COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 41, No. 2 1992

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

On Jacobi Forms and Siegel Modular Forms of Half Integral Weights

by

Tomoyoshi IBUKIYAMA†

(Received July 6, 1992)

A certain good relation between modular forms of one variable of half integral weight and Jacobi forms was given by Maass [5], Zagier [14], Eichler-Zagier [1], and used to prove the Saito-Kurokawa conjecture on the lifting of modular forms of degree one to Siegel modular forms of degree two. Some general theory of Jacobi forms of general degree has been developed in A. Murase [6], [7], G. Shimura [9], T. Yamazaki [13] and Ziegler [15]. In this paper, we shall give a bijection between some $\boldsymbol{C}$-valued Siegel modular forms of half integral weight and Jacobi forms of general degree preserving the action of Hecke operators. More precisely, first, we shall show that the space of Jacobi forms of degree n of weight k with index 1 with respect to the symplectic full modular group is linearly isomorphic to some explicitly described subspace of the space of $\boldsymbol{C}$-valued Siegel modular forms of degree n of weight $k-\frac{1}{2}$ with respect to the discrete subgroup $\Gamma_0(4)\subset Sp(n,\boldsymbol{Z})$. This subspace is a generalization of the *plus* space introduced by Kohnen [4] when $n=1$. This is a kind of refinement of a part of the results of Shimura [9] on isomorphisms between vector valued Siegel modular forms and jacobi forms. Also, a similar kind of correspondence is known by Ziegler ([15] p. 210, 211), but our results are slightly different from his.

Secondly, we shall compare the action of Hecke operators on both spaces. We shall state our main results Theorems 1 and 2 in §1, and prove them in §2, 3.

1. Main results

In this section, after reviewing several definitions and notations, we shall state our main results.

1.1. Linear isomorphisms.

For any natural integer n, we denote by H_n the Siegel upper half space of genus n defined by:

$$H_n=\{X+iY\in M_n(\boldsymbol{C});\ {}^tX=X,\ {}^tY=Y\in M_n(\boldsymbol{R}),\ Y>0\ (Y\text{: positive definite})\}\ .$$

† A part of the results in this paper was obtained while I was supported by SFB 40 and Max Planck Institute für Mathematik in Bonn in 1982–1984

We denote by $Sp(n, \boldsymbol{R})$ the usual real symplectic group of size $2n$:

$$Sp(n, \boldsymbol{R}) = \{g \in M_{2n}(\boldsymbol{R});\ gJ_n {}^t g = J_n\},$$

where

$$J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

The group $Sp(n, \boldsymbol{R})$ acts on $H_n \times \boldsymbol{C}^n$ as usual by:

$$g(\tau, z) = (g\tau, {}^t(c\tau+d)^{-1}z) = ((az+b)(c\tau+d)^{-1}, {}^t(c\tau+d)^{-1}z),$$

for any

$$(\tau, z) \in H_n \times \boldsymbol{C}^n \quad \text{and} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \boldsymbol{R}) \qquad (a, b, c, d \in M_n(\boldsymbol{Z})).$$

Let k be a natural integer. We assume throughout this paper that k is even. For any such k and any holomorphic function $F(\tau, z)$ on $H_n \times \boldsymbol{C}^n$, we consider the following conditions (1), (2):

(1) $F(\tau, z+\tau x+y) = e(-({}^t x\tau x + 2{}^t xz))F(\tau, z)$
for all the column vectors $x, y \in \boldsymbol{Z}^n$.

(2) $F|[g]_k = F$ for all $g = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in Sp(n, \boldsymbol{Z})$,

where we write

$$(F|[g]_k)(\tau, z) = F(g\tau, {}^t(c\tau+d)^{-1}z)\det(c\tau+d)^{-k}e(-{}^t z(c\tau+d)^{-1}cz),$$

and $e(x) = e^{2\pi ix}$. The mapping $F \to F|[g]_k$ defines an action of $Sp(n, \boldsymbol{Z})$ on holomorphic functions on $H_n \times \boldsymbol{C}^n$. If F satisfies the above (1) and (2), then it is easy to see that F has the Fourier expansion of the following form:

$$F(\tau, z) = \sum_{N,r} a(N, r)e(\mathrm{tr}(N\tau))e({}^t rz),$$

where N runs over n by n symmetric half integral matrices, r over all the elements of $\boldsymbol{Z}^n$.

We say that F is a Jacobi form (resp. Jacobi cusp form) of weight k with index 1, if F satisfies the above conditions (1), (2) and besides its Fourier coefficients satisfy

(3) $a(N, r) = 0$, unless $\begin{pmatrix} N & r/2 \\ {}^t r/2 & 1 \end{pmatrix}$ is positive semi-definite (resp. positive definite).

Incidentally, $\begin{pmatrix} N & r/2 \\ {}^t r/2 & 1 \end{pmatrix}$ is positive semi-definite (resp. definite), if and only if $N - \frac{1}{4}r{}^t r$ is positive semi-definite (resp. definite). We denote by $J_{k,1}^{(n)} = J_{k,1}$, or $J_{1,k}^{cusp}$ the whole space of the Jacobi forms, or Jacobi cusp forms, of weight k with index 1, respectively. Now, we shall review some facts on Jacobi forms. For any column vectors $m', m'' \in \boldsymbol{Z}^n$, we define theta functions $\theta_m(\tau, z) = \theta_{m',m''}(\tau, z)$ of characteristic $m = {}^t({}^t m', {}^t m'')$ as usual as follows:

$$\theta_{m',m''}(\tau, z) = \sum_{p \in \mathbf{Z}^n} e\left(\frac{1}{2}\,{}^t\left(p+\frac{m'}{2}\right)\tau\left(p+\frac{m'}{2}\right)+{}^t\left(p+\frac{m'}{2}\right)\left(z+\frac{m''}{2}\right)\right).$$

For each vector $\mu \in \mathbf{Z}^n$, we set

$$\vartheta_\mu(\tau, z) = \theta_{\mu,0}(2\tau, 2z).$$

This function $\vartheta_\mu(\tau, z)$ depends only on μ mod 2. The following fact is known (cf. [1], [5], [10], [13].) For each element $F \in J_{k,1}$, there exists a set of 2^n numbers of holomorphic functions $F_\mu(\tau)$ $(\mu \in (\mathbf{Z}/2\mathbf{Z})^n)$ on H_n such that

$$F(\tau, z) = \sum_{\mu \in (\mathbf{Z}/2\mathbf{Z})^n} F_\mu(\tau)\vartheta_\mu(\tau, z) \tag{1.1}$$

and the function $F_\mu(\tau)$ is uniquely determined by F and μ.

Next, we review on Siegel modular forms of half integral weights. As usual, we denote by $\theta_{m'm''}(\tau)$ the theta constant of characteristic ${}^t({}^tm', {}^tm'')$: $\theta_{m'm''}(\tau) = \theta_{m'm''}(\tau, 0)$. To define an automorphy factor of half integral weight, put

$$\theta(\tau) = \theta_{00}(2\tau, 0) = \sum_{p \in \mathbf{Z}^n} e({}^tp\tau p).$$

We denote by $\Gamma_0(4)$ the subgroup of $Sp(n, \mathbf{Z})$ defined by

$$\Gamma_0(4) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \mathbf{Z});\ c \equiv 0 \bmod 4 \right\}.$$

It is well known and easy to see that

$$\theta(g\tau)^2/\theta(\tau)^2 = sgn(\det d)\det(c\tau + d)$$

for any $g \in \Gamma_0(4)$. (cf. Zhuravrev [11]) For any natural integer k, any function h on H_n, and any $g \in Sp(n, \mathbf{R})$, we write

$$h|[g]_{k-1/2} = h(g\tau)\frac{\theta(g\tau)}{\theta(\tau)}\det(c\tau + d)^{-k}.$$

This defines an action of $\Gamma_0(4)$ on holomorphic functions on H_n. As we have assumed that k is even, we get

$$\left(\frac{\theta(g\tau)}{\theta(\tau)}\right)^{2k-1} = \frac{\theta(\tau)}{\theta(g\tau)}\det(c\tau + d)^k.$$

In this paper, for any even natural integer k, we say that a holomorphic function h on H_n is a Siegel modular form of half integral weight $k-\frac{1}{2}$, if h satisfies the following conditions (1), (2):

(1) $h|[g]_{k-1/2} = h(\tau)$ for any $g \in \Gamma_0(4)$, and hence h has the Fourier expansion:

$$h(\tau) = \sum_T c(T)e(\mathrm{tr}(T\tau)),$$

where T runs over all the symmetric half integral matrices.

(2) The above Fourier coefficients $c(T)=0$, unless T is positive semi-definite. If $c(T)\neq 0$ only when T is positive definite, we say that h is a cusp form. We denote by $M_{k-1/2}(\Gamma_0(4))$, or $S_{k-1/2}(\Gamma_0(4))$, the space of Siegel modular, or Siegel cusp forms of weight $k-\frac{1}{2}$, respectively. Now, we define the analogue of 'Kohnen's plus space' in our case. We define a subspace $M^+_{k-1/2}(\Gamma_0(4))$ of $M_{k-1/2}(\Gamma_0(4))$ by:

$$M^+_{k-1/2}(\Gamma_0(4))=\{h(\tau)\in M_{k-1/2}(\Gamma_0(4));\text{ the coefficients } c(T)=0,\ \text{unless } T\equiv -\mu^t\mu \bmod 4L_n^* \text{ for some } \mu\in \boldsymbol{Z}^n\},$$

where we denote by L_n^* the set of all half integral symmetric matrices of size n. We also define $S^+_{k-1/2}(\Gamma_0(4))$ by:

$$S^+_{k-1/2}(\Gamma_0(4))=M^+_{k-1/2}(\Gamma_0(4))\cap S_{k-1/2}(\Gamma_0(4)).$$

These are analogues for general degree n of the 'plus space' defined when $n=1$ by Kohnen [4].

THEOREM 1. *For any $F\in J_{k,1}$ and $\mu\in(\boldsymbol{Z}/2\boldsymbol{Z})^n$, we define F_μ as in* (1.1), *and define the function $\sigma(F)(\tau)$ on H_n by*:

$$\sigma(F)(\tau)=\sum_{\mu\in(\boldsymbol{Z}/2\boldsymbol{Z})^n} F_\mu(4\tau).$$

Then, we get $\sigma(F)\in M^+_{k-1/2}(\Gamma_0(4))$. Besides, the mapping $\sigma: F\to\sigma(F)$ induces the following linear isomorphisms over $\boldsymbol{C}$.

$$J_{k,1}\cong M^+_{k-1/2}(\Gamma_0(4)),$$

and

$$J^{cusp}_{k,1}\cong S^+_{k-1/2}(\Gamma_0(4)).$$

1.2. Hecke operators.

In this subsection, first we shall review on Hecke operators both on $J_{k,1}$ and $M^+_{k-1/2}(\Gamma_0(4))$, and then we shall state Theorem 2 on comparison of their action on both spaces. For any field $K\subset\boldsymbol{R}$, we define $GSp^+(n,K)$ by:

$$GSp^+(n,K)=\{g\in M_{2n}(K);\ gJ_n{}^tg=n(g)J_n \text{ for some } n(g)\in K^\times,\ n(g)>0\}.$$

According to Murase [6], the Hecke operators on $J_{k,1}$ are defined as follows: For any odd prime p and any natural number δ, we denote by $T(p^{2\delta})$ the subset of $GSp^+(n,\boldsymbol{Q})$ defined by:

$$T(p^{2\delta})=\{g\in GSp^+(n,\boldsymbol{Q})\cap M_{2n}(\boldsymbol{Z});\ gJ_n{}^tg=p^{2\delta}J_n\}.$$

We also denote by ϕ a fixed group homomorphism of $\boldsymbol{Q}_+^\times$ into $\boldsymbol{C}^\times$. For any $F\in J_{k,1}$ and any $Sp(n,\boldsymbol{Z})$-double coset U in $T(p^{2\delta})$, we write

$$F|_kU=\phi(p^\delta)\sum_{x\in(\boldsymbol{Z}/p^\delta\boldsymbol{Z})^{2n}}\ \sum_{g\in Sp(n,\boldsymbol{Z})\backslash U} F|[g]_k[x]_k,$$

where, for any function G on $H_n \times \mathbf{C}^n$, we put

$$F|[g]_k = e(-{}^t z(c\tau+d)^{-1}cz)p^{2\delta kn}\det(c\tau+d)^{-k}F(g\tau, p^{\delta\,t}(c\tau+d)^{-1}z)$$

and

$$F|[x]_k = e({}^t\lambda\tau\lambda + 2{}^t\lambda z)F(\tau, z+\tau\lambda+\mu)\,(x=(\lambda, \mu),\ \lambda, \mu \in (\mathbf{Z}/p^{\delta}\mathbf{Z})^n)\,.$$

We shall call $\phi(p^\delta)$ the normalizing factor of the Hecke operators. It is easy to see that this definition does not depend on the choice of the representatives in the summation. It is also easily seen that, for any $F \in J_{k,1}$, the function $F|_k U$ belongs again to $J_{k,1}$. For integers $i, j \geq 0$ with $i+j=n$, define $Sp(n, \mathbf{Z})$-double cosets $T_{i,j}(p^2)$ in $T(p^2)$ as

$$T_{i,j}(p^2) = Sp(n, \mathbf{Z})e(i, j : p)Sp(n, \mathbf{Z})\,,$$

where we put

$$e(i, j; \mathrm{p}) = \begin{pmatrix} 1_i & 0 & 0 & 0 \\ 0 & p1_j & 0 & 0 \\ 0 & 0 & p^2 1_i & 0 \\ 0 & 0 & 0 & p1_j \end{pmatrix}.$$

Next, we review on Hecke theory on $M_{k-1/2}(\Gamma_0(4))$, according to Zhuravrev [12]. We denote by $\widetilde{GSp}^+(n, \mathbf{R})$ the universal covering group of $GSp^+(n, \mathbf{R})$. Then, it is known that we can embed $\Gamma_0(4)$ into $\widetilde{GSp}^+(n, \mathbf{R})$ by

$$\Gamma_0(4) \ni \gamma \rightarrow (\gamma, \theta(\gamma\tau)\theta(\tau)^{-1}) \in \widetilde{GSp}^+(n, \mathbf{R})\,.$$

We denote by $\tilde{\Gamma}_0(4)$ the image of $\Gamma_0(4)$ by this mapping. For any "upper triangular" matrix

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GSp^+(n, \mathbf{R})\,,$$

we define an element $\tilde{g} \in \widetilde{GSp}^+(n, \mathbf{R})$ by

$$\tilde{g} = (g, n(g)^{-n/4}|\det d|^{1/2})\,.$$

We decompose the $\tilde{\Gamma}_0(4)$-double coset $\tilde{U} = \tilde{\Gamma}_0(4)\tilde{g}\tilde{\Gamma}_0(4)$ into disjoint union:

$$\tilde{U} = \coprod_{i=1}^{d} \tilde{\Gamma}_0(4)\tilde{g}_i\,.$$

We fix a homomorphism ψ of $\mathbf{Q}_+^\times$ into $\mathbf{C}^\times$. For any $h \in M_{k-1/2}(\Gamma_0(4))$, we write

$$(h|_{k-1/2}\tilde{U})(\tau) = \psi(\sqrt{n(g)}) \sum_{i=1}^{d} \varepsilon_i(\tau)^{-2k+1}h(g_i\tau)\,,$$

where we put

$$\tilde{g}_i = (g_i, \varepsilon_i(\tau))\,.$$

Zhuravrev [11] has shown that this is well defined and gives an action of the Hecke operators. We put

$$\tilde{T}_{i,j}(p^2)=\tilde{\Gamma}_0(4)\tilde{e}(i,j;p)\tilde{\Gamma}_0(4)\,.$$

THEOREM 2. *For any even natural number k, the space* $M^+_{k-1/2}(\Gamma_0(4))$ *is stable under the action of the above Hecke operators. Besides, assume that* $F\in J_{k,1}$ *and* $\sigma(F)=h$. *Then, for any integers* $i,j\geq 0$ *with* $i+j=n$, *any odd prime p, and any normalizing factors* ϕ, ψ *in the definition of the Hecke operators, we get*

$$(\phi(p))^{-1}p^{-2kn-j/2}\sigma(F|_kT_{i,j}(p^2))=(\psi(p))^{-1}p^{-n(2k-1)/2}h|_{k-1/2}\tilde{T}_{i,j}(p^2)\,.$$

2. Proof of Theorem 1

2.1. In this subsection, we shall show that, for any $F\in J_{k,1}$, $\sigma(F)(\tau)=\sum_{\mu\in(\boldsymbol{Z}/2\boldsymbol{Z})^n}F_\mu(4\tau)$ belongs to $M^+_{k-1/2}(\Gamma_0(4))$. In order to show that $\sigma(F)$ is automorphic, we must calculate $\sigma(F)(g\tau)$ at least for a set of generators $g\in\Gamma_0(4)$. So, we first give the following Lemma.

LEMMA 2.1. *The group* $\Gamma_0(4)$ *is generated by the following three kinds of elements*:

$$v(4s)=\begin{pmatrix}1_n & 0\\ 4s & 1_n\end{pmatrix},\quad u(s')=\begin{pmatrix}1_n & s'\\ 0 & 1_n\end{pmatrix},\quad \text{and}\quad t(a)=\begin{pmatrix}a & 0\\ 0 & {}^ta^{-1}\end{pmatrix},$$

where s and s' run over any symmetric matrices in $M_n(\boldsymbol{Z})$, *and a over* $GL_n(\boldsymbol{Z})$.

Proof. For each $g=\begin{pmatrix}a & b\\ 4c & d\end{pmatrix}\in\Gamma_0(4)$, we denote by a_{ij} etc. the (i,j)-component of a etc. When we emphasize the dependence on g, we write $a=a(g)$, $a_{ij}=a_{ij}(g)$ and so on. First, we shall show that the last row of c is replaced by the 0-vector. In fact, if the last row of c is not the 0-vector, denote by $c_0(g)$ the greatest common divisor of the components of the n-th row of c. Replacing g by $g\cdot u(s')$ for some $s'={}^ts'\in M_n(\boldsymbol{Z})$, we can assume that $|d_{nn}(g)|\leq 2c_0(g)$. Replacing g by $g'=g\cdot t(a)$ for some $a\in GL_n(\boldsymbol{Z})$, we can assume that the last row of $d(g')$ is $(0,\cdots,0,d_{nn}(g'))$. Since $d_{nn}(g')$ is the greatest common divisor of the components of the last row of $d(g)$, still we have $|d_{nn}(g')|\leq|d_{nn}(g)|\leq 2c_0(g')=2c_0(g)$. By replacing g' by $g''=g'\cdot v(4s)$ for some $s={}^ts\in M_n(\boldsymbol{Z})$, we can assume that $|c_{ni}(g'')|\leq\frac{1}{2}|d_{nn}(g')|$ for every $i=1,\cdots,n$. As g is unimodular, $d_{nn}(g')$ must be odd, and we get $c_{ni}(g'')<c_0(g)$. If the last row of $c(g'')$ is not the 0-vector, this means that $c_0(g'')<c_0(g)$. Repeating the same procedure several times, the last row of c becomes 0. Now, we can assume that $c_{ni}(g)=0$ $(1\leq i\leq n)$ and that the last row of $d(g)$ is $(0,\cdots,0,1)$. As $g\in Sp(n,\boldsymbol{Z})$ and hence $c(g){}^td(g)$ is symmetric, the n-th column of $c(g)$ is also the 0-vector. Hence, repeating the same procedure as above for the first $(n-1,n-1)$-block of $c(g)$ and $d(g)$, we can assume that $c(g)=0$. Then, $g=t(a)u(a^{-1}b)$ and Lemma was proved. q.e.d.

Now, we shall see that $\sigma(F)|[g]_{k-1/2}=\sigma(F)$ for any $g\in\Gamma_0(4)$.

By direct calculation, we get

$$F(u(s)\tau, z)=\sum_{\mu} F_{\mu}(u(s)\tau)\vartheta_{\mu}(\tau+s, z)$$

$$=e\left(\frac{1}{4}\,{}^t\mu s\mu\right)\sum_{\mu} F_{\mu}(u(s)\tau)\vartheta_{\mu}(\tau, z)\,,$$

and on the other hand, as $F\in J_{k,1}$, we get

$$F(u(s)\tau, z)=F(\tau, z)\,.$$

Hence, by the uniqueness of F_{μ}, we get

$$F_{\mu}(4(u(s)\tau))=e({}^t\mu s\mu)F_{\mu}(4\tau)$$

$$=F_{\mu}(4\tau)$$

for each $\mu\in(\boldsymbol{Z}/2\boldsymbol{Z})^n$. Hence, as $\theta(u(s)\tau)=\theta(\tau)$ for any $s={}^ts\in M_n(\boldsymbol{Z})$, we get $\sigma(F)|[u(s)]_{k-1/2}=\sigma(F)$. In the same way, we get

$$F(a\tau{}^ta, az)=(\det a)^{-k}F(\tau, z)$$

$$=\sum_{\mu} F_{\mu a^{-1}}(a\tau{}^ta)\vartheta_{\mu}(\tau, z)\,,$$

and we get

$$\sigma(F)(t(a)\tau)=(\det a)^{-k}\sigma(F)(\tau)\,.$$

As $\theta(t(a)\tau)=\theta(\tau)$, we see $\sigma(F)|[t(a)]_{k-1/2}=\sigma(F)$. Now, the proof for the generators $v(4s)$ is remained. For these generators, we need some formula for $F_{\mu}(4v(4s)\tau)\theta(v(4s)\tau)$. But, as $4v(4s)\tau=4\tau(4s\tau+1_n)^{-1}=v(s)(4\tau)$, and $\theta(v(4s)(\tau/4))=\theta((v(s)\tau)/4)$, all we need is a formula for $F_{\mu}(v(s)\tau)\theta((v(s)\tau)/4)$. This can be obtained by using the theta transformation formula, which we shall quote from J. Igusa [3] pp. 227 below.

For any n by n matrix M, we denote by $(M)_0$ the column vector of degree n whose i-th component is the (i, i)-component of M for each $i=1, \cdots, n$. For any $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in Sp(n, \boldsymbol{Z})$ and $m=\begin{pmatrix} m' \\ m'' \end{pmatrix}\in \boldsymbol{Z}^{2n}$ $(m', m''\in \boldsymbol{Z}^n)$, we denote by $g\cdot m$ the following vector in $\boldsymbol{Z}^{2n}$:

$$g\cdot m=\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}m+\begin{pmatrix} (c{}^td)_0 \\ (a{}^tb)_0 \end{pmatrix}.$$

This is not an action on $\boldsymbol{Z}^{2n}$, but it induces an action of $Sp(n, \boldsymbol{Z})$ on $(\boldsymbol{Z}/2\boldsymbol{Z})^n$ by $m \bmod 2\to g\cdot m \bmod 2$. Now, for each $g\in Sp(n, \boldsymbol{Z})$, we fix, once and for all, a branch of the square root of $\det(c\tau+d)$, and denote it by $\det(c\tau+d)^{1/2}$. (As H_n is simply connected, this is a well defined holomorphic function on H_n.)

PROPOSITION 2.2 (Igusa [3]). *For any $g \in Sp(n, \mathbf{Z})$ and $m \in \mathbf{Z}^{2n}$, there exists an eighth root $\kappa(g)$ of unity such that*

$$\theta_{g\cdot m}(g\tau, {}^t(c\tau+d)^{-1}z)=\kappa(g)e(\phi_m(g))\det(c\tau+d)^{1/2}e\left(\frac{1}{2}\cdot{}^tz(c\tau+d)^{-1}cz\right)\theta_m(\tau, z)$$

for all $\tau \in H_n$ and $z \in \mathbf{C}^n$, where

$$\phi_m(g)=-\frac{1}{8}({}^tm''{}^tbdm'+{}^tm'''{}^tacm''-2{}^tm''{}^tbcm''-2{}^t(a{}^tb)_0(dm'-cm''))\,.$$

The constant $\kappa(g)$ depends only on g and not on m.

Here, we note that there remains ambiguity on signs of $\kappa(g)$, which depends on the choice of the branch of automorphy factor of weight $\frac{1}{2}$. But, we can escape from this ambiguity in our case, because we used $\det(c\tau+d)^k\theta(\tau)/\theta(g\tau)$ as an automorphy factor of the functions in $M_{k-1/2}(\Gamma_0(4))$, and such ambiguity will be absorbed in this automorphy factor, as we shall see later. Now, applying the above theta transformation formula to the functions $\theta(\tau)=\theta_{00}(2\tau)$, and $\vartheta_\mu(\tau, z)=\theta_{\mu,0}(2\tau, 2z)$ for each $\mu \in (\mathbf{Z}/2\mathbf{Z})^n$, we get the following corollary:

COROLLARY 2.3. *For any symmetric matrix $s \in M_n(\mathbf{Z})$, put*

$$v(s)=\begin{pmatrix} 1_n & 0 \\ s & 1_n \end{pmatrix}.$$

Then, we get

$$\theta\left(\frac{1}{4}\cdot v(s)\tau\right)\vartheta_\mu(v(s)\tau, {}^t(s\tau+1_n)^{-1}z)=2^{-n}\det(s\tau+1_n)e({}^tz(s\tau+1_n)^{-1}sz)$$
$$\times\sum_{\nu,\kappa}e\left(-\frac{{}^t\nu\mu}{2}\right)e\left(-\frac{{}^t\nu s\nu}{4}\right)e\left(\frac{{}^t\kappa\nu}{2}\right)\vartheta_\kappa(\tau, z)h\left(\frac{\tau}{4}\right),$$

where ν and κ run over all the elements of $(\mathbf{Z}/2\mathbf{Z})^n$.

Proof. It is easy to see just from the definition that

$$\theta(\tau/4)=\theta_{00}(\tau/2)=\sum_{q\in(\mathbf{Z}/2\mathbf{Z})^n}\theta_{q0}(2\tau)\,.$$

Hence, if we put $\tau'=-(v(s)\tau)^{-1}=-(s\tau+1_n)\tau^{-1}$, we get

$$\theta((v(s)\tau)/4)=\sum_{q\in(\mathbf{Z}/2\mathbf{Z})^n}\theta_{q0}(J_n(\tau'/2))\,.$$

By Proposition 2.2, we get

$$\theta_{q0}(J_n(\tau'/2))=\kappa(J_n)\det(\tau'/2)^{1/2}\theta_{0,-q}(\tau'/2)\,.$$

It is easy to see that

$$\theta_{0,-q}(\tau'/2) = \sum_{\nu\in(\mathbf{Z}/2\mathbf{Z})^n} e(-{}^t\nu q/2)\theta_{\nu 0}(2\tau')$$

and

$$\sum_{q\in(\mathbf{Z}/2\mathbf{Z})^n} e(-{}^t\nu q/2) = \begin{cases} 2^n \cdots \text{if} & \nu=0\,, \\ 0 \;\cdots \text{if} & \nu\neq 0 \text{ in } (\mathbf{Z}/2\mathbf{Z})^n\,. \end{cases}$$

Hence

$$\theta((v(s)\tau)/4) = 2^n\kappa(J_n)\det(\tau'/2)^{1/2}\theta_{00}(2\tau')\,.$$

It is trivial that $\theta_{00}(2\tau') = \theta_{00}(-2\tau^{-1})$, and repeating the same argument for $\theta_{00}(-2\tau^{-1})$, we get

$$\theta((v(s)\tau)/4) = 2^n\kappa(J_n)^2\det(\tau'/2)^{1/2}\det(\tau/2)^{1/2}\theta(\tau/4)\,.$$

Here, the branch of the square root is taken as follows: first, we fix a branch of $\det(\tau)^{1/2}$ on H_n, and substitute τ by $\tau/2$ or $\tau'/2$. In the above calculation, we used Proposition 2.2 twice, and $\kappa(J_n)$ has the same meaning in both cases. Next, if we put $z' = -\tau^{-1}z$, and τ' as before, we get

$$\begin{aligned}\vartheta_\mu(v(s)(\tau, z)) &= \theta_{\mu,0}(J_n(\tau'/2, z')) \\ &= \kappa(J_n)\det(\tau'/2)^{1/2}e({}^tz'(\tau')^{-1}z')\theta_{0,-\mu}(\tau'/2, z')\,.\end{aligned}$$

By easy calculation, we get

$$\theta_{0,-\mu}(\tau'/2, z') = \sum_{\nu\in(\mathbf{Z}/2\mathbf{Z})^n} e(-{}^t\nu\mu/2)\theta_{\nu,0}(2\tau', 2z')\,,$$

$$\theta_{\nu,0}(2\tau', 2z') = e(-{}^t\nu s\nu/4)\theta_{\nu,0}(J_n(\tau/2, -z))\,,$$

and

$$\begin{aligned}\theta_{\nu,0}(J_n(\tau/2, -z)) &= \kappa(J_n)\det(\tau/2)\theta_{0,-\nu}(\tau/2, z)e({}^tz\tau^{-1}z) \\ &= \kappa(J_n)\det(\tau/2)^{1/2}e({}^tz\tau^{-1}z)\times\sum_{\kappa\in(\mathbf{Z}/2\mathbf{Z})^n} e({}^t\kappa\nu/2)\theta_{\kappa,0}(2\tau, 2z)\,.\end{aligned}$$

Summing up, we get

$$\begin{aligned}\vartheta_\mu(v(s)(\tau, z)) &= \kappa(J_n)^2\det(\tau/2)^{1/2}\det(\tau'/2)^{1/2}e({}^tz(s\tau+1_n)^{-1}sz) \\ &\quad\times\sum_{\kappa,\nu\in(\mathbf{Z}/2\mathbf{Z})^n} e(-{}^t\nu\mu/2)e(-{}^t\nu s\nu/4)e({}^t\kappa\nu/2)\vartheta_\kappa(\tau, z)\,.\end{aligned}$$

It is known by Igusa [3] that $\kappa(J_n)^4 = (-1)^n$, and we also get $\det(\tau')\det(\tau) = (-1)^n\det(s\tau+1_n)$. Thus, we get our Corollary. q.e.d.

Now, if $F\in J_{k,1}$, then

$$F(v(s)(\tau, z)) = \det(s\tau+1_n)^k e({}^tz(s\tau+1_n)^{-1}sz)F(\tau, z)\,,$$

and

$$F(v(s)(\tau, z)) = \sum_{\mu \in (\mathbf{Z}/2\mathbf{Z})^n} F_\mu(v(s)\tau)\vartheta_\mu(v(s)(\tau, z)) .$$

Hence, by Corollary 2.3, we get

$$F_\kappa(\tau)\det(s\tau + 1_n)^{k-1} = 2^{-n}\frac{\theta(\tau/4)}{\theta((v(s)\tau)/4)}$$
$$\times \sum_{\mu,\nu \in (\mathbf{Z}/2\mathbf{Z})^n} e\left(-\frac{{}^t\nu\mu}{2}\right)e\left(-\frac{{}^t\nu s\nu}{4}\right)e\left(\frac{{}^t\kappa\nu}{2}\right)F_\mu(v(s)\tau) ,$$

and hence,

$$\sigma(F)(\tau)\det(4s\tau + 1_n)^{k-1}\theta\left(\frac{1}{4}\cdot v(s)(4\tau)\right)\theta(\tau)^{-1}$$
$$= 2^{-n}\sum_{\mu \in (\mathbf{Z}/2\mathbf{Z})^n} F_\mu(4v(4s)\tau)\left(\sum_{\nu,\kappa \in (\mathbf{Z}/2\mathbf{Z})^n} e\left(-\frac{{}^t\nu\mu}{2}\right)e\left(-\frac{{}^t\nu s\nu}{4}\right)e\left(\frac{{}^t\kappa\nu}{2}\right)\right)$$
$$= \sigma(F)(v(4s)\tau) .$$

So, we proved that $\sigma(F)|[g]_{k-1/2} = \sigma(F)$ for any $g \in \Gamma_0(4)$.

Now, we must show the conditions on the Fourier coefficients of $\sigma(F)$. We have

$$F_\mu(4\tau) = \sum_N a(N, \mu)e(\operatorname{tr}((4N - \mu{}^t\mu)\tau)) ,$$

where $a(N, \mu)$ is the Fourier coefficients of F, and N runs over n by n symmetric half integral matrices. Hence, it is obvious that if $F \in J_{k,1}$ (resp. $J_{k,1}^{cusp}$), then $\sigma(F) \in M_{k-1/2}^+(\Gamma_0(4))$ (resp. $S_{k-1/2}^+(\Gamma_0(4))$). So, we proved the first half of Theorem1.

2.2. We shall prove now that σ is a bijection. For each given $h \in M_{k-1/2}^+(\Gamma_0(4))$ such that

$$h(\tau) = \sum_T c(T)e(\operatorname{tr}(T\tau))$$

(where T runs over n by n symmetric half integral matrices), and for each $\mu \in (\mathbf{Z}/2\mathbf{Z})^n$, we denote by $h_\mu(\tau)$ the following holomorphic function on H_n:

$$h_\mu(\tau) = \sum_N c(4N - \mu{}^t\mu)e\left(\operatorname{tr}\left(\left(N - \frac{1}{4}\mu{}^t\mu\right)\tau\right)\right) ,$$

where N runs over n by n symmetric half integral matrices such that $N \equiv -\mu{}^t\mu \bmod 4$. If $F \in J_{k,1}$, then it is clear that $\sigma(F)_\mu = F_\mu$. Hence, we get the injectivity. Now we shall show the surjectivity. For each $h \in M_{k-1/2}^+(\Gamma_0(4))$, define a function G on $H_n \times \mathbf{C}^n$ as follows:

$$G(\tau, z) = \sum_{\mu \in (\mathbf{Z}/2\mathbf{Z})^n} h_\mu(\tau)\vartheta_\mu(\tau, z) .$$

We shall show that $G\in J_{k,1}$. By definition of ϑ_μ, it is trivial that $G(\tau, z+\tau x+y)=e(-{}^t x\tau x-2{}^t xz)G(\tau, z)$. It is well known that $Sp(n, \mathbf{Z})$ is generated by $u(s)$, $v(s)$, and $t(a)$ ($s={}^t s\in M_n(\mathbf{Z})$, and $a\in GL_n(\mathbf{Z})$). By direct calculation, we can easily show that $G|[t(a)]_k=G$ and $G|[u(s)]_k=G$. We must also calculate $G|[v(s)]_k$. To do this, first we shall investigate $h_\mu(v(s)\tau)$.

LEMMA 2.4. *For any* $h\in M^+_{k-1/2}(\Gamma_0(4))$ *and* $\mu\in(\mathbf{Z}/2\mathbf{Z})^n$, *define* $h_\mu(\tau)$ *as above. Then, for any symmetric integral matrix* s *and any* $\kappa\in(\mathbf{Z}/2\mathbf{Z})^n$, *we get*

$$h_\kappa(\tau)\theta(\tau/4)\det(s\tau+1_n)^k$$
$$=2^{-n}\sum_{\nu,\mu\in(\mathbf{Z}/2\mathbf{Z})^n} e\left(-\frac{{}^t\nu\mu}{2}\right)e\left(-\frac{{}^t\nu s\nu}{4}\right)e\left(\frac{{}^t\kappa\nu}{2}\right)h_\mu(v(s)\tau)\theta\left(\frac{1}{4}\cdot v(s)\tau\right).$$

Proof. To use the condition that $h\in M^+_{k-1/2}(\Gamma_0(4))$, first we shall show the following relation:

$$h_\kappa(\tau)=2^{-n}\sum_{s_1\in\Delta} e\left(\frac{{}^t\kappa s_1\kappa}{2}\right)h\left(\frac{\tau+2s_1}{4}\right),$$

where s_1 runs over the set Δ of all diagonal matrices such that each diagonal component is 0 or 1. In fact, for $s_1\in\Delta$, it is easy to see that

$$h\left(\frac{\tau+2s_1}{4}\right)=\sum_\mu h_\mu(\tau+2s_1)=\sum_\mu e\left(-\frac{{}^t\mu s_1\mu}{2}\right)h_\mu(\tau),$$

and

$$\sum_{s_1} e\left(-\frac{{}^t\kappa s_1\kappa}{2}\right)e\left(-\frac{{}^t\mu s_1\mu}{2}\right)=\sum_{s_1} e\left(-\frac{{}^t(\kappa-\mu)s_1(\kappa-\mu)}{2}\right)$$
$$=\begin{cases}2^n\cdots\text{if}\quad \kappa=\mu\ ,\\ 0\ \cdots\text{otherwise}\ .\end{cases}$$

Hence, we get the above relation. Now, for any $s_1\in\Delta$, put

$$\gamma_s(s_1)=\begin{pmatrix}1_n+2s_1 s & -s_1 s s_1\\ 4s & 1_n-2ss_1\end{pmatrix}.$$

Then, it is easy to see that $\gamma_s(s_1)\in\Gamma_0(4)$, and $(v(s)\tau+2s_1)/4=\gamma_s(s_1)((\tau+2s_1)/4)$. So, we get

$$h\left(\frac{v(s)\tau+2s_1}{4}\right)\theta\left(\frac{v(s)\tau+2s_1}{4}\right)=\det(s\tau+1_n)^k h\left(\frac{\tau+2s_1}{4}\right)\theta\left(\frac{\tau+2s_1}{4}\right).$$

But, it is obvious from the definition that

$$\theta\left(\frac{\tau+2s_1}{4}\right)=\sum_{q\in(\mathbf{Z}/2\mathbf{Z})^n} e\left(-\frac{{}^t q s_1 q}{2}\right)\theta_{q0}(2\tau),$$

and by the similar argument as in the proof of Corollary 2.3, we get

$$\sum_{q\in(\mathbf{Z}/2\mathbf{Z})^n} e\left(-\frac{{}^tqs_1q}{2}\right)\theta_{q0}(2v(s)\tau)=2^{-n}\kappa(J_n)^2\det(s\tau+1_n)^{1/2}$$

$$\times\sum_{\mu,\nu,q\in(\mathbf{Z}/2\mathbf{Z})^n} e\left(-\frac{{}^tqs_1q}{2}\right)e\left(-\frac{{}^t\nu q}{2}\right)e\left(-\frac{{}^t\nu s\nu}{4}\right)e\left(-\frac{{}^t\mu\nu}{2}\right)\theta_{\mu0}(2\tau)\,.$$

In the above summation, we first fix μ and ν, and the summation over q does not vanish only when $\nu=\varepsilon$, where we put $\varepsilon=(s_1)_0$: the diagonal vector of s_1 defined in §1. As we get $e(-{}^t\mu\varepsilon/2))=e(-{}^t\mu s_1\mu/2)$, we get

$$\theta\left(\frac{v(s)\tau+2s_1}{4}\right)\theta\left(\frac{\tau+2s_1}{4}\right)^{-1}=\theta\left(\frac{v(s)\tau}{4}\right)\theta\left(\frac{\tau}{4}\right)^{-1}e\left(-\frac{{}^t\varepsilon s\varepsilon}{4}\right).$$

Hence, we get

$$\begin{aligned}&h_\kappa(\tau)\theta(\tau/4)\det(s\tau+1_n)^k\theta((v(s)\tau)/4)^{-1}\\ &=2^{-n}\sum_{s_1\in\Delta} e\left(-\frac{{}^t\varepsilon s_1\varepsilon}{4}\right)e\left(\frac{{}^t\kappa s_1\kappa}{2}\right)h\left(\frac{v(s)\tau+2s_1}{4}\right)\\ &=2^{-n}\sum_{\varepsilon,\mu\in(\mathbf{Z}/2\mathbf{Z})^n} e\left(-\frac{{}^t\varepsilon s\varepsilon}{4}\right)e\left(\frac{{}^t\kappa\varepsilon}{2}\right)e\left(-\frac{{}^t\varepsilon\mu}{2}\right)h_\mu(v(s)\tau)\,.\end{aligned}$$

Thus, we proved Lemma. q.e.d.

End of the proof of Theorem 1. Using Corollary 2.3, we can replace the term $\vartheta_\mu(v(s)\tau)$ in $G|[v(s)]_k$ as follows:

$$G|[v(s)]_k=2^{-n}\det(s\tau+1_n)^{-k}\theta\left(\frac{1}{4}\cdot v(s)\tau\right)\theta\left(\frac{\tau}{4}\right)^{-1}$$

$$\times\sum_{\kappa\in(\mathbf{Z}/2\mathbf{Z})^n}\vartheta_\kappa(\tau,z)\left(\sum_{\nu,\mu\in(\mathbf{Z}/2\mathbf{Z})^n} h_\mu(v(s)\tau)e\left(\frac{{}^t\kappa\nu}{2}\right)e\left(-\frac{{}^t\nu\mu}{2}\right)e\left(-\frac{{}^t\nu s\nu}{4}\right)\right).$$

So, by Lemma 2.4, we get $G|[v(s)]_k=G$. Hence, $G\in J_{k,1}$, and if $h\in S^+_{k-1/2}(\Gamma_0(4))$, then $G\in J^{cusp}_{k,1}$. By definition, it is clear that $\sigma(G)=h$, and hence σ is surjective. Thus, Theorem 1 is completely proved.

3. Proof of Theorem 2

We shall prove Theorem 2 by writing down the action of Hecke operators in terms of the Fourier coefficients. First, we treat Jacobi forms. Denote by U any $Sp(n,\mathbf{Z})$-double coset in $T(p^{2\delta})$, and take a left $Sp(n,\mathbf{Z})$-coset decomposition:

$$U=\coprod_{i=1}^{t} Sp(n,\mathbf{Z})g_i\,,$$

where

$$g_i=\begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}.$$

For any $F\in J_{k,1}$, denote by $a(N,r)$ the Fourier coefficients of F as in §1, and by $a(N,r;U)$ those of $F|_kU$.

LEMMA 3.1. *Notations being as above, we get*

$$a(N,r;U)=\phi(p^\delta)\sum_{\lambda\in(\mathbf{Z}/p^{2\delta}\mathbf{Z})^n}\sum_{i=1}^{t}p^{2\delta kn}\det(d_i)^{-k}a(N_i(\lambda),r_i(\lambda))e(\mathrm{tr}(N_i(\lambda)b_id_i^{-1}),$$

where

$$N_i(\lambda)=\frac{1}{p^{2\delta}}d_i\left[N-\frac{r^tr}{4}+\frac{1}{4}(r-2\lambda)^t(r-2\lambda)\right]^td_i,$$

and

$$r_i(\lambda)=\frac{1}{p^\delta}d_i(r-2\lambda),$$

and we regard $a(N_i(\lambda),r_i(\lambda))=0$, *if* $N_i(\lambda)$ *is not half integral, or if* $r_i(\lambda)$ *is not an integer vector.*

Proof. As we have chosen the upper triangular representatives of $Sp(n,\mathbf{Z})$-coset in U, we can write down the action of U by using the Fourier coefficients. Hence, the proof is straight forward, and the details will be omitted here. q.e.d.

Now, we quote here some results of Zhuravrev [12]. For any integers l, m with $1\le l, m\le n$ and $l+m\le n$, put

$$d_{lm}=\begin{pmatrix} 1_{n-l-m} & 0 & 0 \\ 0 & p1_l & 0 \\ 0 & 0 & p^21_m \end{pmatrix}.$$

We denote by $M_{l,m}(p^\delta)$ a complete set of representatives of matrices of $M_{l,m}(\mathbf{Z})$ modulo p^δ and put $M_l(p^\delta)=M_{l,l}(p^\delta)$. We also denote by B the following set of matrices:

$$B=\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & p^tb_1 \\ 0 & b_1 & b_2 \end{pmatrix};\ a_1={}^ta_1\in M_l(p),\ b_1\in M_{l,m}(p), \text{ and } b_2={}^tb_2\in M_m(p^2)\right\}.$$

Then, a complete set R_{ij} of representatives of the left $\Gamma_0(4)$-cosets in $T_{i,j}(p^2)$ is given by:

$$\begin{pmatrix} p^2d_{lm}^{-1} & b_0 \\ 0 & d_{lm} \end{pmatrix}\times\begin{pmatrix} {}^tu^{-1} & 0 \\ 0 & u \end{pmatrix},$$

where l, m run over integers with $1\le m, j\le l$, and $l+m\le n$, the matrix b_0 runs over all the elements of B such that $\mathrm{rank}(a \bmod p)=l-j$, and the matrix u over representatives of $(GL_n(\mathbf{Z}) \cap d_{lm}^{-1} GL_n(\mathbf{Z}) d_{lm}) \backslash GL_n(\mathbf{Z})$. Besides, the set of representatives of the left $\tilde{\Gamma}_0(4)$-cosets in $\tilde{T}_{i,j}(p^2)$ is given by:

$$\tilde{g}=(g, \varepsilon(g) p^{(l+2m-n)/2})$$

where g runs over $R_{i,j}$ and

$$\varepsilon(g)=\varepsilon_p^{-r(g)}\left(\frac{(-1)^{r(g)} \det a'_1}{p}\right),$$

where $r(g)=\mathrm{rank}(a_1)$, $\varepsilon_p=1$ or i according as $p \equiv 1$, or $3 \bmod 4$, a'_1 is a regular matrix such that ${}^t vav=\begin{pmatrix} a'_1 & 0 \\ 0 & 0 \end{pmatrix}$ for some unimodular matrix v, and $\left(\frac{*}{p}\right)$ is the Legendre symbol. These are due to Zhuravrev (loc. cit.). By using all these, it is fairly easy to calculate the action. For any $h \in M_{k-1/2}^{+}(\Gamma_0(4))$ and any symmetric half integral matrix T, denote by $c(T)$ the Fourier coefficients of h as in §1, and by $c(T; T_{i,j}(p^2))$ those of $h|_{k-1/2} T_{i,j}(p^2)$.

LEMMA 3.2. *We get*

$$c(T; T_{i,j}(p^2))=\psi(p) p^{n(2k-1)/2} \sum_g p^{-(l+2m)(2k-1)/2} c\left(\frac{1}{p^2} dT^t d\right) e\left(\mathrm{tr}\left(\frac{1}{p^2} T^t db\right)\right) \varepsilon(g),$$

where g runs over

$$g=\begin{pmatrix} p^{2t} d^{-1} & b \\ 0 & d \end{pmatrix} \in R_{i,j},$$

and we regard $c(*)=0$, if $*$ is not half integral.

Proof. This is proved by straight forward calculation from the definition and the details will be omitted here. q.e.d.

End of the proof of Theorem 2. Finally, we compare the action of $T_{i,j}(p^2)$ on $F \in J_{k,1}$ and the action of $T_{i,j}(p^2)$ on $\sigma(F) \in M_{k-1/2}^{+}(\Gamma_0(4))$. By definition, we have the following relation between the Fourier coefficients $a(N, r)$ of F and $c(T)$ of $\sigma(F)$:

$$c(4N-r^t r)=a(N, r).$$

By applying Lemma 3.1 and the above relation to the case $U=T_{i,j}(p^2)$, we get

$$a(N, r; T_{i,j}(p^2))=\phi(p^\delta) \sum_g \left[p^{2kn} \det(d)^{-k} c\left(\frac{1}{p^2} d(4N-r^t r)^t d\right) e\left(\mathrm{tr}\left(\frac{1}{p^2}\left(N-\frac{1}{4} r^t r\right)^t db\right)\right.\right.$$
$$\left.\left. \times \sum_\lambda e\left(\frac{1}{4p^2}\, {}^t(r-2\lambda)^t db(r-2\lambda)\right)\right],$$

where $g=\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ runs over $R_{i,j}$, and λ runs over representatives of $(\boldsymbol{Z}/p\boldsymbol{Z})^n$ such that $d(r-2\lambda)\in(p\boldsymbol{Z})^n$. Now, we calculate the exponential sum over λ in the above summation. If $d=d_{lm}u$ $(u\in GL_n(\boldsymbol{Z}))$, and $b=b_0u$ for $b_0\in B$, then we have

$$u(r-2\lambda)\in(p\boldsymbol{Z})^{n-l-m}\times\boldsymbol{Z}^{l+m},$$

and

$${}^t db={}^t u\begin{pmatrix} 0 & 0 & 0 \\ 0 & pa_1 & p^2b_1 \\ 0 & p^2b_1 & p^2b_2 \end{pmatrix}u,$$

where

$$b_0=\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & pb_1 \\ 0 & b_1 & b_2 \end{pmatrix}.$$

Hence, for a fixed g, we get

$$\begin{aligned}\sum_{\lambda} e\left(\frac{1}{4p^2}\,{}^t(r-2\lambda)\,{}^t db(r-2\lambda)\right)&=p^m\sum_{x\in(\boldsymbol{Z}/p\boldsymbol{Z})^l} e\left(\frac{1}{p}\,{}^t x a_1 x\right)\\ &=p^{l+m-r(g)}(\varepsilon_p\sqrt{p}\,)^{r(g)}\left(\frac{\det a_1'}{p}\right)\\ &=p^{l+m-(r(g)/2)}\varepsilon(g)\,.\end{aligned}$$

As $l-r(g)=j$, we get $l+m-\dfrac{r(g)}{2}=\dfrac{j}{2}+\dfrac{l+2m}{2}$. Hence, comparing the above formula and Lemma 3.2, we proved Theorem 2.

References

[1] EICHLER, M. and ZAGIER, D. B.; Theory of Jacobi Forms, Progress in Math. 55, Birkhauser, Boston-Basel-Stuttgart, 1985.

[2] IBUKIYAMA, T.; Construction of half integral weight Siegel modular forms of $Sp(2, \boldsymbol{R})$ from automorphic forms of the compact twist $Sp(2)$, *J. reine angew. Math.*, **359** (1985), 188–220.

[3] IGUSA, J.; On the graded ring of theta-constants, *Amer. J. Math.*, **86** (1964), 219–246.

[4] KOHNEN, W.; Modular forms of half integral weight on $\Gamma_0(4)$, *Math. Ann.*, **248** (1980), 249–266.

[5] MAASS, H.; Über die Spezialschar von Modulformen zweiten Grades (I), (II), (III), *Invent. Math.*, **52**, **53**, **53** (1979), 95–104, 249–253, 255–265.

[6] MURASE, A.; L-functions attached to jacobi forms of degree n. Part I. The basic identity, *J. reine Angew. Math.*, **401** (1989), 122–156.

[7] MURASE, A.; L-functions attached to jacobi forms of degree n. (II) Functional equation, *Math. Ann.*, **290** (1991), 247–276.

[8] SHIMURA, G.; On modular forms of half integral weight, *Ann. of Math.*, **97** (1973), 440–448.

[9] SHIMURA, G.; On certain reciprocity laws for theta functions and modular forms, *Acta math.*, **141** (1978), 35–71.

[10] TAI, Y.; On the Kodaira dimension of the moduli space of abelian varieties, *Invent. Math.*, **68** (1982), 425–439.

[11] ZHURAVREV, V. G.; Euler expansions of theta transforms of Siegel modular forms of half integral weight and their analytic properties, *Math. Sbornik*, **123** (**165**) (1984), 174–194.

[12] ZHURAVREV, V. G.; Hecke rings for a covering of the symplectic group. *Math. Sbornik*, **121** (**163**) (1983), 381–402.

[13] YAMAZAKI, T.; Jacobi forms and a Maass relation for Eisenstein series, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **33** (1986), 295–310.

[14] ZAGIER, D. B.; Sur la conjecture de Saito-Kurokawa (d'après H. Maass). Séminaire de Théorie des nombres, Paris 1979–80, Progress in Math. 12, 371–394, 1981.

[15] ZIEGLER, C.; Jacobi forms of higher degree, *Abh. Math. Sem. Univ. Hamburg*, **59** (1989), 191–224.

Department of Mathematics
College of General Education
Osaka University
Mackikaneyama-machi 1–1
Osaka, 560 Japan