1+\cdots+k_r}\otimes\rho$ by applying this operator and taking the restriction to D_2.

We denote by R_i the $n\times n$ symmetric matrices of variable components. Any V-valued holomorphic linear operator $\mathbb{D}$ on functions on D_1 is give by $\mathbb{D} = Q\left(\frac{\partial}{\partial Z_1},\dots,\frac{\partial}{\partial Z_r}\right)$ where $Q(R_1,\dots,R_r)$ is a V-valued polynomial of the components of R_i, namely a polynomial of $rn(n+1)/2$ variables. Anyway, characterization of such operators are given by the following theorem.

Theorem 2
We assume that $2k_i\geq n$ for all i. The operator $\mathbb{D} = Q\left(\frac{\partial}{\partial Z_1},\dots,\frac{\partial}{\partial Z_r}\right)$ satisfies Condition 2 if and only if Q satisfies the following two conditions.
(1) We have $Q(AR_1^tA,\dots,AR_r^tA) = \rho(A)Q(R_1,\dots,R_r)$ for any $A\in GL_n$.
(2) Denote by X_i $(1\leq i\leq r)$ a $n\times 2k_i$ matrix of variable components. We put $X=(X_1,\dots,X_r)$ and define a polynomial $P(X)$ by $P(X) = Q(X_1^tX_1,\dots,X_r^tX_r)$. Then P is a pluri-harmonic polynomial with respect to X.

As before, we can start from P instead of Q by imposing the condition that

$$P(X_1h_1,\dots,X_rh_r) = P(X_1,\dots,X_r)$$

for any $h_i\in O(2k_i)$ where $O(*)$ is the real compact orthogonal group.

Remark 1. In the above theorems, we assumed that k or k_i are integers. But we can easily generalize theorems to the case of half integral weight just by replacing the action of the symplectic groups in Conditions 1 and 2 into the action of the metaplectic group. As for the details, see [11].

2.3 Multiplicity

The above theorems explains relations between differential operators and certain polyonomials, but did not claim anything about existence or multiplicity (dimension) of such operators. These are essentially reduced to the classical representation theory of the compact orthogonal groups $O(2k)$. We shall see this now. We denote by $\mathcal{H}_{n,d}$ the set of all pluri-harmonic polynomials $P(X)$ on the set of matrices $X \in M_{d,n}$ of size $n \times d$. We can define an action of $GL(n) \times O(d)$ on $\mathcal{H}_{n,d}$ by $P(X) \to P({}^tAXh)$ for $(A,h) \in GL(n) \times O(d)$. So $\mathcal{H}_{n,d}$ is decomposed into irreducible representation of $GL(n) \times O(d)$. The multiplicity of each irreducible representation is one. Besides, for any irreducible representation λ of $O(d)$, there exists at most one irreducible representation τ of $GL(n)$ such that $\tau \otimes \lambda$ appears in $\mathcal{H}_{n,d}$, and vice versa.

In case (I), let us use the formulation by $P(X)$. We fix the weight k. For representations ρ_i of GL_{n_i}, under an assumption $2k \geq n_i$, there exists the irreducible representation λ_i of $O(2k)$ such that $\rho_i \otimes \lambda_i$ appears in $H_{n_i,2k}$. Then the dimension of the vector valued polynomials $P(X) = Q(X^tX)$ in the theorem is clearly equal to the multiplicity of the trivial representation of $O(2k)$ in $\lambda_1 \otimes \cdots \otimes \lambda_r$.

In case (II), we put $d = 2k_1 + \cdots + 2k_r$ and take the irreducible representation λ of $O(d)$ such that $\rho \otimes \lambda$ appears in $\mathcal{H}_{n,d}$. Then the dimension of $P(X) = Q(X_1^tX_1, \ldots, X_r^tX_r)$ is clearly equal to the multiplicity of trivial representation in the restriction of λ to $O(2k_1) \times \cdots \times O(2k_r)$, where this group is regarded as a subgroup of $O(d)$ through the diagonal embedding.

2.4 Examples of dimensions and explicit solutions

Case (I):

1. When $r = 2$, it is easily seen that $\lambda_1 \otimes \lambda_2$ contains the trivial representation if and only if $\lambda_1 = \lambda_2$, Also the multiplicity is one. But if k is very small compared with n_i, the corresponding representation of $GL(n_i)$ in $\mathcal{H}_{n_i,2k}$ is slightly restricted. If k is big enough, for any representation ρ_1 of GL_{n_i} and ρ_2 of $GL(n_2)$ which correspond to the same λ (and hence having the same Young diagram), there exists the unique differential operator (up to constant) which satisfies the condition in Theorem. No we shall see several examples when $r = 2$.

   When $n = 2m$ and $n_1 = n_2 = m$, and $\rho_1 = \rho_2 = \det^k$, we have several results. When $n = 2$ and $n_1 = n_2 = 1$, Q is essentially the Gegenbuaer polynomial. This was first given in Eichler-Zagier [4]. The differential operators in case $n = 4$ was given explicitly in [7] together with a generating function of the polynomials. The general case is being studied by myself jointly with T. Kuzumaki. To obtain the general solution of polynomials Q in question, we first describe the condition on the action of $GL_n \times GL_n$ on Q. We put

$$\begin{pmatrix} R & T \\ {}^tT & S \end{pmatrix} = \sum_{i=0}^{n} P_\alpha(R,S,T)t^\alpha.$$

For a fixed non-negative integer k, we denote by $\mathcal{A}_k$ the vector space of polyonomials Q such that $Q(AR^tA, BS^tB, ATB) = \det(AB)^kQ(R,S,T)$ for any A, $B \in GL_n$. Then the graded ring $\otimes_{k=0}^{\infty}\mathcal{A}_k$ is generated by P_1,...,P_n and $\det(T)$, and we can also show that these are algebraically independent. For an element in $\mathcal{A}_k$, we impose the condition of the pluri-harmonicity, and we get a recursive formula of the coefficients of polynomials $\sum_\nu c_\nu P_\alpha^{e_{\alpha\nu}}$. Then this recursive relation has the unique solution up to constant. This gives the differential operator. The details will appear elsewhere.

If $r = 2$ **and** λ **is spherical**, namely if λ is the representation of $O(d)$ on the harmonic polynomials of a fixed homogeneous degree, then ρ_1 or ρ_2 is the symmetric tensor representation of $GL(n_i)$ of the same degree. The polynomial Q is also described explicitly by substituting variables of Gegenbauer polynomials. As for details, see [7] p. 114.

2. Explicit examples for the case where $r \geq 3$ is treated only for the case $n = r$ and all $n_i = 1$ (See the joint work with Don Zagier [14]), In particular in the case $n = r = 3$, a very explicit result is known. For each integer k, there exists the differential operator if and only if the representation of $GL(1)$ satisfies $\rho_1(a) = a^{\nu_2+\nu_3}$, $\rho_2(a) = a^{\nu_3+\nu_1}$, $\rho_3(a) = a^{\nu_1+\nu_2}$ for some non-negative integers ν_1, ν_2, ν_3. In this case, the polynomial P is a polynomial of three vectors x_1, x_2, $x_3 \in \mathbb{R}^{2k}$. If we put $m_i = (x_i, x_i)$, $r_i = 2(x_i, x_j)$ $((i,j,k) = (1,2,3)$, $(2,3,1)$, or $(3,1,2))$, then Q is the polynomial of six variables m_i and r_i $(1 \leq i \leq 3)$. For each triple $\nu = (\nu_1, \nu_2, \nu_3)$, the polynomial Q_ν exists uniquely up to constant. For a certain normalization, it is given by the following generating function.

$$\sum_\nu Q_\nu(m,r)X_1^{\nu_1}X_1^{\nu_2}X_1^{\nu_3} = \frac{1}{R(T,X)^{k-2}\sqrt{\Delta_0(T,X)^2 - 4d(T)X_1X_2X_3}}$$

where we put

$$\begin{aligned}
\Delta_0(T,X) &= 1 - r_1X_1 - r_2X_2 - r_3X_3 + r_1m_1X_2X_3 + r_2m_2X_3X_1 + r_3m_3X_1X_2 \\
&\quad + m_1m_2X_3^2 + m_2m_3X_1^2 + m_3m_1X_2^2, \\
d(T) &= 4m_1m_2m_3 - m_1r_1^2 - m_2r_2^2 - m_3r_3^2 + r_1r_2r_3 = \frac{1}{2}\det T, \\
R(T,X) &= \frac{1}{2}\left(\Delta_0(X,T) + \sqrt{\Delta_0(T,X)^2 - 4d(T)X_1X_2X_3} \right).
\end{aligned}$$

Case (II): The branching rule of representations of $O(d)$ describes the multiplicity. There are some general description of such theory by Koike and Terada.

1. When $r=2$ and $\rho=\det^{\nu}$, under the assumption $k\geq n$, our differential operator exists if and only if ν is even, and in that case it is unique up to constant. When $\rho=\det^{k}\otimes Sym(j)$ where $Sym(j)$ is the symmetric tensor representation, we have also some result but we omit it here. (See [5]). Explicit differential operators in case $n=1$ and $r=2$ is known as the Rankin-Cohen operators as we explained. When $n=r=2$ and $\rho=\det^{k}$, an explicit formula of the differential operators was first given in Choie and Eholzer [3] in a completely different formulation, not knowing our formulation. The general case (n general and $r=2$, $\rho=\det^{k}$) was treated in [5]. There first it was shown that the polynomials $Q(R_1,R_2)$ such that $Q(AR_1^tA,AR_2^tA)=\det(A)^{2\nu}Q(R_1,R_2)$ is generated by algebraically independent $n+1$ polynomials $P_{\alpha}(R_1,R_2)$ ($0\leq\alpha\leq n$) where P_{α} is defined by
$$\det(R_1+\lambda R_2)=\sum_{\alpha=0}^{n}P_{\alpha}(R_1,R_2).$$
Then the solution we want is a polynomial of P_{α} and imposing the condition on pluri-harmonicity on this, we have a recursive formula of the coefficients of this polynomial. We can show that this has the unique solution up to constant. We do not have a complete closed formula of the coefficients for $n\geq 3$ but this recursive formula gives a constructive way to get the operator. For example, if the parameters k and ν are explicitly given, we can calculate the explicit solution by this formula. As for such examples of explicit operators for small ν for general n, see [5]. Also the result of Choie and Eholzer is an easy corollary of our results. (loc.cit.). Explicit results on vector valued case were also given in [5] and [16]. In particular, Miyawaki gave explicit solutions completely when $n=r=2$ for all the representation $\rho=\det^{k}Sym(j)$ of $GL(2)$.

2. In case of $n=1$, the differential operators for $r\geq 3$ is reduced to the iteration of differential operators for $r=2$ and we cannot get any essentially new operators. But when $n\geq 2$, the situation is more subtle. If we start from Siegel modular forms F of weight k_1 and G of weight k_2, then by Rankin-Cohen type differential operators we can get a Siegel modular form only of weight k_1+k_2+(an even number). But if we start from four Siegel modular forms, we can increase the weight by an odd number as we see later.

3 Application to construction of Siegel modular forms

The Rankin-Cohen type differential operators, namely the differential operators in case (II), are very useful to construct new Siegel modular forms from several known modular forms. There are several advantages to use this method compared with the usual construction using theta functions with spherical harmonics. For example, it can sometimes change the "parity". That is, we can construct modular forms of odd weight starting from those of even weights, or we can link modular forms of Haupt Type and of Neben type. This procedure is very simply described, so we can see easily what we are doing and it also becomes very easy to calculate the Fourier coefficients. For example, for $Z_i = \begin{pmatrix} \tau_i & z_i \\ z_i & w_i \end{pmatrix} \in H_2$ $(1 \le i \le 4)$ and for fixed natural numbers k_1, k_2, k_3, k_4, define $\mathbb{D}_{k_1,k_2,k_3,k_4}$ by

$$\mathbb{D}_{k_1,k_2,k_3,k_4} = \det \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \\ \frac{\partial}{\partial \tau_1} & \frac{\partial}{\partial \tau_2} & \frac{\partial}{\partial \tau_3} & \frac{\partial}{\partial \tau_4} \\ \frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_3} & \frac{\partial}{\partial z_4} \\ \frac{\partial}{\partial w_1} & \frac{\partial}{\partial w_2} & \frac{\partial}{\partial w_3} & \frac{\partial}{\partial w_4} \end{pmatrix}.$$

Then for four Siegel modular forms F_i of weight k_i $(1 \le i \le 4)$, the restriction of $\mathbb{D}(F_1(Z_1)F(Z_2)F(Z_3)F_4(Z_4))$ to $Z = Z_1 = Z_2 = Z_3 = Z_4 \in H_2$ gives a Siegel modular form of weight $k_1 + k_2 + k_3 + k_4 + 3$. For example, the famous Siegel modular form χ_{35} of weight 35 of degree two defined by Igusa is given by

$$\begin{aligned} \chi_{35}(Z) &= Res_{H_2}(\mathbb{D}_{4,6,10,12}(\phi_4(Z_1)\phi_6(Z_2)\chi_{10}(Z_3)\chi_{12}(Z_4))) \\ &= \begin{vmatrix} 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ \frac{\partial \phi_4}{\partial \tau} & \frac{\partial \phi_6}{\partial \tau} & \frac{\partial \chi_{10}}{\partial \tau} & \frac{\partial \chi_{12}}{\partial \tau} \\ \frac{\partial \phi_4}{\partial z} & \frac{\partial \phi_6}{\partial z} & \frac{\partial \chi_{10}}{\partial z} & \frac{\partial \chi_{12}}{\partial z} \\ \frac{\partial \phi_4}{\partial w} & \frac{\partial \phi_6}{\partial w} & \frac{\partial \chi_{10}}{\partial w} & \frac{\partial \chi_{12}}{\partial w} \end{vmatrix}, \end{aligned}$$

where ϕ_l is the Eisenstein series of weight l and χ_{10} or χ_{12} is the unique normalized cusp form of weight 10 or 12 with respect to $Sp(2, \mathbb{Z})$. By this formula, it is very easy to calculate Fourier coefficients of χ_{35} since ϕ_4, ϕ_6, χ_{10} and χ_{12} are all obtained by Saito-Kurokawa lifting and their Fourier coefficients are easily described, and we can make an experiment on L function very easily(cf. [1]). We have many results of this sort. See [1], [9], [10], [11]. We pick up the results very shortly below. All the results are on Siegel modular forms of degree two.

Scalar valued case: For any natural number N, we define the discrete subgroup $\Gamma_0(N)$ of $Sp(2,\mathbb{Z})$ as usual by

$$\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Z}); C \equiv 0 \bmod N \right\}.$$

For $\Gamma_0(N) \subset Sp(2,\mathbb{Z})$ with $N = 3$ or 4, we define a character χ_N of $\Gamma_0(N)$ by $\chi(\gamma) = \left(\frac{-N}{\det(D)}\right)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $\left(\frac{*}{*}\right)$ is the Kronecker symbol. We denote by $\Gamma_0^{\chi_N}(N)$ the kernel of χ_N which is an index two subgroup of $\Gamma_0(N)$. Each graded ring of Siegel modular forms belonging to $Sp(2,\mathbb{Z})$, $\Gamma_0(2)$, $\Gamma_0^{\chi_3}(3)$, or $\Gamma_0^{\chi_4}(4)$ is generated by four algebraically independent Siegel modular forms and the fifth Siegel modular form obtained by the above Rankin-Cohen type differential operators. All are explicitly given. The weights of the generators are given as follows.

Discrete Group	weights	fifth weight by RC-op.
$Sp(2,\mathbb{Z})$	$4,6,10,12$	35
$\Gamma_0(2)$	$2,4,4,6$	19
$\Gamma_0^{\chi_3}(3)$	$1,3,3,4$	14
$\Gamma_0^{\chi_4}(4)$	$1,2,2,3$	11

Incidentally, the fifth one in the above has also a Borcherds product expression.

Vector valued case of integral weight For natural numbers k and j, we denote by $A_{k,j}(Sp(2,\mathbb{Z}))$ the vector space of vector valued Siegel modular forms of weight $\det^k Sym(j)$ of $Sp(2,\mathbb{Z})$. For a fixed j, we put

$$\begin{aligned} A_{*,j}(Sp(2,\mathbb{Z})) &= \sum_{k=0}^{\infty} A_{k,j}(Sp(2,\mathbb{Z})), \\ A_{*,j}^{even}(Sp(2,\mathbb{Z})) &= \sum_{k=0}^{\infty} A_{2k,j}(Sp(2,\mathbb{Z})), \text{ and} \\ A_{*,j}^{odd}(Sp(2,\mathbb{Z})) &= \sum_{k=0}^{\infty} A_{2k+1,j}(Sp(2,\mathbb{Z})). \end{aligned}$$

These are modules over the graded ring $A^{even}(Sp(2,\mathbb{Z})) = \sum_{k=0}^{\infty} A_{2k}(Sp(2,\mathbb{Z}))$ of scalar valued Siegel modular forms of even weight. Generators of these modules are known in the following cases.

- $A^{even}_{*,2}(Sp(2,\mathbb{Z}))$. T. Satoh described Rankin-Cohen type differential operators for $\rho = Sym_2$ with $n = r = 2$ acting on two Siegel modular forms. He applied these operators to the scalar valued generators ϕ_4, ϕ_6, χ_{10}, χ_{12} of even weight and constructed generators of $A^{even}_{*,2}(Sp(2,\mathbb{Z}))$. These generators are not free. We omit the result here (See [17]).
- $A^{odd}_{*,2}(Sp(2,\mathbb{Z}))$. This is in [9]. The Rankin-Cohen operator acting on two Siegel modular forms of even weights(i.e. the case $r = 2$) produces only Siegel modular forms in $A^{even}_{*,j}(Sp(2,\mathbb{Z}))$. This was the reason that Satoh did not treat $A^{odd}_{*,2}(Sp(2,\mathbb{Z}))$. But we can show that there is a Rankin-Cohen operator acting on three Siegel modular forms F, G, H of weight k_1, k_2, k_3 which produces a Siegel modular form of weight $\det^{k_1+k_2+k_3+1} Sym(2)$. We denote by $\{F, G, H\}_{\det Sym(2)}$ the form constructed by this operator. Hence this can change the parity. By using these operators, we can give the following generators.

$$\{\phi_4, \phi_6, \chi_{10}\}_{\det Sym(2)} \in A_{21,2}, \quad \{\phi_4, \phi_6, \chi_{12}\}_{\det Sym(2)} \in A_{23,2},$$
$$\{\phi_4, \chi_{10}, \chi_{12}\}_{\det Sym(2)} \in A_{27,2}, \quad \{\phi_6, \chi_{10}, \chi_{12}\}_{\det Sym(2)} \in A_{29,2}.$$

  These are not free and satisfy one linear relation over $A^{even}(Sp(2,\mathbb{Z}))$. We omit the details.
- $A^{even}_{*,4}(Sp(2,\mathbb{Z}))$:
  For any even non-negative k and an even natural number j, there exists a Rankin-Cohen type differential operator which produces a Siegel modular form of weight $\det^{k_1+k_2+k} Sym(j)$ from Siegel modular forms F of weight k_1 and G of weight k_2. (As for existence, see [5] p.460. As for explicit shapes of the operators, see [5] p.460–461 and [16].) We denote this new Siegel modular form by $\{F, G\}_{\det^k Sym(j)}$. If $k = 0$ we write $\{F, G\}_{Sym(j)}$. Then we can show that $A^{even}_{*,4}(Sp(2,\mathbb{Z}))$ is a free $A^{even}(Sp(2,\mathbb{Z}))$ module and generators are given by

$$\{\phi_4, \phi_4\}_{Sym(4)} \in A_{8,4}, \qquad \{\phi_4, \phi_6\}_{Sym(4)} \in A_{10,4},$$
$$\{\phi_4, \phi_6\}_{\det^2 Sym(4)} \in A_{12,4} \qquad \{\phi_4, \chi_{10}\}_{Sym(4)} \in A_{14,4},$$
$$\{\phi_6, \phi_{10}\}_{Sym(4)} \in A_{16,4}.$$

  As for details see [10].

- $A^{odd}_{*,4}(Sp(2,\mathbb{Z}))$:
  This is a free $A^{even}(Sp(2,\mathbb{Z}))$ module and the generators are given by

$$\{\phi_4,\phi_4,\psi_6\}_{\det Sym(4)} \in A_{15,4}, \quad \{\phi_4,\phi_6,\psi_6\}_{\det Sym(4)} \in A_{17,4},$$
$$\{\phi_4,\psi_4,\chi_{10}\}_{\det Sym(4)} \in A_{19,4}, \quad \{\phi_4,\psi_4,\chi_{12}\}_{\det Sym(4)} \in A_{21,4},$$
$$\{\phi_4,\psi_6,\chi_{12}\}_{\det Sym(4)} \in A_{23,4}.$$

  where $\{F,G,H\}_{Sym(4)}$ is a Rankin-Cohen type differential operator. As for details, see [10].

- $A^{even}_{*,6}(Sp(2,\mathbb{Z}))$:
  This is a free $A^{even}(Sp(2,\mathbb{Z}))$ module and the generators are given by

$$\begin{aligned} \phi_{6,6} &\in A_{6,6}, & X_8 &\in A_{8,6}, X_{10} \in A_{10,6} \\ \{\phi_4,\chi_6\}_{\det^2 Sym(6)} &\in A_{12,2}, & \{\phi_4,\chi_{10}\}_{Sym(6)} &\in A_{14,2}, \\ \{\phi_4,\chi_{12}\}_{Sym(6)} &\in A_{16,2}, & \{\phi_6,\chi_{12}\}_{Sym(6)} &\in A_{18,2}. \end{aligned}$$

  where $\phi_{6,6}$ is the Eisenstein series of weight $\det^6 Sym(6)$, and X_8 and X_{10} are certain Siegel modular forms explicitly given by theta functions with spherical functions. For details, see [10].

Remark If we consider the direct sum of all $A_{k,j}(Sp(2,\mathbb{Z}))$, namely if we take $\otimes_{k=0,j=0}^{\infty} A_{k,j}(Sp(2,\mathbb{Z}))$, then this becomes a ring by taking the tensor product as a product. (We note that each irreducible representation of $GL(2)$ in $Sym(j)\otimes Sym(j')$ has always multiplicity one, so the product is well defined.) But this big ring is not finitely generated over $A^{even}(Sp(2,\mathbb{Z}))$. In case of Jacobi forms on $H_1\times\mathbb{C}$, a similar situation happens and if we replace the notion of Jacobi forms by "weak Jacobi forms", then the ring becomes finitely generated and the generators are explicitly described (Eichler-Zagier [4]). So, it would be an interesting problem to find a definition of a ring of "weak Siegel modular forms" which contain this ring and is finitely generated.

Vector valued with half-integral weight: We review the definition very briefly. Put $\theta(Z)=\sum_{p\in\mathbb{Z}^2} exp(2\pi i^t pZp)$. Let ψ be the character of $\Gamma_0(4)$ defined by $\psi(\gamma)=\chi_4(\gamma)=\left(\frac{-4}{\det(D)}\right)$. Now let χ be the trivial character or ψ of $\Gamma_0(4)$. For any non-negative integer k and j, we say that a $\mathbb{C}^{j+1}$-valued holomorphic function F on H_2 is a Siegel modular form of weight $\det^{k-1/2} Sym(j)$ with character χ if F satisfies

$$F(\gamma Z)=\chi(\gamma)(\theta(\gamma Z)/\theta(Z))^{2k-1} Sym(j)(CZ+D)F(Z)$$

for all $\gamma \in \Gamma_0(4)$. We denote by $A_{k-1/2,j}(\Gamma_0(4))$ or by $A_{k-1/2,j}(\Gamma_0(4),\psi)$ the space of vector valued Siegel modular forms of weight $\det^{k-1/2} Sym(j)$ of respect to $\Gamma_0(4)$ with the trivial character or with the character ψ. We call the former "Haupt type" and the latter "Neben type", imitating Hecke's naming. We put $A = \sum_{k=0}^{\infty} A_k(\Gamma_0(4),\psi^k)$, where $A_k(\Gamma_0(4),\psi^k)$ is the space of scalar valued Siegel modular forms of $\Gamma_0(4)$ of integral weight k with character ψ^k. Obviously this becomes a ring. For a fixed j, we put $A_{half,j}(\Gamma_0(4)) = \sum_{k=0}^{\infty} A_{k+1/2,j}(\Gamma_0(4))$ and $A_{half,j}(\Gamma_0(4),\psi) = \sum_{k=0}^{\infty} A_{k+1/2,j}(\Gamma_0(4),\psi)$. These are modules over the ring A. The ring A is generated as a ring over $\mathbb{C}$ by four generators f_1, g_2, x_2, f_3 of weight 1, 2, 2, 3, respectively(See [6]). All are given explicitly by using theta constants but we omit the details. Now, we would like to describe the module structure of the space of vector valued Siegel modular forms of half integral weight over A. We have the same sort of Rankin-Cohen type differential operators on three modular forms which "increases" weight by $\det \otimes Sym(2)$ or $\det \otimes Sym(4)$. These will be denoted by $\{F,G,H\}_{\det Sym(j)}$ as before.

- $A_{half,2}(\Gamma_0(4))$. This is known by Tsushima. He imitated the paper by T. Satoh [17]. We omit the details here (cf. [18].)
- $A_{half,2}(\Gamma_0(4),\psi)$. This is a free A module of rank 3 and generated by

$$\begin{aligned} \{\theta, g_2, x_2\}_{\det Sym(2)} &\in A_{11/2,2}(\Gamma_0(4),\psi), \\ \{\theta, g_2, f_3\}_{\det Sym(2)} &\in A_{13/2,2}(\Gamma_0(4),\psi), \\ \{\theta, x_2, f_3\}_{\det Sym(2)} &\in A_{13/2,2}(\Gamma_0(4),\psi), \end{aligned}$$

  where $\theta = \theta(Z)$. As for the details, see [11].
- $A_{half,4}(\Gamma_0(4),\psi)$. This is not a free A module. It is generated by 6 elements which have one (explicitly written) linear relation over A. These generators are given by

$$\begin{array}{lll} \{f_1, g_2, \theta\}_{\det Sym(4)}, & \{f_1, x_2, \theta\}_{\det Sym(4)}, & \{f_1, g_2, \theta\}_{\det Sym(4)}|U(4), \\ \{g_2, g_2, \theta\}_{\det Sym(4)}, & \{g_2, \theta, x_2\}_{\det Sym(4)}, & \{x_2, x_2, \theta\}_{\det Sym(4)}, \end{array}$$

  where $f_1 = \theta^2$ and $U(4)$ is the Hecke operator (with respect to the image of $\Gamma_0(4)$ in the metaplectic group) defined by $F(Z)|U(4) = \sum_T a(4T) exp(2\pi i Tr(TZ))$ for $F(Z) = \sum_T a(T) exp(2\pi i Tr(TZ))$. As for details, see [11].

4 Holonomic system and Special functions

Here we treat only the case (I). As we explained shortly already, when $n = r = 2$ and $n_1 = n_2 = 1$, then the differential operators are obtained from the Gegenbauer polynomials. Hence there is a large possibility that we can develop a new kind of theory of special functions in other cases too. This is partly excuted for $n = r = 3$ and for $(n, r) = (2m, 2)$.

First we explain our prototype (Gegenbauer polynomial case) more closely. When $n = r = 2$ and $n_1 = n_2 = 1$, the polynomial P in Theorem 1 for $\rho(a) = a^\nu$ is described as follows.
(1)P is a polynomial $P(x, y)$ of x, $y \in \mathbb{R}^d$. $(d = 2k)$.
(2)$P(ax, ay) = a^\nu P(x, y)$.
(3)$\Delta_x P = \Delta_y P = 0$, where Δ_x or Δ_y is the usual Laplacian with respect to x or y, respectively.
Now, let us recall the usual theory of the spherical representation of $O(d)$ and the Gegenbauer polynomials. The real compact orthogonal group $O(d)$ of size d acts on the space of homogenous harmonic polynomials of d variables of degree ν. For a fixed ν, this is an irreducible representation and of class one, namely it has one-dimensional fixed vectors by $O(d-1)$, where $O(d-1)$ is embedded naturally in $O(d)$. The $O(d-1)$ fixed vector can be regarded as a function on $O(d-1)\backslash O(d)/O(d-1) \cong S^{d-1}/O(d-1)$ (the $d-1$ dimensional unit sphere.) Also this is a function of the first component x_1 of $x \in \mathbb{R}^d$. This is the Gegenbauer polynomial of degree ν. This is the usual explanation.

Now we change the formulation a little. We start from a polynomial $P(x, y)$ of $2d$ variables (x, y) where x, $y \in \mathbb{R}^d$. We assume that $P(x, y)$ is homogenous of degree ν as a polynomial of each x or y, respectively, namely we have $P(ax, by) = (ab)^\nu P(x, y)$. We also assume that $P(x, y)$ is harmonic with respect to each x or y: $\Delta_x P = \Delta_y P = 0$, where $\Delta_x = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ and $\Delta_y = \sum_{i=1}^d \frac{\partial^2}{\partial y_i^2}$. We denote the space of such polynomials by $H(d, \nu)$. Then $O(d) \times O(d)$ acts on $H(d, \nu)$ by $P(x, y) \to P(xh_1, yh_2)$. We embed $O(d)$ diagonally into $O(d) \times O(d)$. Then vectors in $H(d, \nu)$ fixed by $O(d)$ forms one dimensional subspace. For such invariant polynomial $P(x, y)$, there exists a polynomial $Q(m, r)$ of two variables such that $Q(n(x)n(y), (x, y)) = P(x, y)$, where $n(x)$, $n(y)$ are norms and (x, y) is the usual inner product. Then $Q(1, r)$ (with a certain normalization) is the classical Gegenbauer polynomial of degree ν. Now by homogenuity condition, the polynomial $Q(m, r)$ is essentially a polynomial of $r/\sqrt{m}$, namely we have $Q(m, r) = m^{\nu/2} Q(1, r/\sqrt{m})$. In order to get the differential operator of Gegenbauer polynomials $Q(1, \xi)$, we change variables of the Laplacian Δ_x from x to $(n(x), \xi)$ where $\xi = r/\sqrt{m}$.

Then we can separate these variables in the differential equation $\Delta_x P = 0$. Indeed we have

$$\begin{aligned} & (m)^{-\nu} n(x)\Delta_x P(x,y) \\ = \;& (1-\xi^2)\frac{\partial^2 Q(1,\xi)}{\partial \xi^2} - (d-1)\xi\frac{\partial Q(1,\xi)}{\partial \xi} + \nu(\nu+d-2)Q(1,\xi) = 0. \end{aligned}$$

This gives the Gegenbauer differential equation. This is of Fuchsian type and has two linearly independent solutions, one of which is the Gegenbauer polynomial when ν is a positive integer. Also for a fixed d, the Gegenbauer polynomials are orthogonal polynomials on the interval $[-1,1]$ with respect to the measure $(1-\xi^2)^{(d-3)/2}d\xi$. There are a lot of classical theories of special functions about the solutions of this equation for complex parameters ν.

Now we imitate this in the case $n = r = 3$ and $n_1 = n_2 = n_3 = 1$ (For the details, see [14]). We consider polynomials $P(x,y,z)$ (x, y, $z \in \mathbb{R}^d$) such that

(1) $P(a_1x, a_2y, a_3z) = a_1^{\nu_2+\nu_3}a_2^{\nu_3+\nu_1}a_3^{\nu_1+\nu_2}P(x,y,z)$ where ν_i are fixed non-negative integers, and

(2) P is harmonic for each x, y, z, namely $\Delta_x P = \Delta_y P = \Delta_z P = 0$.

(3) P is invariant by $O(d)$, i.e. $P(xh, yh, zh) = P(x,y,z)$ for any $h \in O(d)$.

Then we have a polynomial $Q(m_1, m_2, m_3, r_1, r_2, r_3)$ of 6 variables $m_1 = n(x)$, $m_2 = n(y)$, $m_3 = n(z)$, $r_1 = 2(y,z)$, $r_2 = 2(z,x)$, $r_3 = 2(x,y)$ such that $Q(m_1, m_2, m_3, r_1, r_2, r_3) = P(x,y,z)$. We put $\xi_i = r_i/(2\sqrt{m_j m_k})$ where $(i,j,k) = (1,2,3)$, $(2,3,1)$ or $(3,1,2)$. We can show that for each $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$, there exists the unique polynomial P as above up to constant(this fact has been already explained in the previous section), and the polynomials $Q_{\boldsymbol{\nu}}(1,1,1,2\xi_1,2\xi_2,2\xi_3)$ for various multi-indices $\boldsymbol{\nu} = (\nu_1,\nu_2,\nu_3)$ are orthogonal polynomials of three variables (ξ_1,ξ_2,ξ_3) on the domain

$$\left\{ T_0 = \begin{pmatrix} 1 & \xi_3 & \xi_2 \\ \xi_3 & 1 & \xi_1 \\ \xi_2 & \xi_1 & 1 \end{pmatrix} ; T_0 \text{ is positive definite.} \right\}$$

with the measure $det(T_0)^{(d-4)/2}d\xi_1 d\xi_2 d\xi_3$.

Now we would like to get a similar (partial) differential equations as the Gegenbauer ordinary differential equation.

Rewriting the Laplacian by these new variables m_i and ξ_i and separating the variables, we get the following system of differential equations with respect to three variables ξ_1, ξ_2, ξ_3.

$$\begin{aligned}
\mathbb{D}_1 &= (1-\xi_2^2)\frac{\partial^2}{\partial \xi_2^2} + (1-\xi_3^2)\frac{\partial^2}{\partial \xi_3^2} + 2(\xi_1 - \xi_2\xi_3)\frac{\partial^2}{\partial \xi_2 \partial \xi_3} \\
&\quad -(d-1)(\xi_2\frac{\partial}{\partial \xi_2} + \xi_3\frac{\partial}{\partial \xi_3}) + (\nu_2+\nu_3)(\nu_2+\nu_3+d-2), \\
\mathbb{D}_2 &= (1-\xi_3^2)\frac{\partial^2}{\partial \xi_3^2} + (1-\xi_1^2)\frac{\partial^2}{\partial \xi_1^2} + 2(\xi_2 - \xi_3\xi_1)\frac{\partial^2}{\partial \xi_3 \partial \xi_1} \\
&\quad -(d-1)(\xi_3\frac{\partial}{\partial \xi_3} + \xi_1\frac{\partial}{\partial \xi_1}) + (\nu_3+\nu_1)(\nu_3+\nu_1+d-2). \\
\mathbb{D}_3 &= (1-\xi_1^2)\frac{\partial^2}{\partial \xi_1^2} + (1-\xi_2^2)\frac{\partial^2}{\partial \xi_2^2} + 2(\xi_3 - \xi_1\xi_2)\frac{\partial^2}{\partial \xi_1 \partial \xi_2} \\
&\quad -(d-1)(\xi_1\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial \xi_2}) + (\nu_1+\nu_2)(\nu_1+\nu_2+d-2).
\end{aligned}$$

Usually, systems of partial differential equations have so many solutions, and often it has meaning only after giving an boundary condition and so on. But there are rare cases where the space of solutions of the system is a finite dimensional vector space without adding any condition. There is a very good theory on such holonomic systems (cf. Ochiai's article in this volume), but it is not easy at all to give simple explicit examples of such system. Also such system would be a clue to a new theory of special functions and we could say that any new examples are precious models for new mathematics. In our case, by computer calculation, we can check that this system is holonomic, equivalent to a Pfaffian system of rank 8, and regular singular. (As for the meaning of these terminology, see the article by H. Ochiai in this volume.) Hence there are 8 linearly independent solutions. It will be a very interesting question to solve this equation explicitly in any sense for general (not necessarily integer) parameters d and ν_i. Several such attempt has been already done. For example, we can write down all the solutions by elementary functions when d is even and $(\nu_1, \nu_2, \nu_3) = 0$. The general case is still in progress. (See [14] Part II.)

In the above we explained the case of $n = r = 3$. But it is also very interesting to consider the case $r = 2$ and $n = 2m$, $n_1 = n_2 = m$ and $\rho = \det^k$, since this is in a sense a direct generalization of the classical Gegenbauer polynomial case. This is my recent joint work with T. Kuzumaki. In this case, the polynomial Q which gives the differential operator is a polynomial $Q(R, S, T)$ of $R =^t R$, $S =^t S$, $T \in M_2$. By the relation

$$(*) \qquad Q(AR^tA, BS^tB, AR^tB) = \det(AB)^\nu Q(R, S, T),$$

we see that P depends essentially only on m parameters. There are several choices for these parameters. For example we can take equare roots of eigenvalues λ_i $(1 \leq i \leq n)$ of $R^{-1/2}TS^{-1}\,{}^tTR^{-1/2}$ as parameters. Then the polynomial Q is written by $\det(RS)^\nu$ and λ_i. But this time, if we try to rewrite the operator $\Delta_{i,j}$ for the condition of pluri-harmonicity by $\det(RS)$ and λ_i, then there remains terms which cannot be expressed by $\det(RS)$ and λ_i. As we already explained, the polynomials which satisfy (*) is generated by P_α and $\det(T)$. The ring R generated by these are not stable under the action of $\Delta_{ij}(X)$ and images are in the free module over R of rank m, where the generators are outside R. Hence for the image of polynomials Q under differential operators, each coefficient of this module should vanish. Since each coefficient is an image of some differential operator, we can get m numbers of differential operators from $\Delta_{ij}(X)$. We get essentially the same differential operators from $\Delta_{ij}(X)$ and $\Delta_{ij}(Y)$ with $1 \leq i, j \leq n$. So we get a system of m differential operators. We can show that for $m = 1$, 2 and 3, this system is holonomic of rank 2^m. We believe that for any m the system should be holonomic and of rank 2^m, though the study is still in progress. (See [13].)

5 Vector valued Jacobi forms and scalar valued Jacobi forms

This section is a joint work with Kyomura (and talked by him in the workshop) and this is not directly related to the other sections. But since here appears other types of differential operators, we include this in this article. As for details, see [12].

Scalar valued Jacobi forms are usually defined as holomorphic functions which satisfy the same relations satisfied by the coefficients at a fixed half-integral matrix S of the Fourier-Jacobi expansion of a scalar valued Siegel modular form of weight k satisfy. So they have two parameters, i.e. weight k and index S. If we want to generalize the definition of Jacobi forms to a vector valued case, one way to do so is to change the automorphy factor $\det(CZ + D)^k$ of weight k to $\rho(CZ + D)$ for a higher dimensional representation ρ of GL_n. (cf. Ziegler [19]). This definition has good meaning since still it has a close connection with vector valued Siegel modular forms of half-integral weight. But if we consider the Fourier-Jacobi coefficients of vector valued Siegel modular forms, this definition is not enough even in the case of Jacobi forms on $H_1 \times \mathbb{C}$. So, we give slightly more complicated definition of vector valued Jacobi forms which can appear as coefficients of vector valued Siegel modular forms. Our theory is to connect this new definition and the usual definition by differential operators at least in the simplest case $H_1 \times \mathbb{C}$. The general case (i.e. the case of functions on $H_n \times \mathbb{C}^n$ etc.) still remains to be solved.

Let $\rho_j = Sym(j)$ and $\rho_{k,j} = \det^k Sym(j)$ be the irreducible representation of GL_2, notation being as before. For any natural numbers k and m, any $\mathbb{C}^{j+1}$-valued function ϕ on $\mathcal{H}_1 \times \mathbb{C}$, any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and any $(X, \kappa) = ([\lambda, \mu], \kappa) \in \mathbb{R}^3$, we write

$$\begin{aligned} (\phi|_{(k,j),m}[g])(\tau, z) &= \rho_{k,s} \begin{pmatrix} c\tau + d & cz \\ 0 & 1 \end{pmatrix}^{-1} e^m \left(-\frac{cz^2}{c\tau + d} \right) \phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right), \\ (\phi|_{j,m}(X, \kappa))(\tau, z) &= \rho_j \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^{-1} e^m(\lambda^2\tau + 2\lambda z + \lambda\mu + \kappa)\phi(\tau, z + \lambda\tau + \mu). \end{aligned}$$

If F is a Siegel modular form of degree two of weight $\rho_{k,j}$ with respect to $Sp(2, \mathbb{Z})$ with $F(Z) = \sum_{m=0}^{\infty} \theta_m(\tau, z) exp(2\pi i m w)$, then its Fourier-Jacobi coefficient θ_m (which is a vector valued function) is invariant by the Jacobi group $SL_2(\mathbb{Z})^J$ by this action. So, let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. We say that a $\mathbb{C}^{j+1}$-valued holomophic function ϕ on $H_1 \times \mathbb{C}$ is a vector valued Jacobi forms of weight $\rho_{k,j}$ belonging to Γ^J if it satisfies the following three conditions.

(1) $\phi|_{(k,j),m}\gamma = \phi$ for all $\gamma \in \Gamma$.

(2) $\phi|_{j,m}[X] = \phi$ for all $X \in \mathbb{Z}^2$.

(3) For each $\gamma \in SL_2(\mathbb{Z})$, the function $\phi|_{(k,j),m}\gamma$ has the Fourier expansion of the form

$$\phi = \sum_{\substack{r^2 \le 4mn \\ n, r \in \mathbb{Z}}} C(n, r) exp(2\pi i \tau) exp(2\pi i z).$$

We denote the space of these functions by $J_{(k,j),m}(\Gamma^J)$. Directly by definition, we see that the last component of $\phi \in J_{(k,j),m}(\Gamma^J)$ belongs to the space $J_{k,m}(\Gamma^J)$ of scalar valued Jacobi forms of weight k and of index m with respect to $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$. The condition for other components contains automorphy factor of $J_{k+l}(\Gamma^J)$ in some part but also has "error terms". We would like to know if these "error terms are erased somehow. We can correct this by taking derivatives of other components. More precisely the answer to this is as follows. We have a linear isomorphism

$$J_{(k,j),m}(\Gamma^J) \to J_{k,m}(\Gamma^J) \times J_{k+1,m}(\Gamma^J) \times \cdots \times J_{k+j,m}(\Gamma^J)$$

and this mapping can be given explicitly by differential operators. The essence of this assertion is the existence of a differential operator commuting with actions of Jacobi group over the real field on both sides with automorphy factor. Namely, the above assertion is a direct corollary of the following theorem.

Theorem 3 (joint with Kyomura) *For any natural numbers k, j and m, there exists a linear holomorphic differential operator $\mathbb{D}_{(k,j),m}$ with constant coefficients from the space W of $j+1$ dimensional vectors of functions on $H_1 \times \mathbb{C}$ to W itself such that*

$$\left(\mathbb{D}_{(k,j),m}(f|_{(k,j),m}[M])\right)_\mu = (\mathbb{D}_{(k,j),m}(f))_\mu|_{k+\mu,m}[M]$$

for both $M \in SL_2(\mathbb{R})$ and $M = [X,\kappa] \in \mathbb{R}^3$, where suffices μ means the $\mu+1$-th components of the vector for $0 \le \mu \le j$. (In short it commutes with the action of the real Jacobi group.)

We can write down $\mathbb{D}_{(k,j),m}$ explicitly but it involves much notation. (See [12]). We give here only the example for $j=2$. We put $\partial_z = \frac{1}{2\pi i}\frac{\partial}{\partial z}$, $\partial_\tau = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$ and $L_m = 4m\partial_\tau - \partial_z^2$. For $j=2$, the above differential operator is given by

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \phi_0 \\ \phi_1 - m^{-1}\partial_z\phi_0 \\ \phi_2 - (2m)^{-1}\partial_z\phi_1 + (2m)^{-2}\left(\partial_z^2\phi_0 + \frac{1}{2k-1}L_m\phi_0\right) \end{pmatrix}$$

The inverse mapping is also calculated easily. For example, if we take $f_i \in J_{k+i,m}(\Gamma^J)$ for $i = 0$, 1 and 2, then the following vector

$$\begin{pmatrix} f_2 + (2m)^{-1}\partial_z f_1 + (2m)^{-2}\left(\partial_z^2 f_0 - \frac{1}{2k-1}L_m f_0\right) \\ f_1 + m^{-1}\partial_z f_0 \\ f_0 \end{pmatrix}$$

belongs to $J_{(k,j),m}(\Gamma^J)$.

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