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In the previous paper [3] (II), we announced an explicit formula for the class number of any genera of maximal lattices in the positive definite binary quaternion hermitian spaces over definite quaternion algebras B over the rational number field. In order to obtain the formula, we only need some local data (some numbers of 'optimal embeddings' and some local masses), as it has been shown in [2], [3] (I). Actually, large part of these data has already been given in [3] (I). More precisely, regard B^2 as the quaternion hermitian space with metric $n(x)+n(y)$ for $(x, y)\in B^2$. Let O be a maximal order of B, and L be a maximal left O-lattice of B^2. Put $L_p=L\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$ and $O_p=O\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$. At such a prime p that $L_p\cong O_p^2$, these local data have already been obtained in [3] (I).

In this paper, we shall calculate these local data in the remaining cases, thus giving all that are necessary for completing the proof of the class number formula announced in [3] (II). In fact, by using these data, the announced formula can be obtained in the same way as in [3] (I).

§1. Preliminaries.

Take B, O, L as in the introduction. Put $B_p=B\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$. Put

$$G=\{g\in M_2(B);\ g{}^t\bar{g}=n(g)1_2,\ n(g)\in\boldsymbol{Q}_+^\times\},$$

$$G_p=\{g\in M_2(B_p);\ g{}^t\bar{g}=n(g)1_2,\ n(g)\in\boldsymbol{Q}_p^\times\},$$

and

$$G_p^*=\left\{g\in M_2(B_p);\ g\begin{pmatrix}0&1\\1&0\end{pmatrix}{}^t\bar{g}=n(g)\begin{pmatrix}0&1\\1&0\end{pmatrix},\ n(g)\in\boldsymbol{Q}_p^\times\right\}.$$

Then, $G_p^*=\xi G_p\xi^{-1}$ for any $\xi\in GL_2(B_p)$ such that $\xi{}^t\bar{\xi}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. When $B_p=M_2(Q_p)$, we have $L_p=(O_p, O_p)g$ for some $g\in G_p^*$. So, from now on till the end of this paper, we assume that B_p is division. Then, $L_p=(O_p, O_p)g$, or $L_p=(\pi O_p, O_p)g$ for some $g\in G_p^*$, where π is a prime element of O_p. So, assume that $L_p=(\pi O_p, O_p)$. Let R_p be the right order of L_p, that is,

$$R_p=\begin{pmatrix}O_p&\pi^{-1}O_p\\\pi O_p&O_p\end{pmatrix}.$$

Put $V_p^* = G_p^* \cap R_p^\times$. The following (1.1) is known as the Iwasawa decomposition:

$$(1.1)^{1)} \qquad G_p^* = \coprod_{n \in \boldsymbol{Q}_p^\times} \bigcup_{\substack{\alpha \in B_p^\times \\ \beta \in B(\alpha)}} \begin{pmatrix} n\bar{\alpha}^{-1} & \beta \\ 0 & \alpha \end{pmatrix} V_p^*,$$

where β runs over the set $B(\alpha) = \{\beta \in B;\ \mathrm{tr}(\bar{\alpha}\beta) = 0\}$. For $g \in G_p^*$, let $Z(g)_p$ be the commutor algebra of $\boldsymbol{Q}_p(g)$ in $M_2(B_p)$. For any $\boldsymbol{Z}_p$-order Λ_1, Λ_2 of $Z(g)_p$, we write $\Lambda_1 \sim \Lambda_2$ when $\Lambda_1 = a\Lambda_2 a^{-1}$ for some $a \in Z(g)_p \cap G_p^*$. For any $\boldsymbol{Z}_p$-order Λ_p of $Z(g)_p$ and any torsion element $g \in G_p^*$, we define

$$c_p(g, R_p, \Lambda_p) = \sharp((Z(g)_p \cap G_p^*) \backslash M_p(g, R_p, \Lambda_p)/V_p^*),$$

where

$$M_p(g, R_p, \Lambda_p) = \{x \in G_p^*;\ x^{-1}gx \in V_p^*,\ Z(g)_p \cap xR_px^{-1} \sim \Lambda_p\}.$$

§ 2. Calculation of local data.

In this section, we calculate $c_p(g, R_p, \Lambda_p)$ and some local index of G-Masses for p such that $L_p \cong (\pi O_p, O_p)$. Next lemma is useful to show that $c_p = 0$ for many g in question.

LEMMA 2.1. *Fix* $\xi = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in GL_2(O_p)$ *so that* $\xi^t\bar{\xi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Let* $g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ *be an element of* $G_p \cap GL_2(O_p)$. *Then, there exists an element* γ *of* G_p^* *such that* $\gamma^{-1}\xi g \xi^{-1}\gamma \in V_p^*$ *if and only if* $za\bar{z} + wb\bar{w} \in \pi O_p$.

PROOF. We have $\xi g \xi^{-1} = \begin{pmatrix} xa\bar{z} + yb\bar{w} & xa\bar{x} + yb\bar{y} \\ za\bar{z} + wb\bar{w} & za\bar{x} + wb\bar{y} \end{pmatrix}$. So, the sufficiency is obvious. Conversely, assume that $za\bar{z} + wb\bar{w} \in O_p^\times$. For simplicity, put $\xi g \xi^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Assume that $\begin{pmatrix} n & \beta \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} n & \beta \\ 0 & 1 \end{pmatrix} \in V_p^*$ for some $n \in \boldsymbol{Q}_p^\times$ and $\beta \in B_p$ with $\mathrm{tr}(\beta) = 0$. Then, $n \in p\boldsymbol{Z}_p$ and $\beta \in O_p$, since $C \in O_p^\times$. So, $n\pi^{-1}O_p \subset \pi O_p$, and we have $A\beta - \beta C\beta + B - \beta D \in \pi O_p$. Let F be an unramified quadratic subfield of B_p and O_F be the maximal order of F. Then, $O_p = O_F \oplus \pi O_F$ as modules. For any $u \in O_p$, denote the projection of u to O_F by u_0 (the same letter with suffix zero). Then, $A\beta - \beta C\beta + B - \beta D \equiv (\beta_0^2 N(w_0) + \beta_0(z_0\bar{x}_0 - x_0\bar{z}_0) - N(y_0))(a_0 - b_0) \equiv 0 \bmod \pi$. But $a_0 - b_0 \in O_F^\times$, since $z\bar{z} + w\bar{w} = 0$, $C \in O_p^\times$, and $za\bar{z} + wa\bar{w} \in O_p^\times$. Besides, it is easy to see that $w \in O_p^\times$. Therefore, $\beta_0^2 - \beta_0(\bar{w}_0^{-1}\bar{y}_0 - w_0^{-1}y_0) - N(w_0^{-1}y_0) = (\beta_0 + w_0^{-1}y_0) \times (\beta_0 - \bar{w}_0^{-1}\bar{y}_0) \equiv 0 \bmod pO_F$, so, $\mathrm{tr}(w_0^{-1}y_0) \equiv 0 \bmod p$. Since ${}^t\bar{\xi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $\mathrm{tr}(y\bar{w}) = 1$, and $\mathrm{tr}(w_0^{-1}y_0) \equiv \mathrm{tr}(w^{-1}y) = N(w)^{-1}\mathrm{tr}(\bar{w}y) = N(w)^{-1} \bmod p$. This is a contradiction. Thus, the lemma is proved by virtue of (1.1). q.e.d.

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1) There is a misprint in [3] (I): $\begin{pmatrix} n\bar{\alpha}^{-1} & \beta \\ 1 & \alpha \end{pmatrix}$ in (29) should be $\begin{pmatrix} n\bar{\alpha}^{-1} & \beta \\ 0 & \alpha \end{pmatrix}$.

COROLLARY 2.2. *Notations being as in Lemma 2.1, let $f_1(x)$, $f_2(x) \in \boldsymbol{Z}[x]$ be the characteristic polynomial of a, b, respectively. If the ideal spanned by f_1 and f_2 in $\boldsymbol{Z}[x]$ contains an integer prime to p, then $c_p(g, R_p, \Lambda_p)=0$ for any Λ_p. (Note that the converse is not necessarily true.)*

Recall that the elements of G are classified into six cases (I), $\cdots$, (VI), and if the elements are of finite order, their principal polynomials are as follows:

Case (I) $f_1(x)=(x-1)^4$, $f_1(-x)$,
Case (II) $f_2(x)=(x-1)^2(x+1)^2$,
Case (III) $f_3(x)=(x-1)^2(x^2+1)$, $f_3(-x)$,
$f_4(x)=(x-1)^2(x^2+x+1)$, $f_4(-x)$,
$f_5(x)=(x-1)^2(x^2-x+1)$, $f_5(-x)$,
Case (IV) $f_6(x)=(x^2+1)^2$,
$f_7(x)=(x^2+x+1)^2$, $f_7(-x)$,
Case (V) $f_8(x)=(x^2+1)(x^2+x+1)$, $f_8(-x)$,
$f_9(x)=(x^2+x+1)(x^2-x+1)$,
Case (VI) $f_{10}(x)=(x^4+x^3+x^2+x+1)$, $f_{10}(-x)$,
$f_{11}(x)=(x^4+1)$,
$f_{12}(x)=(x^4-x^2+1)$.

First, we treat the elements belonging to the case (I), (II), or (III). In these cases, the elements of G whose principal polynomials are (a fixed) $f_i(x)$ form a single G-conjugacy class, and a representative of this class can be taken to be of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G$.

PROPOSITION 2.3. *Put $g=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then,*

$$c_p(g, R_p, \Lambda_p)=\begin{cases} 1 \cdots \text{if } \Lambda_p \sim R_p, \\ 0 \cdots \text{otherwise.} \end{cases}$$

Let Λ be a right order of a lattice $L \in \mathcal{L}(D_1, D_2)$, where $\mathcal{L}(D_1, D_2)$ is defined as in [3] (II). Then,

$$M_G(\Lambda)=\frac{1}{2^7 3^2 5} \prod_{p \mid D_1} (p-1)(p^2+1) \prod_{p \mid D_2} (p^2-1).$$

PROOF. The first assertion is obvious. Put $U_p^* = GL_2(O_p) \cap G_p^*$. The assertion for $M_G(\Lambda)$ is reduced to [3] (I) Prop. 9, noting that $[V_p^*: V_p^* \cap U_p^*]=p^2+1$ and $[U_p^*: V_p^* \cap U_p^*]=p+1$.

PROPOSITION 2.4. *Put $g=\pm\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$, where $\omega \in O_p$ is a root of unity of order 2, 3, 4, or 6. Then, the local data we need are as follows:*

(i) *if* $p=2$ *or* 3, *put* $\xi=\begin{pmatrix}1 & \delta \\ \delta & 1\end{pmatrix}$, *where* $\delta=(1+\varepsilon)/2$, $\varepsilon\in O_p$, *and* $\varepsilon^2=5$. *Then,*

(a) *if* $p=2$ *and* ω *is of order* 2, $c_p(\xi g\xi^{-1}, R_p, \Lambda_p)=1$ *for*

$$\Lambda_p\sim\xi\left\{\begin{pmatrix}x & 0 \\ 0 & y\end{pmatrix};\ x,\ y\in O_p,\ x\equiv y \mod. \pi\right\}\xi^{-1}$$

(b) *if either* $p=2$ *and* ω *is of order* 4, *or* $p=3$ *and* ω *is of order* 3, $c_p(\xi g\xi^{-1}, R_p, \Lambda_p)=1$ *for*

$$\Lambda_p\sim\xi\left\{\begin{pmatrix}x & 0 \\ 0 & y\end{pmatrix};\ x\in O_p,\ y\in o_p,\ x\equiv y \mod. \pi\right\}\xi^{-1}$$

where $o_p=\boldsymbol{Z}_p[\omega]$,

(ii) *in the other cases,* $c_p(g, R_p, \Lambda_p)=0$, *where* g *is considered as an element of* G_p^* *by an isomorphism* $G_p\cong G_p^*$.

PROOF. First, we prove (a). Put $h=\xi\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\xi^{-1}=\begin{pmatrix}-\varepsilon & 2 \\ -2 & \varepsilon\end{pmatrix}$. Then, $Z(h)_p\cap G_p^*=\xi\left\{\begin{pmatrix}a & 0 \\ 0 & b\end{pmatrix};\ a,\ b\in B_p^\times,\ N(a)=N(b)\right\}\xi^{-1}$. For any x_1, $x_2\in G_p^*$, write $x_1\sim x_2$ if $x_1\in(Z(h)_p\cap G_p^*)x_2V_p^*$. Then, by (1.1), $x\sim\begin{pmatrix}n & \beta \\ 0 & 1\end{pmatrix}$, where $n\in\boldsymbol{Q}_p^\times$ and $\mathrm{tr}(\beta)=0$. Assume that $\begin{pmatrix}n & \beta \\ 0 & 1\end{pmatrix}^{-1}h\begin{pmatrix}n & \beta \\ 0 & 1\end{pmatrix}\in V_p^*$. Then, $2\beta\in O_p$, $n\in\boldsymbol{Z}_p$, and $-\varepsilon\beta+2\beta^2+2-\beta\varepsilon\in n\pi^{-1}O_p$. Take a prime element $\pi\in O_p$ so that $\pi\varepsilon=-\varepsilon\pi$. Put $2\beta=a\varepsilon+b\pi$, where $b\in\boldsymbol{Z}_p[\delta]$ and $a\in\boldsymbol{Z}_p$. If $n\in\boldsymbol{Z}_p^\times$, then $\begin{pmatrix}n & \beta \\ 0 & 1\end{pmatrix}\sim\begin{pmatrix}1 & \beta \\ 0 & 1\end{pmatrix}\sim\begin{pmatrix}1 & a\varepsilon/2 \\ 0 & 1\end{pmatrix}\sim\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$. If $n\in p\boldsymbol{Z}_p^\times$, then $-\varepsilon\beta+2\beta^2+2-\beta\varepsilon\equiv 0 \mod. \pi$. Therefore, $-5a+5a^2/2+N(b)+atr(\varepsilon b\pi)/2\equiv 0 \mod. \pi$, and by this, we have $a\in 2\boldsymbol{Z}_p$ and $b\in 2\boldsymbol{Z}_p[\delta]$, that is, $\beta\in O_p$. So, $\begin{pmatrix}n & \beta \\ 0 & 1\end{pmatrix}\sim\begin{pmatrix}2 & 0 \\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}2 & \varepsilon \\ 0 & 1\end{pmatrix}$. But $\begin{pmatrix}2 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \delta\pi/2 \\ -\delta\pi & 0\end{pmatrix}$, $\begin{pmatrix}2 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}-\delta^2 & 0 \\ 2(3\delta+1) & -\delta^2\end{pmatrix}\begin{pmatrix}2 & \varepsilon \\ 0 & 1\end{pmatrix}^{-1}\in Z(g)_p\cap G_p^*$. Therefore, $\begin{pmatrix}n & \beta \\ 0 & 1\end{pmatrix}\sim\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$ also in this case. If $n\in p^2\boldsymbol{Z}_p$, we can show similarly as above that $(5a/2)^2+(5a/2)+1\equiv 0 \mod. \pi$, which is a contradiction. Thus, we proved (a). We can prove (b) in the similar way.

(ii) is obvious by virtue of Cor. 2.2. q. e. d.

Next, let $g\in G$ belong to the case (IV) and the principal polynomial of g be $(x^2+ax+b)^2$. We review briefly the classification of conjugacy classes in this case. Put $F=\boldsymbol{Q}(g)$. Then, $Z(g)=B\otimes_{\boldsymbol{Q}}F$. Let τ be the nontrivial automorphism F over $\boldsymbol{Q}$, and put $\sigma=1\otimes\tau$. Put $h=\begin{pmatrix}0 & 1 \\ -b & a\end{pmatrix}$ and $w=\begin{pmatrix}2b & a \\ a & 2\end{pmatrix}$. We can take $\xi\in GL_2(B)$ so that $\xi^{-1}h\xi=g$. Put $v={}^t\bar{\xi}w\xi$ and $Z_0(g)=\{x\in Z(g);\ vx^\sigma=xv\}$. Let g_1, $g_2\in G$ belong to the case (IV) and assume that they have the same principal polynomial. Then, $Z_0(g_1)\cong Z_0(g)$ if and only if g_1 and g_2 belong to the same G-conjugacy class (cf. [3] Prop. 3). We recall some more notations. Let Λ be a $\boldsymbol{Z}$-order of $Z(g)$ and put $\Lambda_0=Z_0(g)\cap\Lambda$. Let $\Lambda_{0,\max}$ be any maximal order of $Z_0(g)$ containing Λ_0. Then, we write $d_p(\Lambda_p)=[(\Lambda_{0,\max})_p^\times : \Lambda_{0p}^\times][o_p^\times : \boldsymbol{Z}_p[g]^\times]$ and $e_p(\Lambda_p)=[\Lambda_p^\times\cap G_p^* : \boldsymbol{Z}_p[g]^\times\Lambda_{0p}^\times]$, where o is the maximal order of F and $o_p=o\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$, and so on. The values of $d_p(\Lambda_p)$ and $e_p(\Lambda_p)$ are needed to calculate $M_G(\Lambda)$ ([3] Prop. 12). Now, we calculate local data. First, we treat the case where $Z_0(g)_p=Z_0(g)\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$ is split. In this case, $F_p=F\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$ must be a field (since we have assumed that B_p is division), and we can take $g=\begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}\in G_p^*$ as a representative of the conjugacy class, where $\omega\in O_p$ and $f_i(\omega)=0$ ($i=6$ or 7), and $Z_0(g)_p=\begin{pmatrix}\boldsymbol{Q}_p & \rho\boldsymbol{Q}_p\\ \rho\boldsymbol{Q}_p & \boldsymbol{Q}_p\end{pmatrix}$, where $\rho=\sqrt{-3}$, $\sqrt{-1}$, according as $F_p=\boldsymbol{Q}_p(\sqrt{-3})$, $\boldsymbol{Q}_p(\sqrt{-1})$.

PROPOSITION 2.5. *Notations being as above, we have:*

(i) *if* $\left(\frac{F_p}{p}\right)=-1$,

$$c_p(g, R_p, \Lambda_p)=\begin{cases}1\cdots \textit{if } \Lambda_p\sim M_2(F_p)\cap R_p,\\ 0\cdots \textit{otherwise},\end{cases}$$

(ii) *if* $p=3$ *and* $F_p=\boldsymbol{Q}_p(\sqrt{-3})$,

$$c_p(g, R_p, \Lambda_p)=\begin{cases}1\cdots \textit{if } \Lambda_p\sim\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}M_2(o_p)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}^{-1}=\Lambda_1,\\ 1\cdots \textit{if } \Lambda_p\sim\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in M_2(o_p);\ a\equiv d,\ b\equiv\varepsilon^2c \bmod \rho\right\}=\Lambda_2,\\ 0\cdots \textit{otherwise},\end{cases}$$

$$d_p(\Lambda_1)=e_p(\Lambda_1)=1,\quad d_p(\Lambda_2)=8,\quad e_p(\Lambda_2)=2,$$

where ε *is an element of* $O_p^\times$ *such that* $\varepsilon^2=-1$, $\varepsilon\rho=-\rho\varepsilon$.

(iii) *if* $p=2$ *and* $F_p=\boldsymbol{Q}_p(\sqrt{-1})$,

$$c_p(g, R_p, \Lambda_p)=\begin{cases}1\cdots \textit{if } \Lambda_p\sim\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}M_2(o_p)\begin{pmatrix}1 & 0\\ 0 & \pi^{-1}\end{pmatrix}=\Lambda_1,\\ 1\cdots \textit{if } \Lambda_p\sim M_2(F_p)\cap\begin{pmatrix}1 & \varepsilon y\\ 0 & 1\end{pmatrix}R_p\begin{pmatrix}1 & -\varepsilon y\\ 0 & 1\end{pmatrix}=\Lambda_2,\\ 0\cdots \textit{otherwise},\end{cases}$$

$$d_p(\Lambda_1)=3,\quad e_p(\Lambda_1)=1,\quad d_p(\Lambda_2)=3,\quad e_p(\Lambda_2)=2,$$

where ε *is an element of* $O_p^\times$ *such that* $\varepsilon^2=5$, $\varepsilon\rho=-\rho\varepsilon$, *and* $\pi=1+\rho$, $y=(3-\rho)/4$.

PROOF. This proposition can be obtained virtually in the same way as in [3] Prop. 15, and the proof will be omitted here. q.e.d.

Next, we treat the case where $Z_0(g)_p$ is division. If $F_p=\boldsymbol{Q}_p(g)$ is a field, a representative of the G_p-conjugacy class in this case is given by

$$g=\begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}\cdots \text{if } -1\notin N_{F/\boldsymbol{Q}}(F_p^\times), \quad \text{and}$$

$$g=\begin{pmatrix}\omega & 0\\ 0 & \eta^{-1}\omega\eta\end{pmatrix}\cdots \text{if } -1\in N_{F/\boldsymbol{Q}}(F_p^\times),$$

where $\omega\in O_p$ is a root of $x^2+ax+b=0$, and $\eta\in B_p^\times$ is any element such that $N(\eta)\notin N_{F/\boldsymbol{Q}}(F_p^\times)$.

PROPOSITION 2.6. *Notations and assumptions being as above,*

(i) *if* $\left(\frac{F}{p}\right)=1$, *then*

$$c_p(g, R_p, \Lambda_p)=\begin{cases} 1\cdots \textit{if } \Lambda_p\sim\xi\begin{pmatrix}O_p & 0\\ 0 & O_p\end{pmatrix}\xi^{-1}, \ d_p=e_p=1,\\ 0\cdots \textit{otherwise},\end{cases}$$

where $\xi\in GL_2(O_p)$ *is any element such that* $\xi^t\bar{\xi}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$,

(ii) *if* $\left(\frac{F}{p}\right)=-1$, *then* $c_p(g, R_p, \Lambda_p)=0$ *for any* Λ_p.

(iii) *if* $p=3$ *and* $F_p=\boldsymbol{Q}_p(\sqrt{-3})$, *then*

$$c_p(g, R_p, \Lambda_p)=\begin{cases} 1\cdots \textit{if } \Lambda_p\sim\xi M_2(F_p)\xi^{-1}\cap R_p=\Lambda_1,\\ 0\cdots \textit{otherwise},\end{cases}$$

$$d_p(\Lambda_1)=4,\quad e_p(\Lambda_1)=2,$$

where $\xi\in GL_2(O_p)$ *is any element such that* $\xi^t\bar{\xi}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

(iv) *if* $p=2$ *and* $F_p=\boldsymbol{Q}_p(\sqrt{-1})$, *put* $g=\begin{pmatrix}2 & -\varepsilon\\ \varepsilon & -2\end{pmatrix}\in G_p^*$. *Then we have*

$$c_p(g, R_p, \Lambda_p)=\begin{cases} 1\cdots \textit{if } \Lambda_p\sim\xi M_2(F)\xi^{-1}\cap\begin{pmatrix}2 & \varepsilon\\ 0 & 1\end{pmatrix}R_p\begin{pmatrix}2 & \varepsilon\\ 0 & 1\end{pmatrix}^{-1}=\Lambda_1,\\ 1\cdots \textit{if } \Lambda_p\sim\xi M_2(F)\xi^{-1}\cap\begin{pmatrix}4 & \beta\\ 0 & 1\end{pmatrix}R_p\begin{pmatrix}4 & \beta\\ 0 & 1\end{pmatrix}^{-1}=\Lambda_2,\\ 0\cdots \textit{otherwise},\end{cases}$$

$$d_p(\Lambda_1)=1,\ e_p(\Lambda_1)=2,\ d_p(\Lambda_2)=6,\ e_p(\Lambda_2)=2,$$

where ξ *is any element such that* $\xi^t\bar{\xi}=\begin{pmatrix}o & 1\\ 1 & o\end{pmatrix}$, *and* $\beta=\varepsilon(1+\pi)$, $\varepsilon^2=5$, $\pi^2=-2$, $\varepsilon\pi=-\pi\varepsilon$.

PROOF. (i) A representative of G_p^*-conjugacy class is given by $\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix}$, where $\omega_1, \omega_2\in\boldsymbol{Z}_p$ are distinct roots of $x^2+ax+b=0$. Therefore, the proof is virtually the same as in [3], Prop. 15. (ii) This is easily proved by Cor. 2.2.

(iii), (iv) The proof is lengthy and will be omitted here. q. e. d.

Next, let $g\in G$ belong to the case (V) and, $f_i(g)=0$ or $f_i(-g)=0$ for $i=8$ or 9. By virtue of Cor. 2.2, $c_p(g, R_p, \Lambda_p)=0$ for any Λ_p, unless $p=2$ and $f_9(g)=0$. When $p=2$ and $f_9(g)=0$, let $\zeta\in O_p$ be a root of unity with order 3. Then, g is G_p-conjugate to $h_1=\begin{pmatrix}\zeta & 0\\ 0 & -\zeta\end{pmatrix}\in G_p$, or $h_2=\begin{pmatrix}\zeta & 0\\ 0 & -\bar{\zeta}\end{pmatrix}\in G_p$.

PROPOSITION 2.7. *Notations being as above,*
(i) *if* $p=2$ *and* $f_9(g)=0$, *then* $c_p(h_2, R_p, \Lambda_p)=0$ *for any* Λ_p, *and*

$$c_p(h_1, R_p, \Lambda_p)=\begin{cases} 2 \cdots \text{if } \Lambda_p\sim\xi\left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix};\ a,\ b\in \boldsymbol{Z}_p[\zeta],\ a\equiv b \bmod. p\right\}\xi^{-1}, \\ 0 \cdots \text{otherwise,}\end{cases}$$

(ii) *in other cases,* $c_p(g, R_p, \Lambda_p)=0$ *for any* Λ_p.

The proof is virtually the same as in Prop. 2.4, and will be omitted here.

Finally, we assume that $g\in G$ belong to the case (VI). Put $F=\boldsymbol{Q}(g)$. If g is of finite order, then $\boldsymbol{Z}_p[g]$ is the maximal order of F_p. Therefore, $c_p(g, R_p, \Lambda_p)=0$ unless $\Lambda_p=\boldsymbol{Z}_p[g]$. Now, we put $c_p(g)=c_p(g, R_p, \boldsymbol{Z}_p[g])$. Now, the natural injection of G into $G_p\cong G_p^*$ induces a mapping of G-conjugacy classes to G_p^*-conjugacy classes, and the image of the set $\{g\in G;\ f_i(g)=0\}$ by this mapping decomposes into at most two G_p^*-conjugacy classes. We denote by t the number of these G_p^*-conjugacy classes.

PROPOSITION 2.8. *Let* $g\in G$ *be of order* 5 *or* 10.
(i) *If* $p\equiv 2$ *or* 3 *mod.* 5, *then* $t=1$ *and* $c_p(g)=2$,
(ii) *If* $p\equiv -1$ *mod.* 5, *then* $t=2$ *and we denote by* g, g' *the two representatives of the* G_p^**-conjugacy classes. Then,* $c_p(g)=c_p(g')=0$.
(iii) *If* $p=5$, *then* $t=1$ *and* $c_p(g)=1$.

PROOF. (ii) can be easily proved by Cor. 2.2. (i) and (iii) can be proved virtually in the same way as in [3] Prop. 19 by using Theorem of Chevalley (-Hasse-Noether). q. e. d.

Next, if $g\in G$ is of order 8 or 12, then g^2 belongs to the case (IV), and the G_p-conjugacy class containing g is uniquely determined by the G_p-conjugacy class containing g^2. As we have shown in [3] Prop. 8, we easily describe the local-global correspondence of the conjugacy classes by using the structure of $Z_0(g^2)$. For simplicity, we write $Z_0(g^2)=1$ (resp. -1) if $Z_0(g^2)$ is split (resp. division).

PROPOSITION 2.9. *Assume that* g *is of order* 8.

(i) *If* $p\equiv 7$ *mod.* 8, *then* $t=2$, *and* $c_p(g)=0$ *for both* G_p^**-conjugacy classes.*
(ii) *If* $p\equiv 3$ *or* 5 *mod.* 8, *then* $t=1$ *and* $c_p(g)=2$.
(iii) *If* $p=2$, *then* $t=2$, *and* $c_p(g)=1$ *for both* G_p^**-conjugacy classes.*

PROPOSITION 2.10. *Assume that* g *is of order* 12.
(i) *If* $p\equiv -1$ *mod.* 12, *then* $t=2$, *and* $c_p(g)=0$ *for both* G_p^**-conjugacy classes.*
(ii) *If* $p\equiv 5$ *mod.* 12, *then* $t=1$, $c_p(g)=2$, *and* $Z_0(g^2)_p=+1$.
(iii) *If* $p\equiv 7$ *mod.* 12, *then* $t=1$, $c_p(g)=2$, *and* $Z_0(g^2)_p=-1$.
(iv) *If* $p=2$, *then* $t=2$, *and we denote by* g, g' *the two representatives of the conjugacy classes. Then we have*:

$$c_p(g)=0, \quad Z_0(g^2)_p=-1, \quad \textit{and}$$
$$c_p(g')=1, \quad Z_0(g'^2)_p=+1.$$

(v) *If* $p=3$, *then* $t=2$, *and we take* g, g' *similarly as above. Then we have*:

$$c_p(g)=0, \quad Z_0(g^2)_p=-1, \quad \textit{and}$$
$$c_p(g')=1, \quad Z_0(g'^2)_p=+1.$$

The proofs of these two propositions are similar to that of [3] Prop. 19, and will be omitted here.

References

[1] Hashimoto, K., On Brandt matrices associated with the positive definite quaternion hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 227-245.
[2] Hashimoto, K., On class numbers of positive definite ternary quaternion hermitian forms, in preparation.
[3] Hashimoto, K. and T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 549-601, (II), J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), 695-699.
[4] Ibukiyama, T., On symplectic Euler factors of genus two, Doctoral thesis, Univ. Tokyo, 1980.
[5] Ihara, Y., On certain arithmetical Dirichlet series, J. Math. Soc. Japan **16** (1964), 214-235.
[6] Satake, I., Theory of spherical functions on reductive algebraic groups over 𝔭-adic fields, Publ. Math. IHES **18** (1964), 5-69.
[7] Shimura, G., Arithmetic of alternating forms and quaternion hermitian forms, J. Math. Soc. Japan **15** (1963), 33-65.

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