ENDOSCOPY ON GSP(4)

HIRAGA, KAORU

This is a brief summary on endoscopies corresponding to $GSp(4)$ [1].

1. Introduction

In this article, F will be a number field or a local field of characteristic 0. We write W_F for the Weil group. Let

$$\tilde{G} = GL_4 \times GL_1$$

defined over F. Then the dual group of $\tilde{G}$ is

$$\hat{\tilde{G}} = GL_4(\mathbb{C}) \times \mathbb{C}^\times.$$

Since $\tilde{G}$ is split over F, the action of W_F on $\hat{\tilde{G}}$ is trivial, and the L-group ${}^L\tilde{G}$ of $\tilde{G}$ is $\hat{\tilde{G}} \times W_F$. We define an outer-automorphism θ of $\tilde{G}$ by

$$\theta(g,z) = (J\,{}^t g^{-1} J^{-1}\,,\, \det g \cdot z), \qquad (g,z) \in GL_4 \times GL_1,$$

where [2]

$$J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

Then θ is defined over F. We define an outer-automorphism $\hat{\theta}$ of $\hat{\tilde{G}}$ by

$$\hat{\theta}(g,z) = (J\,{}^t g^{-1} J^{-1} \cdot z\,,\, z), \qquad (g,z) \in GL_4(\mathbb{C}) \times \mathbb{C}^\times.$$

Then $\hat{\theta}$ is the dual automorphism of $\hat{\tilde{G}}$ preserving the standard splitting of $\hat{\tilde{G}}$. We put

$$\hat{G} = \hat{\tilde{G}}^{\hat{\theta}} = \{(g,z) \in \hat{\tilde{G}} \mid \hat{\theta}(g,z) = (g,z)\},$$

[1] Arthur's good article [Art04] on this subject is available from " Arthur archive ": http://www.claymath.org/cw/arthur/index.php

[2] This formulation might be different from the one in the other articles in this book.

and we make W_F act on $\hat{G}$ trivially. Then $\hat{G} = GSp(4, \mathbb{C})$ and it is the dual group of $GSpin(5) = GSp(4)$ (split over F). We put[3]

$$G = GSp(4) = \{(g, z) \in \tilde{G} | \, (J\,{}^t g^{-1} Jz \, , \, z) = (g, z)\}.$$

(This automorphism of $\tilde{G}$ is defferent from θ). Then we may regard $\hat{G}$ as the dual group of G [4]. The L-group ${}^L G$ of G is $\hat{G} \times W_F$.

By the above formulation, we may regard G as a twisted endoscopic group of $\tilde{G}$. More precisely, the group G is the first component of twisted endoscopic data $(G, {}^L G, 1, \xi)$ for $(\tilde{G}, \theta)$, where $1 \in \hat{\tilde{G}}$ is the identity element and ξ is the natural embedding of ${}^L G$ to ${}^L \tilde{G}$. (See [KS99, §2.1] for the definition of the endoscopic data).

2. Standard endoscopy for $GSp(4)$

We put

$$T_0 = \left\{ \left(\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_2^{-1} z & \\ & & & x_1^{-1} z \end{pmatrix}, z \right) \right\} \subset G$$

$$B_0 = \left\{ \left(\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, z \right) \in G \right\}$$

Then T_0 is a split maximal torus of G and B_0 is a Borel subgroup of G. The standard parabolic subgroups of G are G, B_0, P_S, and P_J, where

$$P_S = \left\{ \left(\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, z \right) \in G \right\},$$

$$P_J = \left\{ \left(\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, z \right) \in G \right\}.$$

[3]We could define $GSp(4)$ by putting $J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$, but in order to avoid the confusion, we use the same J for G and $\hat{G}$.

[4]We have to keep in mind that the correspondences between G and $\tilde{G}$ (in the theory of endoscopy) are defined through the dual groups.

The Levi factors of P_S and P_J are

$$M_S = \left\{ \left(\begin{pmatrix} A & 0 \\ 0 & J_2{}^t A^{-1} J_2^{-1} z \end{pmatrix}, z \right) \middle| \; A \in GL_2, \, z \in GL_1 \right\},$$

$$M_J = \left\{ \left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ 0 & 0 & 0 & a^{-1}z \end{pmatrix}, z \right) \middle| \; z \in GL_1, \, \det \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = z \right\},$$

where $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The Levi factors T_0, M_S, M_J, and G are standard endoscopic groups of G. There exists one more standard endoscopic group for G. Let

$$s = \left(\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, 1 \right) \in \hat{G}.$$

Then

$$\mathrm{Cent}(s, \hat{G}) = \left\{ \left(\begin{pmatrix} a_{11} & & & a_{12} \\ & b_{11} & b_{12} & \\ & b_{21} & b_{22} & \\ a_{21} & & & a_{22} \end{pmatrix}, z \right) \in \hat{G} \right\}.$$

(This matrix is contained in $\hat{G}$ if and only if

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \det \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = z \in \mathbb{C}^{\times}).$$

We put

$$\hat{H}_{ell} = \mathrm{Cent}(s, \hat{G}).$$

and make W_F act on $\hat{H}_{ell}$ trivially. Then

$${}^L H_{ell} = \hat{H}_{ell} \times W_F$$

is the L-group of

$$H_{ell} = GL_2 \times GL_2 / \{ (x, x^{-1}) \, | \, x \in GL_1 \},$$

and

$$(H_{ell}, {}^L H_{ell}, s, \xi)$$

is a set of elliptic endoscopic data for G, where ξ is the natural embedding of ${}^L H_{ell}$ to ${}^L G$.

3. Transfer

In this section, F will be a local field of characteristic 0. We say that $\gamma \in G(F)$ is *strongly regular semisimple* if $T = \mathrm{Cent}(\gamma, G)$ is a connected maximal torus of G. We write $G(F)_{\mathrm{sreg}}$ for the set of strongly regular semisimple elements in $G(F)$. For $\gamma \in G(F)_{\mathrm{sreg}}$ and $f \in C_c^\infty(G(F))$, the orbital integral $J(\gamma, f)$ and the stable orbital integral $J^{st}(\gamma, f)$ are defined by

$$J(\gamma, f) = \int_{G(F)/T(F)} f(g\gamma g^{-1})\, dg$$

$$J^{st}(\gamma, f) = \int_{G/T(F)} f(g\gamma g^{-1})\, dg.$$

We write $\mathcal{D}(G)$ for the space of invariant distributions on $G(F)$. We say that $J \in \mathcal{D}(G)$ is *stable* if

$$J(C_c^{\infty,0}(G(F))) = 0,$$

where

$$C_c^{\infty,0}(G(F)) = \{f \in C_c^\infty(G(F)) |\, J^{st}(\gamma, f) = 0,\ {}^\forall\gamma \in G(F)_{\mathrm{sreg}}\},$$

and denote the space of stable distributions on $G(F)$ by $\mathcal{D}^{st}(G)$. Similarly, we define $\mathcal{D}(H_{ell})$, $\mathcal{D}^{st}(H_{ell})$, $\mathcal{D}(M_S)$, $\mathcal{D}^{st}(M_S)$, ...[5] If π is an irreducible admissible representation of $G(F)$, we write the distribution character of π by $J(\pi, \cdot)$.

We say that $\gamma \in \tilde{G}(F)$ is *strongly θ-regular θ-semisimple* if

$$G_{\gamma\theta} = \{g \in \tilde{G} |\, g\gamma\theta(g)^{-1} = \gamma\}$$

is an abelian torus, and write $\tilde{G}(F)_{\theta-\mathrm{sreg}}$ for the set of strongly θ-regular θ-semisimple elements in $\tilde{G}(F)$. For $\gamma \in \tilde{G}(F)_{\theta-\mathrm{sreg}}$ and $f \in C_c^\infty(\tilde{G}(F))$, we define the twisted orbital integral by

$$J^\theta(\gamma, f) = \int_{G(F)/G_{\gamma\theta}(F)} f(g\gamma\theta(g)^{-1})\, dg.$$

If π is an irreducible admissible θ-invariant representations of $\tilde{G}(F)$, we denote the twisted character of π by $J^\theta(\pi, \cdot)$. We write $\mathcal{D}^\theta(\tilde{G})$ for the space of θ-invariant distributions on $\tilde{G}(F)$. Then $J^\theta(\gamma, \cdot), J^\theta(\pi, \cdot) \in \mathcal{D}^\theta(\tilde{G})$.

[5] We have $\mathcal{D}(H) = \mathcal{D}^{st}(H)$ for $H = H_{ell}, M_S, M_J, T_0$.

We want to define the endoscopic transfer of distributions

$$\mathrm{Tran}_G^{\tilde{G}} : \mathcal{D}^{st}(G) \longrightarrow \mathcal{D}^{\theta}(\tilde{G}),$$
$$\mathrm{Tran}_H^G : \mathcal{D}^{st}(H) \longrightarrow \mathcal{D}(G), \quad \text{for } H = H_{ell}, T_0, M_S, M_J, G.$$

For $H = T_0, M_S, M_J, G$, the endoscopic transfer Tran_H^G is defined by the parabolic induction[6]. So it remains to define $\mathrm{Tran}_G^{\tilde{G}}$ and $\mathrm{Tran}_{H_{ell}}^G$. Let

$$\mathcal{A}_{G/\tilde{G}} : Cl_{ss}(G) \longrightarrow Cl_{ss}(\tilde{G}, \theta),$$
$$\mathcal{A}_{H_{ell}/G} : Cl_{ss}(H) \longrightarrow Cl_{ss}(G),$$

be the map defined in [KS99, Theorem 3.3A], where $Cl_{ss}(\tilde{G}, \theta)$ is the set of θ-conjugacy classes of θ-semisimple elements in $G(\overline{F})$ and $Cl(G)$ (resp. $Cl(H_{ell})$) is the set of conjugacy classes of semisimple elements in $G(\overline{F})$ (resp. $H_{ell}(\overline{F})$). We say that $\gamma_G \in G(F)_{\mathrm{sreg}}$ is a *norm* of $\gamma_{\tilde{G}} \in \tilde{G}(F)_{\theta-\mathrm{sreg}}$ if $\gamma_{\tilde{G}}$ is contained in the image of the $G(\overline{F})$-conjugacy class of γ_G under $\mathcal{A}_{G/\tilde{G}}$. Similarly, we say that $\gamma_{H_{ell}} \in H_{ell}(F)_{\mathrm{sreg}}$ is a *norm*[7] of $\gamma_G \in G(F)_{\mathrm{sreg}}$ if γ_G is contained in the image of the $H_{ell}(\overline{F})$-conjugacy class of $\gamma_{H_{ell}}$ under $\mathcal{A}_{H_{ell}/G}$. We say that $\gamma_G \in G(F)_{\mathrm{sreg}}$ is *strongly $\tilde{G}$-regular semisimple* if γ_G is a norm of some $\gamma_{\tilde{G}} \in \tilde{G}(F)_{\theta-\mathrm{sreg}}$, and $\gamma_{H_{ell}} \in H_{ell}(F)_{\mathrm{sreg}}$ is *strongly G-regular semisimple* if $\gamma_{H_{ell}}$ is a norm of some $\gamma_G \in G(F)_{\mathrm{sreg}}$. We write $G(F)_{\tilde{G}-\mathrm{sreg}}$ for the set of strongly $\tilde{G}$-regular semisimple elements in $G(F)_{\mathrm{sreg}}$, and $H_{ell}(F)_{G-\mathrm{sreg}}$ for the set of strongly G-regular semisimple elements in $H_{ell}(F)_{\mathrm{sreg}}$. In [LS87], Langlands–Shelstad defined the transfer factor $\Delta(\gamma_{H_{ell}}, \gamma_G)$ for $\gamma_G \in G(F)_{\mathrm{sreg}}$ and its norm $\gamma_{H_{ell}} \in H_{ell}(F)_{G-\mathrm{sreg}}$, and in [KS99, Chap.4–5], Kottwitz–Shelstad defined the transfer factor $\Delta(\gamma_G, \gamma_{\tilde{G}})$ for $\gamma_{\tilde{G}} \in \tilde{G}(F)_{\theta-\mathrm{sreg}}$ and its norm $\gamma_G \in G(F)_{\tilde{G}-\mathrm{sreg}}$. If $\gamma_G \in G(F)_{\tilde{G}-\mathrm{sreg}}$ is not a norm of $\gamma_{\tilde{G}} \in \tilde{G}(F)_{\theta-\mathrm{sreg}}$, then we put $\Delta(\gamma_G, \gamma_{\tilde{G}}) = 0$, and if $\gamma_{H_{ell}} \in H_{ell}(F)_{G-\mathrm{sreg}}$ is not a norm of $\gamma_G \in G(F)_{\mathrm{sreg}}$, then we put $\Delta(\gamma_{H_{ell}}, \gamma_G) = 0$.

In order to define the endoscopic transfer, we need the following conjectures.

[6]Endoscopic transfer Tran_H^G does depend on the fourth component ξ of the endoscopic data. For $H = T_0, M_S, M_J, G$, we take ξ to be the natural embedding, then Tran_H^G matches with the transfer defined by the parabolic induction.

[7]Usually, the term "image" is used.

Conjecture 3.1 (Transfer conjecture for $(\tilde{G}, G)$)**.** *For any function $f^{\tilde{G}} \in C_c^\infty(\tilde{G}(F))$, there exists $f^G \in C_c^\infty(G(F))$ which satisfies*

$$J^{st}(\gamma_G, f^G) = \sum_{\gamma_{\tilde{G}}} \Delta(\gamma_G, \gamma_{\tilde{G}}) J^\theta(\gamma_{\tilde{G}}, f^{\tilde{G}}), \quad {}^\forall\gamma_G \in G(F)_{\tilde{G}-\mathrm{sreg}},$$

where the sum runs over the θ-conjugacy classes in $\tilde{G}(F)_{\theta-\mathrm{sreg}}$.

If $f^{\tilde{G}}$ and f^G satisfy the above equation, then we say that $f^{\tilde{G}}$ and f^G *have matching orbital integrals.*

Conjecture 3.2 (Transfer conjecture for (G, H_{ell}))**.** *For any function $f^G \in C_c^\infty(G(F))$, there exists $f^{H_{ell}} \in C_c^\infty(H_{ell}(F))$ which satisfies*

$$J^{st}(\gamma_{H_{ell}}, f^{H_{ell}}) = \sum_{\gamma_G} \Delta(\gamma_{H_{ell}}, \gamma_G) J(\gamma_{H_{ell}}, f^G), \quad {}^\forall\gamma_{H_{ell}} \in H_{ell}(F)_{G-\mathrm{sreg}},$$

where the sum runs over the θ-conjugacy classes in $G(F)_{\mathrm{sreg}}$.

If f^G and $f^{H_{ell}}$ satisfy the above equation, then we say that f^G and $f^{H_{ell}}$ *have matching orbital integrals.*

For $F = \mathbb{R}, \mathbb{C}$, the transfer conjecture for standard endoscopy is proved by Shelstad (see [She82]), and the transfer conjecture for twisted endoscopy is proved by Renard [Ren03]. If F is a p-adic field, then the transfer conjecture for (G, H_{ell}) is proved by Hales [Hal89], and the transfer conjecture for $(\tilde{G}, G)$ is essentially proved by Hales [Hal94] (see [Walpp] for descent).

By using the transfer conjecture, we can define the endoscopic transfer by

$$(\mathrm{Tran}_G^{\tilde{G}} J_G)(f^{\tilde{G}}) = J_G(f^G), \quad J_G \in \mathcal{D}^{st}(G),$$

where $f^{\tilde{G}}$ and f^G have matching orbital integrals, and

$$(\mathrm{Tran}_{H_{ell}}^G J_{H_{ell}})(f^G) = J_{H_{ell}}(f^{H_{ell}}), \quad J_{H_{ell}} \in \mathcal{D}^{st}(H_{ell}),$$

where f^G and $f^{H_{ell}}$ have matching orbital integrals.[8]

4. Local A-packets

In this section, we still assume that F is a local field of characteristic 0. We put

$$L_F = \begin{cases} W_F, & F = \mathbb{R}, \mathbb{C}, \\ W_F \times SU_2, & F = \ p\text{-adic field}. \end{cases}$$

[8]For $H = T_0, M_S, M_J, G$, we can also define Tran_H^G by using the functions with matching orbital integrals, and this matches with the definition using the parabolic induction.

We fix a 1 dimensional character

$$\chi : L_F \longrightarrow \mathbb{C}^\times.$$

We say that a semisimple representation[9]

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$$

is χ-*self dual* if

$${}^t\psi^{-1} \cdot \chi \simeq \psi,$$

and χ-self dual representation ψ is called *symplectic* (resp. *orthogonal*) if there exists $A \in GL_n(\mathbb{C})$ such that

$$A {}^t\psi^{-1}\chi A^{-1} = \psi,$$
$${}^tA = -A \quad (\text{resp. } {}^tA = A).$$

If

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow {}^LG$$

is an A-parameter such that

$$L_F \xrightarrow{\psi} {}^LG \longrightarrow {}^L\tilde{G} \xrightarrow{pr_2} \mathbb{C}^\times$$

is equal to χ, where pr_2 is a projection of ${}^L\tilde{G} = GL_4(\mathbb{C}) \times \mathbb{C}^\times \times W_F$ to $\mathbb{C}^\times$, then

$$\psi_0 : L_F \xrightarrow{\psi} {}^LG \longrightarrow {}^L\tilde{G} \xrightarrow{pr_1} GL_4(\mathbb{C})$$

is a χ-self dual symplectic semisimple representation of degree 4, where pr_1 is a projection of ${}^L\tilde{G} = GL_4(\mathbb{C}) \times \mathbb{C}^\times \times W_F$ to $GL_4(\mathbb{C})$.

Lemma 4.1. *If ψ and ψ' are A-parameters on $G = GSp(4)$, then ψ and ψ' are equivalent as A-parameters if and only if ψ_0 and ψ'_0 are equivalent as 4 dimensional representations of $L_F \times SL_2(\mathbb{C})$, where ψ_0 and ψ'_0 are defined as above.*

Remark 4.2. Although the existence of the global Langlands group is still hypothetical, we can prove the global analogue of Lemma 4.1 under the assumption that the Langlands group exists.

By Lemma 4.1, we may identify A-parameters on G with 4 dimensional semisimple χ-self dual symplectic representations. In the follwing, we will identify A-parameters with 4 dimensional semisimple χ-self dual symplectic representations.

For an A-parameter

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow {}^LG,$$

[9] In this article, we always assume that the restrictin of ψ to SL_2 is algebraic.

we put

$$S_\psi = \{s \in Sp(4,\mathbb{C}) |\ \mathrm{Int}\, s \circ \psi = \psi\},$$
$$\mathcal{S}_\psi = S_\psi / S_\psi^0,$$

where $\cdot^0$ means the identity component. We denote by $\mathcal{Z}_\psi$ the image of the center $\{\pm I_4\}$ of $Sp(4,\mathbb{C})$ in $\mathcal{S}_\psi$. For $s \in S_\psi$, we put

$${}^L H_s = \mathrm{Cent}(s, \hat{G})^0 \psi(W_F) \subset {}^L G.$$

Then ${}^L H_s$ is isomorphic to the dual group of a quasi-split reductive group H_s [10]. Since ψ factors through ${}^L H_s$, we can define an A-parameter

$$\psi_{H_s} :\ L_F \times SL_2(\mathbb{C}) \longrightarrow {}^L H_s.$$

If F is a p-adic field, we define a homomorphism $\alpha :\ L_F \longrightarrow L_F \times SL_2(\mathbb{C})$ by

$$\alpha(w \times t) = w \times t \times \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{pmatrix}, \quad w \times t \in W_F \times SU_2.$$

Similarly, if $F = \mathbb{R}, \mathbb{C}$, we define $\alpha :\ L_F \longrightarrow L_F \times SL_2(\mathbb{C})$ by

$$\alpha(w) = w \times \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{pmatrix}, \quad w \in W_F.$$

For an A-parameter ψ, we put

$$\phi_\psi = \psi \circ \alpha :\ L_F \xrightarrow{\alpha} L_F \times SL_2(\mathbb{C}) \xrightarrow{\psi} {}^L G.$$

Then ϕ_ψ is an L-parameter. Similarly, we get an L-parameter $\phi_{\psi_{H_s}}$ from ψ_{H_s}. By the Local Langlands correspondence for GL_n [HT01, Hen00], the L-parameter ϕ_ψ defines an irreducible admissible representation $\tilde{\pi}_\psi$ of $\tilde{G}(F)$, since ϕ_ψ defines an L-parameter $L_F \xrightarrow{\phi_\psi} {}^L G \longrightarrow {}^L \tilde{G}$. Similarly, if $H_s \neq GSp(4)$, in other words $s \neq \pm I_4$, the L-parameter $\phi_{\psi_{H_s}}$ defines an irreducible admissible representation $\pi_{\psi_{H_s}}$ of $H_s(F)$. We put $\Pi_\psi(\tilde{G}) = \{\tilde{\pi}_\psi\}$. If $H_s \neq GSp(4)$, we also put $\Pi_{\psi_{H_s}}(H_s) = \{\pi_{\psi_{H_s}}\}$.

Conjecture 4.3 (Conjecture for $GSp(4)$). *For an A-parameter*

$$\psi :\ L_F \times SL_2(\mathbb{C}) \longrightarrow {}^L G,$$

there exists a finite set $\Pi_\psi(G)$ of irreducible admissible representations of $G(F)$ satisfying the following conditions.

(1) There exists a non-zero stable virtual character $J(\psi)$ which is contained in the subspace of $\mathcal{D}(G)$ spanned by $\{J(\pi) |\ \pi \in \Pi_\psi(G)\}$.

[10] Since $G = GSp(4)$, this holds.

(2) There exists a non-zero constant $c \in \mathbb{C}^\times$ *such that*

$$\mathrm{Tran}_G^{\tilde{G}}\, J(\psi) = c \cdot J^\theta(\tilde{\pi}_\psi).$$

(3) There exists a map

$$\Pi_\psi(G) \longrightarrow \Pi(\mathcal{S}_\psi/\mathcal{Z}_\psi),$$

where $\Pi(\mathcal{S}_\psi/\mathcal{Z}_\psi)$ *is the set of equivalence classes of irreducible representations of* $\mathcal{S}_\psi/\mathcal{Z}_\psi$*. We write* $\langle \,\cdot\,, \pi\rangle$ *for the irreducible character of* $\mathcal{S}_\psi/\mathcal{Z}_\psi$ *which is defined by* $\pi \in \Pi_\psi(G)$ *through the above map.*

(4) For $s \in S_\psi$*, there exists a non-zero constant* $c \in \mathbb{C}^\times$ *such that*

$$\mathrm{Tran}_{H_s}^{G}\, J(\psi_{H_s}) = c \cdot \sum_{\pi \in \Pi_\psi(G)} \langle s s_\psi, \pi\rangle J(\pi),$$

where $s_\psi = \psi\left(1 \times \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right)$, $(1 \times \left(\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right) \in L_F \times SL_2(\mathbb{C}))$, *and* $J(\psi_{H_s}) = J(\pi_{\psi_{H_s}})$ *if* $H_s \neq GSp(4)$*.*

(5) The conjectural L-packet $\Pi_{\phi_\psi}(G)$ *of* ϕ_ψ *is contained in* $\Pi_\psi(G)$*.*

(6) There exists a unique π_{gen} *in* $\Pi_{\phi_\psi}(G)$ *whose associated standard representation is generic*[11]*.*

(7) The irreducible character $\langle \,\cdot\,, \pi_{gen}\rangle$ *is the trivial character of* $\mathcal{S}_\psi/\mathcal{Z}_\psi$*.*

Remark 4.4. The condition (4) implies that

$$\sum_{\pi \in \Pi_\psi(G)} \langle s_\psi, \pi\rangle J(\pi)$$

is a stable distribution.

In the following, we assume that χ is trivial. It is easy to see that an A-parameter ψ is elliptic if and only if ψ is a 4 dimensional irreducible χ-self dual symplectic representation or the direct sum of two 2 dimensional irreducible χ-self dual symplectic representations ψ_1, ψ_2 such that $\psi_1 \not\simeq \psi_2$. If ψ is a 4 dimensional irreducible χ-self dual symplectic representation, then $\mathcal{S}_\psi/\mathcal{Z}_\psi = \{1\}$. If ψ is the direct sum of two 2 dimensional irreducible χ-self dual symplectic representations ψ_1, ψ_2 such that $\psi_1 \not\simeq \psi_2$, then $\mathcal{S}_\psi/\mathcal{Z}_\psi \simeq \mathbb{Z}/2\mathbb{Z}$.

Example 4.5 (Yoshida lift type)**.** Let $\psi = \psi_1 \oplus \psi_2$, where ψ_1 and ψ_2 are irreducible 2 dimensional χ-self dual symplectic representations of L_F such that $\psi_1 \not\simeq \psi_2$. (Therefore $\psi(SL_2(\mathbb{C}))$ is trivial). Then $\Pi_\psi(G)$ should consist of two square integrable representations π_{gen}, π_h, where

[11] Because there exists a only one regular unipotent orbit in $GSp(4, F)$, it is not necessary to choose a generic character.

π_{gen} is a generic representation. (If $F = \mathbb{R}$ then π_h is a holomorphic discrete series representation). The irreducible character $\langle \cdot, \pi_{gen} \rangle$ should be trivial, and $\langle \cdot, \pi_h \rangle$ should be the other (non-trivial) character of $\mathbb{Z}/2\mathbb{Z}$. If the image of $s \in S_\psi$ in $\mathcal{S}_\psi/\mathcal{Z}_\psi$ is not the identity element, then $H_s \simeq H_{ell}$ and ψ_{H_s} defines an irreducible square integrable representation $\pi_{\psi_{H_s}}$ of $H_s(F)$. The virtual character

$$J(\pi_{gen}) + J(\pi_h)$$

should be stable, and

$$\mathrm{Tran}_{H_s}^{G} J(\pi_{\psi_{H_s}}) = J(\pi_{gen}) - J(\pi_h)$$

for a suitable choice of the scalar factor of the transfer factor.

In the following, we take a suitable scalar factor of the transfer factor so that the above equation holds.

Example 4.6 (Saito–Kurokawa lift type)**.** Let $\psi = \psi_1 \oplus \psi_2$, where ψ_1 is a 2 dimensional irreducible χ-self dual symplectic representation of L_F and ψ_2 is the 2 dimensional irreducible (algebraic) representation of $SL_2(\mathbb{C})$. Then $\mathcal{S}_\psi/\mathcal{Z}_\psi \simeq \mathbb{Z}/2\mathbb{Z}$, and $\Pi_\psi(G)$ should consist of two irreducible admissibile representations of $G(F)$. First, we consider the L-packet $\Pi_{\phi_\psi}(G)$. Since ϕ_ψ factors through ${}^L M_S$, the L-parameter ϕ_ψ determines an irreducible essentially square integrable representation π_M of $M_S(F)$. Then the Langlands subquotient π_{SK} of $\mathrm{Ind}_{P_S}^{G} \pi_M$ should be the unique element in $\Pi_{\phi_\psi}(G)$. Hence

$$\{\pi_{SK}\} = \Pi_{\phi_\psi}(G) \subset \Pi_\psi(G).$$

What is the other representation in $\Pi_\psi(G)$? Let ψ_2' be the irreducible 2 dimensional representations of L_F which corresponds to the Steinberg representation of $GL_2(F)$ if F is a p-adic field, and the weight 2 discrete series representation of $GL_2(F)$ if $F = \mathbb{R}$. Put $\psi' = \psi_1 \oplus \psi_2'$. Then $S_\psi = S_{\psi'}$. Since ψ' is of Yoshida lift type, we should have $\Pi_{\psi'}(G) = \{\pi_{gen}, \pi_h\}$. Let $s \in S_\psi$ be an element whose image in $\mathcal{S}_\psi/\mathcal{Z}_\psi$ is not the identity element. Then, as in the case of Yoshida lift type, we have ${}^L H_s \simeq {}^L H_{ell}$, and ψ' (resp. ψ) determines an irreducible representation $\pi_{\psi'_{H_s}}$ (resp. $\pi_{\psi_{H_s}}$). Since

$$\mathrm{Tran}_{M_S}^{H_s}(\pi_M) = J(\pi_{\psi_{H_s}}) + J(\pi_{\psi'_{H_s}}),$$
$$\mathrm{Tran}_{M_S}^{G} J(\pi_M) = J(\pi_{SK}) + J(\pi_{gen}),$$

should hold, we should have[12]

$$\begin{aligned}\mathrm{Tran}_{H_s}^{G} J(\psi_{H_s}) &= \mathrm{Tran}_{H_s}^{G} J(\pi_{\psi_{H_s}}) \\ &= \mathrm{Tran}_{H_s}^{G} \left(\mathrm{Tran}_{M_S}^{H_s} J(\pi_M) - J(\pi_{\psi'_{H_s}})\right) \\ &= \mathrm{Tran}_{M_S}^{G} J(\pi_M) - \mathrm{Tran}_{H_s}^{G} J(\pi_{\psi'_{H_s}}) \\ &= J(\pi_{SK}) + J(\pi_{gen}) - (J(\pi_{gen}) - J(\pi_h)) \\ &= J(\pi_{SK}) + J(\pi_h).\end{aligned}$$

Because the image of s_ψ in $\mathcal{S}_\psi/\mathcal{Z}_\psi$ is not the identity element, this means that

$$\Pi_\psi(G) = \{\pi_{SK}, \pi_h\}$$

and $\langle\,\cdot\,, \pi_{SK}\rangle$ should be the trivial representation of $\mathcal{S}_\psi/\mathcal{Z}_\psi \simeq \mathbb{Z}/2\mathbb{Z}$ and $\langle\,\cdot\,, \pi_h\rangle$ should be the other (non-trivial) representation of $\mathcal{S}_\psi/\mathcal{Z}_\psi$. (If $F = \mathbb{R}$, this follows from the result of Shelstad [She82]).

5. Global multiplicity formula

In this section, F will be a number field. Let L_F be the conjectural Langlands group. We fix a 1 dimensional character χ of W_F. Then χ determines a 1 dimensional character of the center of $G(F)$, which we also denote by χ. We denote by r_{disc} the right regular representation of $G(\mathbb{A}_F)$ on $L^2_{disc}(G(F)\backslash G(\mathbb{A}_F)), \chi)$, where $L^2_{disc}(G(F)\backslash G(\mathbb{A}_F)), \chi)$ is the subspace of $L^2(G(F)\backslash G(\mathbb{A}_F)), \chi)$ consisting of the discrete spectrum. Let

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow {}^L G$$

be an elliptic A-parameter such that $pr_2 \circ \psi$ matches with the character $L_F \longrightarrow W_F \xrightarrow{\chi} \mathbb{C}^\times$, where $pr_2 : {}^L\tilde{G} \longrightarrow \mathbb{C}^\times$ is the projection from ${}^L\tilde{G} = GL_4(\mathbb{C}) \times \mathbb{C}^\times \times W_F$ to $\mathbb{C}^\times$. We denote by $\Psi_{ell}(G, \chi)$ the set of equivalence classes of such elliptic A-parameters. As in the case of the local A-parameters, we define $\mathcal{S}_\psi/\mathcal{Z}_\psi$. For any place v of F, the A-parameter ψ should give a local A-parameter

$$\psi_v : L_{F_v} \times SL_2(\mathbb{C}) \longrightarrow {}^L G.$$

Let $\pi_v \in \Pi_\psi(G)$. Then, since there exist natural homomorphisms

$$S_\psi \longrightarrow S_{\psi_v},$$
$$\mathcal{S}_\psi/\mathcal{Z}_\psi \longrightarrow \mathcal{S}_{\psi_v}/\mathcal{Z}_{\psi_v},$$

we can define $\langle s, \pi_v\rangle$ for $s \in S_\psi$. For each place v of F, we take an irreducible admissible representation π_v of $G(F_v)$ so that π_v are

[12] Since $\mathrm{Tran}_{M_S}^{H_s}$ matches with the parabolic induction, we have $\mathrm{Tran}_{M_S}^{G} = \mathrm{Tran}_{H_s}^{G} \circ \mathrm{Tran}_{M_S}^{H_s}$. In general, we cannot composite the endoscopic transfers.

unramified for almost all places v of F. Then, for $\pi = \otimes'_v \pi_v$ and $s \in S_\psi$, we put

$$\langle s, \pi \rangle_\psi = \begin{cases} \prod_v \langle s, \pi_v \rangle, & \text{if } \pi_v \in \Pi_{\psi_v}(G) \text{ for all places } v \text{ of } F, \\ 0, & \text{otherwise.} \end{cases}$$

As in [Art90, (4.5)], we define a character

$$\epsilon_\psi : \mathcal{S}_\psi / \mathcal{Z}_\psi \longrightarrow \{\pm 1\}.$$

Conjecture 5.1. *The multiplicity of* $\pi = \otimes' \pi_v$ *in* r_{disc} *is*

$$\sum_{\psi \in \Psi_{ell}(G,\chi)} \frac{1}{\sharp \mathcal{S}_\psi / \mathcal{Z}_\psi} \sum_{s \in \mathcal{S}_\psi / \mathcal{Z}_\psi} \epsilon_\psi(s) \langle s, \pi \rangle_\psi.$$

Remark 5.2. Since $\mathcal{S}_{\psi_v} / \mathcal{Z}_{\psi_v}$ are abelian groups, Remark 4.2 and Conjecture 5.1 imply that r_{disc} is multiplicity free.

Remark 5.3. By conjectural correspondence between the irreducible n dimensional representations of L_F and the irreducible cuspidal automorphic representations of $GL_n(F)$, we can formulate Conjecture 5.1 without using the conjectural Langlands group L_F.

In the following, we assume that χ is the trivial character.

Example 5.4. If ψ is an 4 dimensional irreducible χ-self dual symplectic representation of $L_F \times SL_2(\mathbb{C})$, then $\mathcal{S}_\psi / \mathcal{Z}_\psi = \{1\}$. Therefore any irreducible representation $\pi = \otimes' \pi_v$ of $G(\mathbb{A}_F)$ such that $\pi_v \in \Pi_{\psi_v}(G)$ for all places v of F should appear in r_{disc}.

Example 5.5 (Yoshida lift type)**.** Let $\psi = \psi_1 \oplus \psi_2$, where ψ_1 and ψ_2 are 2 dimensional irreducible χ-self dual symplectic representations of L_F such that $\psi_1 \not\simeq \psi_2$. Then

$$\mathcal{S}_\psi / \mathcal{Z}_\psi \simeq \mathbb{Z}/2\mathbb{Z},$$

and the character ϵ_ψ is the trivial character of $\mathcal{S}_\psi / \mathcal{Z}_\psi$. Assume that there exist two places v_1, v_2 such that ψ_{v_1}, ψ_{v_2} are elliptic. Then for $i = 1, 2$, we have $\mathcal{S}_{\psi_{v_i}} / \mathcal{Z}_{\psi_{v_i}} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\Pi_{\psi_{v_i}}(G) = \{\pi_{gen,v_i}, \pi_{h,v_i}\}$. Moreover, we assume that $\mathcal{S}_{\psi_v} / \mathcal{Z}_{\psi_v} = \{1\}$ for $v \neq v_1, v_2$. Then for $v \neq v_1, v_2$, the A-packet $\Pi_{\psi_v}(G)$ consists of a single representation π_v. For such an A-parameter ψ, Conjecture 5.1 says that

$$\pi_{gen,v_1} \otimes \pi_{gen,v_2} \bigotimes \otimes'_{v \neq v_1, v_2} \pi_v,$$

$$\pi_{h,v_1} \otimes \pi_{h,v_2} \bigotimes \otimes'_{v \neq v_1, v_2} \pi_v$$

appear in r_{disc} and

$$\pi_{gen,v_1} \otimes \pi_{h,v_2} \bigotimes \otimes'_{v \neq v_1,v_2} \pi_v,$$

$$\pi_{h,v_1} \otimes \pi_{gen,v_2} \bigotimes \otimes'_{v \neq v_1,v_2} \pi_v$$

do not appear in r_{disc}.

Example 5.6 (Saito–Kurokawa type)**.** Let $\psi = \psi_1 \oplus \psi_2$, where ψ_1 is a 2 dimensional irreducible χ-self dual symplectic representation of L_F and ψ_2 is the 2 dimensional irreducible (algebraic) representation of $SL_2(\mathbb{C})$. Then

$$\mathcal{S}_\psi/\mathcal{Z}_\psi \simeq \mathbb{Z}/2\mathbb{Z},$$

and

$$\epsilon_\psi = \begin{cases} \text{trivial character}, & \text{if } \epsilon(\frac{1}{2}, \psi_1) = 1, \\ \text{non-trivial character}, & \text{if } \epsilon(\frac{1}{2}, \psi_1) = -1. \end{cases}$$

We assume that ψ satisfies the similar conditions in Example 5.5. Hence ψ_{v_1}, ψ_{v_2} are elliptic, and

$$\mathcal{S}_{\psi_v}/\mathcal{Z}_{\psi_v} = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } v = v_1, v_2, \\ \{1\} & \text{otherwise}, \end{cases}$$

and

$$\Pi_{\psi_v}(G) = \begin{cases} \{\pi_{SK,v}, \pi_{h,v}\}, & \text{if } v = v_1, v_2, \\ \{\pi_v\}, & \text{otherwise}. \end{cases}$$

In this case, Conjecture 5.1 says that

$$\pi_{SK,v_1} \otimes \pi_{SK,v_2} \bigotimes \otimes'_{v \neq v_1,v_2} \pi_v,$$

$$\pi_{h,v_1} \otimes \pi_{h,v_2} \bigotimes \otimes'_{v \neq v_1,v_2} \pi_v$$

appear in r_{disc} if and only if $\epsilon(\frac{1}{2}, \psi_1) = 1$, and

$$\pi_{SK,v_1} \otimes \pi_{h,v_2} \bigotimes \otimes'_{v \neq v_1,v_2} \pi_v,$$

$$\pi_{h,v_1} \otimes \pi_{SK,v_2} \bigotimes \otimes'_{v \neq v_1,v_2} \pi_v$$

appear in r_{disc} if and only if $\epsilon(\frac{1}{2}, \psi_1) = -1$.

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