PERIODS, DUAL REDUCTIVE PAIRS, AND SEESAW IDENTITIES

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Introduction.

In the strict sense of the word a period is the integral of a differential form on some manifold over a cycle on that manifold. In particular, the central topic of this workshop are periods of arithmetic interest that arise by considering arithmetically defined differential forms on abelian varieties or on Shimura varieties. In a more loose sense, the word period (or period integral) is used for the integral of an automorphic form on an adelic algebraic group $H(\mathbb{A})$ over an embedded subgroup $H'(\mathbb{A})$ (or $H'(\mathbb{A})/H'(\mathbb{Q})$); such period integals have often (but not always) a close connection to arithemetic periods in the more narrow sense above; like the latter they also tend to have connections to special values of L-functions.

We survey here a technique for treating such period integrals that has been used by several people and been systematized by Kudla. The central tool here is the oscillator or Weil representation of the metaplectic group, which is used to derive identities between period integrals for seemingly unrelated group-subgroup pairs.

1. Weil representation and dual reductive pairs.

Let F be a number field, W a finite dimensional vector space over F with nondegenerate alternating bilinear form $\langle\ \rangle$, $G = \mathrm{Sp}(W)$ the symplectic group of $(W, \langle\ \rangle)$, $\psi = \prod_v \psi_v$ a fixed additive character of the adele group $\mathbb{A} = F_{\mathbb{A}}$, $\tilde{G}_v = \widetilde{\mathrm{Sp}}(W_v)$ resp. $G_{\mathbb{A}} = \widetilde{\mathrm{Sp}}(W_{\mathbb{A}})$ the local resp. adelic metaplectic extension of $\mathrm{Sp}(W)$ and $\omega = \omega_\psi$ the oscillator representation of $\widetilde{\mathrm{Sp}}(W_{\mathbb{A}})$, realized in the Schrödinger model on the space $S(X_{\mathbb{A}})$ of Schwartz-Bruhat functions on the adelization $X_{\mathbb{A}}$ of a maximal totally isotropic subspace (Lagrangian) of W (for details see [11]).

A dual reductive pair in $G = \mathrm{Sp}(W)$ is a pair H_1, H_2 of reductive subgroups $H_i \subseteq G$ that are mutual centalizers, i.e., such that $C_G(H_1) = H_2$, $C_G(H_2) = H_1$. It is known that then also the preimages $\tilde{H}_1(\mathbb{A})$, $\tilde{H}_2(\mathbb{A})$ in the metaplectic group $\tilde{G}_{\mathbb{A}} = \widetilde{\mathrm{Sp}}(W_{\mathbb{A}})$ commute. Standard examples are orthogonal-symplectic pairs $H_1 = O(V)$, $H_2 = \mathrm{Sp}(W_0)$ with $V \otimes W_0 \cong W$, unitary pairs $H_1 = U(V_1)$, $H_2 = U(V_2)$ with

$V_1 \otimes_{\mathbb{R}} V_2 \cong W$ and pairs $H_1 = \mathrm{GL}(V_1)$, $H_2 = \mathrm{GL}(V_2)$ with

$$W \cong (V_1 \otimes V_2) \oplus (V_1 \otimes V_2)^*.$$

Considering for each Schwartz-Bruhat function $\varphi \in S(X_{\mathbb{A}})$ the theta kernel

$$\Theta_{\varphi,\psi}(g) = \sum_{x \in X_{\mathbb{Q}}} (\omega_\psi(g)\varphi)(x) \quad (g \in \widetilde{\mathrm{Sp}}(W_{\mathbb{A}}))$$

and a dual reductive pair H_1, H_2 as above we have (up to questions of convergence) maps $f_1 \longmapsto \Theta_{\varphi,\psi}(f_1)$ with

$$\Theta_{\varphi,\psi}(f_1)(\tilde{h}_2) = \int\limits_{H_1(\mathbb{Q})\backslash H_1(\mathbb{A})} f_1(\tilde{h}_1)\Theta_{\varphi,\psi}(\tilde{h}_1\tilde{h}_2)d\tilde{h}_1,$$

$f_2 \longmapsto \Theta_{\varphi,\psi}(f_2)$ with

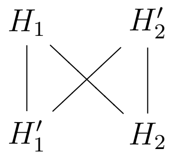
$$\Theta_{\varphi,\psi}(f_2)(\tilde{h}_1) = \int\limits_{\tilde{H}_2(\mathbb{Q})\backslash \tilde{H}_2(\mathbb{A})} f_2(\tilde{h}_2)\Theta_{\varphi,\psi}(\tilde{h}_1\tilde{h}_2)d\tilde{h}_2,$$

mapping an automorphic form f_1 on $\tilde{H}_1(\mathbb{A})$ to an automorphic form $\Theta_{\varphi,\psi}(f_1)$ on $\tilde{H}_2(\mathbb{A})$ and vice versa.

We will omit the (fixed) character ψ in the sequel and just write w, Θ_φ etc.

2. SEESAW DUAL PAIRS AND SEESAW IDENTITIES.

Definition 1. *(Kudla)*
A seesaw dual pair in $\mathrm{Sp}(W)$ *is a pair of dual reductive pairs* (H_1, H_2), $H_1', H_2')$ *in* $\mathrm{Sp}(W)$ *wih* $H_1' \subseteq H_1$, $H_2 \subseteq H_2'$*:*



(where the vertical lines are inclusions and the slanted lines connect members of a dual pair).

Proposition 2. *(Kudla)*
In a seesaw dual pair as above let automorphic forms f_1' *on* $\tilde{H}_1'(\mathbb{A})$, f_2 *on* $\tilde{H}_2(\mathbb{A})$ *and a Schwartz-Bruhat function* $\varphi \in S(X_{\mathbb{A}})$ *be given and assume that the integrals defining functions* $\Theta_\varphi(f_1')$ *on* $\tilde{H}_2'(\mathbb{A})$ *and* $\Theta_\varphi(f_2)$ *on* $\tilde{H}_1(\mathbb{A})$ *converge.*

Then, if the Petersson products $\langle \Theta_\varphi(f_1'), f_2\rangle$ *and* $\langle f_1', \Theta_{\overline{\varphi}}(f_2)\rangle$ *are defined and the integrals defining* θ_φ *resp. the Petersson product can be interchanged, one has*

$$\langle \Theta_\varphi(f_1'), f_2\rangle = \langle f_1', \Theta_{\overline{\varphi}}(f_2)\rangle,$$

and any identity of this type is called a seesaw identity.

Proof. If everything is defined the proof is almost trivial:

$$\begin{aligned}\langle\Theta_\varphi(f_1'),f_2\rangle &= \int\limits_{H_2(\mathbb{Q})\backslash\tilde H_2(\mathbb{A})}\left(\int\limits_{H_1'(\mathbb{Q})\backslash\tilde H_1'(\mathbb{A})} f_1'(\tilde h_1')\Theta_\varphi(\tilde h_1'\tilde h_2)d\tilde h_1'\right)\overline{f_2(\tilde h_2)}d\tilde h_2\\ &= \langle f_1',\Theta_{\overline{\varphi}}(f_2)\rangle\end{aligned}$$

after interchanging the order of integration. □

Remark.

a) The local version of the seesaw principle is representation theoretic: Let the groups in the previous situation be local groups and denote by S the oscillator representation of $\widetilde{\mathrm{Sp}}(W_v)$. Let π_1' be an irreducible admissible representation of $\tilde H_1'$ with

$$\mathrm{Hom}_{\tilde H_1'}(S,\pi_1')\neq\{0\}.$$

Then by Howe duality there is (at least if the residue characteristic is odd) a unique irreducible admissible representation $\pi_2'=\Theta(\pi_1')$ of $\tilde H_2'$ associated to π_1'. In the same way we consider irreducible admissible representations π_2 of $\tilde H_2$, $\Theta(\pi_2)$ of $\tilde H_1$. Then the seesaw identity becomes

$$\mathrm{Hom}_{\tilde H_1'}(\Theta(\pi_2),\pi_1')=\mathrm{Hom}_{\tilde H_2}(\Theta(\pi_1'),\pi_2).$$

b) As mentioned in the introduction, integrals of an automorphic form on some adelic group $H(\mathbb{A})$ over an embedded subgroup $H'(\mathbb{A})$ (also as integrals against another automorphic form on $H'(\mathbb{A})$) are often called period integrals. They do not always have a clear arithmetic meaning or an interpretation in terms of periods in the stricter sense of the word.
Obviously, seesaw identities yield identities between period integrals in the broader sense. Although in the setup given above these identities look rather trivial, they contain often highly nontrivial information.

c) Although the seesaw principle seems to give a very simple machine for the production of identities, there are usually some obstacles in the way.
One reason for this is the following: If (H_1,H_2) is a dual reductive pair and one wants to get a good interpretation of the lifting $\Theta_\varphi(f_1)$ of an automorphic form f_1 on $\tilde H_1(\mathbb{A})$ one usually has to choose a splitting $W=X\oplus X'$ of the underlying symplectic space into Lagrangians X,X' that is adapted to the situation. For example, in a symplectic-orthogonal pair $(\mathrm{Sp}(W_0),O(V))$ one looks at

$$X=X_0\otimes W,\ \text{where } X_0 \text{ is a Lagrangian in } W_0.$$

For the seesaw identity, however, one has to work with one Lagrangian of W for both pairs (H_1', H_2'), (H_1, H_2), which will usually not be adapted to both pairs in question. One method to deal with this problem is to work with the standard splitting for the big symplectic group into which all participating groups are embedded and trace computations in that model explicitly back through the embeddings of the members of the seesaw pair into that big symplectic group. Alternatively, one has to change models in the middle of the comparison and recompute everything with respect to the new splitting.

In addition, it is often necessary to choose a specific test function $\varphi \in S(X_{\mathbb{A}})$ in order to give the desired interpretation to the lifted form $\Theta_\varphi(f_1)$. Again it is not likely that the same φ will do the job for both pairs considered.

As a consequence of these difficulties, the seesaw principle is often mainly used as a motivating explanation whereas the actual computations don't seem to use it explicitly.

3. Hecke's example with Kudla's generalization.

We consider the following classical example of Hecke [9], which is of course formulated without the terminology introduced above.

Let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field of discriminant $|D|$ with ring of integers $\mathfrak{o}_k$, $\mathfrak{a} \subseteq \mathfrak{o}_k$ an integral ideal of norm A, $\rho \in \mathfrak{a}$ fixed, $Q \in \mathbb{N}$ fixed and put (for $\tau \in \mathbb{H}$)

$$\vartheta(\tau, \rho, \mathfrak{a}, Q\sqrt{D}) := \sum_{\substack{\mu \equiv \rho \bmod \mathfrak{a}Q\sqrt{D} \\ \mu \in \mathfrak{o}_K}} \mu e^{2\pi i \tau \frac{\mu\mu'}{AQ|D|}},$$

where μ' is the Galois conjugate of μ; ϑ_2 is a theta series of weight 2 with Größencharakter (which in this case is a harmonic polynomial of degree 1).

Put further (for $\rho \neq 0$)

$$\begin{aligned} j(\tau; \rho, \mathfrak{a}, Q\sqrt{D}) &:= \frac{2\pi i}{AQ|D|} \int_\infty^\tau \vartheta_2(\tau; \rho, \mathfrak{a}, Q\sqrt{D}) d\tau \\ &= \sum_{\mu \equiv \rho \bmod \mathfrak{a}Q\sqrt{D}} \frac{1}{\mu'} e^{2\pi i \tau \frac{\mu\mu'}{AQ|D|}} \end{aligned}$$

The (nowadays) usual transformation formulas for theta series imply:

$$j(-\frac{1}{\tau}; \rho, \mathfrak{a}, Q\sqrt{D}) = \frac{1}{Q\sqrt{D}} \sum_{\substack{\alpha \bmod \mathfrak{a}Q\sqrt{D} \\ \alpha \equiv 0 \bmod \mathfrak{a}}} \epsilon^{-\mathrm{tr}(\frac{\alpha'\rho}{A})} \cdot j(\tau; \alpha, \mathfrak{a}, Q\sqrt{D}) + P(\rho),$$

where the period $P(\rho)$ is independent of τ and where we put $\epsilon = \exp(\frac{2\pi i}{Q|D|})$.

Letting τ tend to $i\infty$ we obtain

$$\begin{aligned} P(\rho) &= j(0;\rho,\mathfrak{a},Q\sqrt{D}) \\ &= \lim_{x\to 1} \sum_{\mu\equiv\rho \bmod \mathfrak{a}Q\sqrt{D}} \frac{x^{\mu\mu'}}{\mu'} \\ &= \sum_{\mu\equiv\rho \bmod \mathfrak{a}Q\sqrt{D}} \frac{1}{\mu'|\mu|^s}\Bigg|_{s=0}. \end{aligned}$$

Here the last series (being the value of the zeta function with Größencharakter $\frac{\mu}{|\mu|}$ for $\mathbb{Q}(\sqrt{D})$ at $\frac{s+1}{2}$) is convergent for $\mathrm{Re}(s) > 1$ and the value at $s = 0$ is defined by analytic continuation.

We rewrite the last series (for s in the region of convergence) as

$$\sum_{\substack{m_2\equiv r_2 \bmod QD \\ m_1\equiv r_1 \bmod QD}} \frac{-\sqrt{D}|D|^{s/2}}{(m_1\omega_1' + m_2\omega_2')|m_1\omega_1' + m_2\omega_2'|^s}$$

where $\{\omega_1, \omega_2\}$ is a $\mathbb{Z}$-basis of $\mathfrak{a}$ and $\rho\sqrt{D} = r_1\omega_1 + r_2\omega_2$.

Writing

$$G_1(\tau, 1; r_1, r_2, N) = \sum_{m_i\equiv r_i \bmod N} \frac{1}{(m_1\tau + m_2)|m_1\tau + m_2|^s}\Bigg|_{s=0}$$

for the value at $s = 0$ of the Eisenstein series occurring here and using the fact (proven by Hecke) that $G_1(\tau, 1; r_1, r_2, N)$ is a holomorphic modular form of weight 1 and level N, we have obtained the identity

$$P(\rho) = -\sqrt{D}G_1(\omega_1', \omega_2', r_1, r_2, Q|D|)$$

of which Hecke [9] says:

> So erscheinen die Perioden $P(\rho)$ als Werte von elliptischen Modulformen für Argumente, welche dem imaginär-quadratischen Zahlkörper $K(\sqrt{D})$ angehören, und auf diese Weise ist der Zusammenhang mit der komplexen Multiplikation gegeben.[1]

We have now to see why this example of Hecke is in fact a seesaw identity, following Kudla [10].

Consider the following more general situation:

With D as above let $D = \sqrt{D}$, let a positive definite matrix $Q_0 \in$

[1]In this way the periods $P(\rho)$ appear as values of elliptic modular forms at arguments which belong to the imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$, and in this way the connection with complex multiplication is given.

$M_n^{\mathrm{sym}}(\mathbb{Q})$ and a matrix $Q_1 \in M_{n+1}^{\mathrm{sym}}(\mathbb{Q})$ of signature $(n,1)$ be given (e.g. $Q_0 = 1_n, \quad Q_1 = \begin{pmatrix} 0 & & 1/2 \\ & 1_{n-1} & \\ 1/2 & & 0 \end{pmatrix}$), denote by $U(n)$ the unitary group of the hermitian form with matrix Q_0 over $\mathbb{Q}(\sqrt{D})$, by $U(n,1)$ the unitary group of the hermitian form with matrix Q_1, by $O(n,1) \subseteq U(n,1)$ the orthogonal group of the (rational) symmetric bilinear form with matrix Q_1.

$U(n)(\mathbb{R})$ is embedded into $\mathrm{Sp}_n(\mathbb{R})$ as the stabilizer of the point $\tau_0 = -\delta^{-1}Q_0^{-1}$.

We have then the dual reductive pairs

$$U(n) \times U(n,1) \text{ and } O(n,1) \times \mathrm{Sp}_n \text{ in } \mathrm{Sp}_{n(n+1)}$$

which form the seesaw pair

$$\begin{array}{ccc} U(n,1) & & \mathrm{Sp}_n \\ | & \times & | \\ O(n,1) & & U(n) \end{array}$$

Roughly, the identity to study will arise as follows: On $U(n)$ we take the trivial representation which (upon lifting to $U(n,1)$) gives a theta series for $U(n,1)$, on $O(n,1)$ we take the trivial representation, which lifts to (a special value in the variable s of) an Eisenstein series on Sp_n by the Siegel-Weil theorem. Integration of the theta series on $U(n,1)$ over the embedded $O(n,1)$ will then give a period that relates to evaluation of the Eisenstein series on Sp_n at a special point stabilized by $U(n)$.

Before we can make this more precise we look at the relevant symmetric spaces and their embeddings:

We fix matrices $T_0 \in \mathrm{GL}_n^+(\mathbb{R})$, $T_1 \in \mathrm{GL}_{n+1}^+(\mathbb{R})$ with

$$^tT_0Q_0T_0 = 1_n, \quad {}^tT_1Q_1T_1 = \begin{pmatrix} & & 1/2 \\ & 1_n & \\ 1/2 & & \end{pmatrix}.$$

Let

$$\begin{aligned} \mathbb{D} &= \{\mathbf{z} = (z_0, \mathbf{z}_1) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid (2\delta)^{-1}(z_0 - \overline{z}_0) - {}^t\overline{\mathbf{z}}_1\mathbf{z}_1 > 0\}, \\ \mathbb{B} &= \{\mathbf{x} = (x_0, \mathbf{x}_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_0 - {}^t\mathbf{x}_1\mathbf{x}_1 > 0\}, \end{aligned}$$

so that $\mathbb{D}$ resp. $\mathbb{B}$ is the symmetric space associated to $U(n,1)(\mathbb{R})$ resp. $O(n,1)(\mathbb{R})$ if Q_1 is the standard matrix $\begin{pmatrix} 0 & & 1/2 \\ & 1_{n-1} & \\ 1/2 & & 0 \end{pmatrix}$, whereas for general Q_1 as above with associated matrix T_1 the vector

$$T_1 \begin{pmatrix} z_0 \\ \mathbf{z}_1 \\ \delta^{-1} \end{pmatrix} \in \mathbb{C}^{n+1}$$

spans for $(z_0, \mathbf{z}_1) \in \mathbb{D}$ a negative line with respect to the hermitian form with matrix Q_1.

We denote by D_1 the set of all lines in $\mathbb{C}^{n+1}$ obtained in this way and by B_1 the subset coming in the same way from point of $\mathbb{B}$; D_1 resp. B_1 is then (considered as subset of $\mathbb{P}^n(\mathbb{C})$ resp. $\mathbb{P}^n(\mathbb{R})$) the symmetric space of $U(n,1)(\mathbb{R})$ resp. $O(n,1)(\mathbb{R})$; on these spaces we have natural actions of the respective groups.
We have then embeddings

$$\kappa : \mathbb{B} \hookrightarrow \mathbb{D} \quad (x_0, \mathbf{x}_1) \longmapsto (-\delta^{-1}x_0, \delta^{-1}\mathbf{x}_1)$$

(for $n = 1$, $\mathbb{B} = \mathbb{R}_{>0}$ embeds as the imaginary axis into $\mathbb{D} = \mathbb{H}$),

$$\kappa' : \mathbb{B} \longrightarrow \mathbb{H} \times \mathbb{B} \quad (x_0, \mathbf{x}_1) \longmapsto (-\delta^{-1}Q_0^{-1}, x_0, \mathbf{x}_1)$$

and $\epsilon : \mathbb{D} \longrightarrow \mathbb{H}_{n(n+1)}$ given for $\mathbf{z} = (z_0, \mathbf{z}_1)$ as follows: Put

$$\omega_2(\mathbf{z}) = T_0 \otimes T_1 \begin{pmatrix} z_0 & -2{}^t\mathbf{z}_1 & -z_0 \\ \mathbf{z}_1 & \delta^{-1}1_{n-1} & 0 \\ \delta^{-1} & 0 & \delta^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ & -1_n \end{pmatrix} \in M_{n(n+1)}(\mathbb{C})$$

$$\omega_1(\mathbf{z}) = Q_0 T_0 \otimes \delta Q_n T_1 \begin{pmatrix} z_0 & -2{}^t\mathbf{z}_1 & -z_0 \\ \mathbf{z}_1 & \delta^{-1}1_{n-1} & 0 \\ \delta^{-1} & 0 & \delta^{-1} \end{pmatrix} \in M_{n(n+1)}(\mathbb{C}),$$

where $\otimes$ denotes the Kronecker product of matrices. Then

$$\epsilon(\mathbf{z}) = \omega_2(z)\omega_1(z)^{-1} \in \mathbb{H}_{n(n+1)}$$

is an embedding $\mathbb{D} \longrightarrow \mathbb{H}_{n(n+1)}$.
For $\mathbf{x} = (x_0, \mathbf{x}_1)$ and $\tau \in \mathbb{H}_n$ we define in a similar way

$$\omega_1'(\mathbf{x}) = 1_n \otimes Q_1 T_1 \begin{pmatrix} -x_0 & -2{}^t\mathbf{x}_1 & x_0 \\ \mathbf{x}_1 & 1_{n-1} & 0 \\ 1 & 0 & 1 \end{pmatrix} \in M_{n(n+1)}(\mathbb{C}),$$

$$\begin{aligned} \omega_2'(\mathbf{x}, \tau) = {} & \tfrac{1}{2}(\tau + \overline{\tau}) \otimes T_1 \begin{pmatrix} -x_0 & -2{}^t\mathbf{x}_1 & x_0 \\ \mathbf{x}_1 & 1_{n-1} & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ & + \tfrac{1}{2}(\tau - \overline{\tau}) \otimes T_1 \begin{pmatrix} -x_0 & -2{}^t\mathbf{x}_1 & x_0 \\ \mathbf{x}_1 & 1_{n-1} & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & \\ & 1_n \end{pmatrix} \end{aligned}$$

and

$$\epsilon' : \mathbb{B} \times \mathbb{H} \longrightarrow \mathbb{H} : (\mathbf{x}, \tau) \longmapsto \omega_2'(\mathbf{x}, \tau)\omega_1'(\mathbf{x})^{-1}.$$

Notice here that ϵ', even when restricted to $\{\mathbf{x}\} \times \mathbb{H}_n$ for some fixed $\mathbf{x} \in \mathbb{B}$, is not holomorphic.

We have then a commutative diagram

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\kappa} & \mathbb{D} \\ \downarrow \kappa' & & \downarrow \epsilon \\ \mathbb{H}_n \times \mathbb{B} & \xrightarrow{\epsilon'} & \mathbb{H}_{n(n+1)} \end{array}$$

which is checked to be equivariant with respect to the actions of $O(n,1)$ on $\mathbb{B}$, of $U(n,1)$ on $\mathbb{D}$, of $\mathrm{Sp}_n(\mathbb{R})$ on $\mathbb{H}_n$ and of $\mathrm{Sp}_{n(n+1)}(\mathbb{R})$ on $\mathbb{H}_{n(n+1)}$ (where the actions of $O(n,1)$ and $U(n,1)$ are transported from B_1, D_1 to $\mathbb{B}, \mathbb{D}$ by the isomorphisms given above) and with the embeddings of the groups considered above.

We consider now a Siegel modular form F on $\mathbb{H}_{n(n+1)}$ which arises as the function on $\mathbb{H}_{n(n+1)}$ corresponding to the theta kernel θ_φ on the metaplectic cover of $\mathrm{Sp}_{n(n+1)}(\mathbb{A})$ for a suitable test function $\varphi \in S(X_\mathbb{A})$, where X is a suitable Lagrangian of the standard alternating space W of dimension $2n(n+1)$.

Let $\tilde{\Gamma} \subseteq \mathrm{Sp}_{n(n+1)}(\mathbb{Z})$ be the congruence subgroup with respect to which F is a modular form, $\Gamma_U \subseteq U(n,1)$ its inverse image under the embedding $U(n,1)(\mathbb{R}) \hookrightarrow \mathrm{Sp}_{n(n+1)}(\mathbb{R})$, $\Gamma_O = \Gamma_U \cap O(n,1)$.

Assuming that all integrals occurring below are defined, the seesaw identity becomes

$$\begin{aligned} \int\limits_{\Gamma_O \backslash \mathbb{B}} F(\epsilon \circ \kappa(\mathbf{x})) d\mathbf{x} &= \int\limits_{\Gamma_O \backslash \mathbb{B}} F(\epsilon' \circ \kappa'(\mathbf{x})) d\mathbf{x} \\ &= \int\limits_{\Gamma_O \backslash \mathbb{B}} F(\epsilon'(\underbrace{\delta^{-1} E_n}_{=\tau_0}, \mathbf{x}) d\mathbf{x} \\ &= G(\tau_0) \end{aligned}$$

with

$$G(\tau) := \int\limits_{\Gamma_O \backslash \mathbb{B}} F(\epsilon'(\tau, \mathbf{x})) d\mathbf{x} \ .$$

Here the first integral

$$\int\limits_{\Gamma_O \backslash \mathbb{B}} F(\epsilon \circ \kappa(\mathbf{x})) d\mathbf{x}$$

corresponds to the Petersson product

$$\langle \theta_\varphi(\mathbf{1}_{U_n}), \mathbf{1} \rangle$$

on $O(n,1)(\mathbb{Q})\setminus O(n,1)(\mathbb{A})$ of the restriction to $O(n,1)(\mathbb{A})$ of the theta lift to $U(n,1)(\mathbb{A})$ of the identity $\mathbf{1}_{U_n}$ on $U_n(\mathbb{A})$,

$$G(\tau) = \int\limits_{\Gamma_O\backslash\mathbb{B}} F(\epsilon'(\tau,\mathbf{x}))d\mathbf{x}$$

corresponds to the theta lift $\theta_\varphi(\mathbf{1}_{O(n,1)})$ to $\mathrm{Sp}_n(\mathbb{A})$ of the identity $\mathbf{1}_{O(n,1)}$ on $O(n,1)(\mathbb{A})$ and $G(\tau_0)$ to the Petersson product

$$\langle\theta_\varphi(\mathbf{1}_{O(n,1)}),\mathbf{1}_{U_n}\rangle$$

of the restriction to $U(n)(\mathbb{A})\hookrightarrow \mathrm{Sp}_n(\mathbb{A})$ of $\theta_\varphi(\mathbf{1}_{O(n,1)})$ with $\mathbf{1}_{U_n}$ as functions on the (compact) quotient $U(n)(\mathbb{Q})\backslash U(n,\mathbb{A})$; notice that this last integral appears just as an evaluation in the classical computation since $U(n)(\mathbb{R})$ is the stabilizer of the point $\tau_0\in\mathbb{H}_n$.

We see that the identity above is indeed nothing but the seesaw identity

$$\langle\theta_\varphi(\mathbf{1}_{U_n}),\mathbf{1}_{O(n,1)}\rangle = \langle\theta_\varphi(\mathbf{1}_{O(n,1)}),\mathbf{1}_{U_n}\rangle.$$

We see also that in this classical context the identity looks even more trivial than in the adelic context; as there the interesting feature is that for appropriate F one has to interpret both sides of the identity in a meaningful way.

We specialize now th the case of Hecke's example, i.e., to $n=1$ with $Q_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $Q_0=1$. The group $U(1,1)$ is then isomorphic to SL_2 and $\mathbb{D}$ is just the upper half plane $\mathbb{H}$.

The embedding $\kappa:\ \mathbb{B}\hookrightarrow\mathbb{D}$ becomes

$$\kappa(x)\longmapsto -\delta^{-1}x\in\mathbb{H}\quad (x\in\mathbb{R}_{>0}),$$

so the integral over $\mathbb{B}$ becomes the Mellin transform (notice that Γ_O is trivial in this case).

The other embeddings beome now

$$\begin{aligned}\epsilon(z_0) &= \begin{pmatrix} z_0 & 0\\ 0 & -\delta^{-1}z_0^{-1}\end{pmatrix} && \in\mathbb{H}_2\quad (z_0\in\mathbb{D})\\ \epsilon'(\tau,x_0) &= \begin{pmatrix} (\tau-\overline{\tau}) & \tau+\overline{\tau}\\ \tau+\overline{\tau} & (\tau-\overline{\tau})/x_0\end{pmatrix} && \in\mathbb{H}_2\quad ((\tau,x_0)\in\mathbb{H}\times\mathbb{B}).\end{aligned}$$

There remains the task of finding the right test function φ, so that the modular form F on $\mathbb{H}_2$ associated to $\theta_\varphi(g)$ gives Hecke's identity.

The basic idea is to choose the Lagrangian in the alternating space W of dimension 4 according to the standard basis used in the standard matrix representation of $\mathrm{Sp}(W)$ as Sp_2, choose the finite components φ_p of the test function as characteristic function of a coset $\mathbf{r}+L_p$ of a suitable lattice L on X and let the embeddings described above take care of the rest; in particular, one can choose $\mathbf{r}$ and L such that upon pulling back θ_φ to $\mathrm{Sp}_1=\mathrm{SL}_2$ one obtains the Eisenstein series G_1 (at $s=0$) and upon pulling back to $U(1,1)\cong\mathrm{SL}_2$ one obtains a version of the theta series ϑ_2. The main problem arises at the infinite place, where taking φ_∞ as the standard Gaussian doesn't work; one of the reasons for this is the fact that this natural choice leads to the function

$$F_0(\tau) = \sum_{\mathbf{y}\in\mathbf{r}+L} \exp(\pi i {}^t\mathbf{y}\tau\mathbf{y})$$

whose pullback under the nonholomorphic embedding ϵ' gives something on $\mathbb{H}\times\mathbb{B}$, which even as function of the $\mathbb{H}$-component is not holomorphic. In fact, at this point it turns out to be more convenient to modify the approach a little: Instead of pulling back a fixed F on $\mathbb{H}_{n(n+1)}$ one creates functions F_1 on $\mathbb{H}_n \times \mathbb{B}$ and F_2 on $\mathbb{D}$ such that one has

$$F_1 \circ \kappa' = F_2 \circ \kappa \text{ on } \mathbb{B},$$

replacing in our seesaw machine described above

$$\begin{aligned} F\circ\epsilon' &\quad \text{by } F_1\\ F\circ\epsilon &\quad \text{by } F_2 \end{aligned}$$

one can still derive the identity as intended. To construct F_1, F_2 one puts

$$\tilde{F}_0(\mathbf{v},\tau) = \exp(\pi i {}^t\mathbf{v}(\tau-\overline{\tau})^{-1}\mathbf{v}) \sum_{\mathbf{y}\in\mathbf{r}+L} \exp(\pi i({}^t\mathbf{y}\tau\mathbf{y} + 2{}^t\mathbf{y}\mathbf{v}) \text{ for } \mathbf{v}\in\mathbb{C}^2$$

and

$$\begin{aligned} F_2(z) &= (\det\ \omega_1(z))^{1/2}\tfrac{\partial}{\partial v_2}(\tilde{F}_0(\check{\omega}_1(z)\mathbf{v},\epsilon(z))\Big|_{\mathbf{v}=0}\\ F_1(\tau_1,x) &= \det(\mathrm{Im}_\delta\tau_1)^{1/2}\det\omega_1'(x)^{-1/2}\tfrac{\partial}{\partial v_2}F_0(\check{\omega}_1'(x)\mathbf{v},\epsilon'(\tau_1,x))\Big|_{\mathbf{v}=0}, \end{aligned}$$

Here $\check{\omega}_1 = {}^t\omega_1{}^{-1}$ and $\mathrm{Im}_\delta(\tau_1) = \frac{\tau_1-\overline{\tau}_1}{2\delta}$ as usual.

It is then checked that this choice does indeed give $F_1\circ\kappa' = F_2\circ\kappa$ and that evaluation of the terms leads to Hecke's identity. Interesting points of this (admittedly rather complicated looking) setup are that upon replacing $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ by an anisotropic Q_1 the Eisenstein series is replaced by Hecke's theta series attached to real quadratic fields from [8] and the Mellin transform is replaced by integrals over geodesics connecting two real points in the hyperbolic plane $\mathbb{H}$. Another interesting feature is that one gets similar results for any n.

4. Triple product L-function and Gross-Prasad conjecture

We want to sketch briefly how seesaw identities can be applied to the problems mentioned in the title of this section.

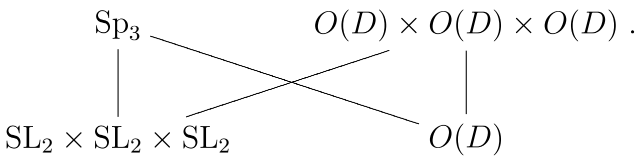
Let f_1, f_2, f_3 be three (holomorphic) cusp forms for the group $\mathrm{SL}_2(\mathbb{Z})$ of weights k_1, k_2, k_3 and put $w = \frac{k_1+k_2+k_3}{2}$. It has been proved by Garrett

[3] that the completed triple product L-function $\Lambda(s, f_1 \otimes f_2 \otimes f_3)$ has an integral representation which can be written as a Petersson product

$$\Lambda(s+\frac{1}{2}(\omega+1), f_1\otimes f_2\otimes f_3) = \langle f_1(z_1)f_2(z_2)f_3(z_3), E(\begin{pmatrix} z_1 & & \\ & z_2 & \\ & & z_3 \end{pmatrix}, s)\rangle_{z_1,z_2,z_3}$$

of $f_1(z_1)f_2(z_2)f_3(z_3)$ with the pullback to the diagonal of a suitable Eisenstein series; this has been generalized by Rallis and Piatetskii–Shapiro [12] to tensor products of three arbitrary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$. This integral representation has been used by Harris and Kudla [7] to relate the central critical value at $s = \frac{w+1}{2}$ of the triple product L-function with the square of a value of a trilinear form on $\pi_1'\otimes\pi_2'\otimes\pi_3'$ for the representations π_i' of the adelic multiplicative group of a suitable quaternion algebra which are associated to the π_i by the Jacquet Langlands correspondence. More explicit versions of this relation have been proved by Gross and Kudla [4] and by Böcherer and the author [1]. The proof of this relation by use of a seesaw identity proceeds as follows:

We consider the following seesaw dual pair



where D is a suitable quaternion algebra and $O(D)$ the orthogonal group of the norm form on D.

The Eisenstein series E in the integral representation is shown to contain a summand E' that is the Eisenstein series whose value at $s = 0$ is obtained as theta lift of the identity on $O(D)$ by the Siegel-Weil theorem. It is then shown that the Petersson product of the restriction to the diagonal of E with the product $f_1(z_1)f_2(z_2)f_3(z_3)$ is in fact, after selection of the right quaternion algebra, equal to the Petersson product of the restriction of E' with $f_1(z_1)f_2(z_2)f_3(z_3)$. Which quaternion algebra is the right one depends on the ϵ-factors attached to the π_i (or in classical language on the Atkin-Lehner eigenvalues of the modular forms f_i considered). One obtains then

$$L(\frac{w+1}{2}, f_1 \otimes f_2 \otimes f_3) = C\langle f_1 \otimes f_2 \otimes f_3, \theta_\varphi(\mathbf{1}_{O(D)})\rangle$$

for a suitable test function φ and some constant C.

Writing

$$\theta_\varphi(f_1 \otimes f_2 \otimes f_3) = \Psi_1 \otimes \Psi_2 \otimes \Psi_3$$

with automorphic forms Ψ_i on $O(D)(\mathbb{A})$, analysis of the theta lifting from SL_2 to $O(D)$ shows that one can write $\Psi_i = \psi_i \otimes \psi_i$, where ψ_i arises from f_i by the correspondence of Jacquet–Langlands (or of

Eichler in the classical context); one uses here that for the similitude group $GO(D)$ one has

$$GO(D) \cong (D^\times \times D^\times)/Z(D^\times).$$

The seesaw identity gives then

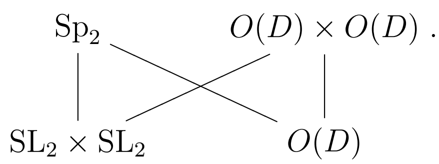
$$L(\frac{w+1}{2}, f_1 \otimes f_2 \otimes f_3) = C\langle(\psi_1 \otimes \psi_2 \otimes \psi_3) \otimes (\psi_1 \otimes \psi_2 \otimes \psi_3).1 \otimes 1\rangle,$$

This last Petersson product is on $D^\times(\mathbb{A}) \times D^\times(\mathbb{A})$, where we view $\psi_1 \otimes \psi_2 \otimes \psi_3$ as function on $D^\times(\mathbb{A})$ by inserting the same variable into all three functions. It is evaluated as

$$T(\psi_1 \otimes \psi_2 \otimes \psi_3)^2$$

for a certain trilinear form.

We put the trilinear form $T(\psi_1 \otimes \psi_2 \otimes \psi_3)$ into another seesaw diagram:



Here we put $\Psi_1 = \psi_1 \otimes \psi_1'$ on the $O(D)$ in the lower right and cusp forms f_2, f_3 corresponding to $\Psi_2 = \psi_2 \otimes \psi_2$, $\Psi_3 = \psi_3 \otimes \psi_3$ on $O(D)$ as above on $\mathrm{SL}_2 \times \mathrm{SL}_2$

When D is definite, the theta lift of Ψ_1 is the Yoshida lifting $Y^{(2)}(\psi_1, \psi_1')$, a holomorphic Siegel modular form of degree 2 obtained as a linear combination of theta series of degree 2 of the ideals associated to orders of a specified type (e.g. maximal orders) in D.

The seesaw identity gives in this case

$$\langle Y^{(2)}(\psi_1, \psi_1'), f_2 \otimes f_3\rangle = C'T(\psi_1, \psi_2, \psi_3)T(\psi_1', \psi_2, \psi_3) \ .$$

Squaring both sides one obtains [2] that the square of the integral on the left hand side is proportional to the product of the central critical values

$$L(\frac{w+1}{2}, f_1 \otimes f_2 \otimes f_3)L(\frac{w+1}{2}, f_1' \otimes f_2 \otimes f_3).$$

In view of the facts that $\mathrm{PGSp}_2 \cong SO(5)$, $GO(4) \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/Z(\mathrm{GL}_2)$, this identity can be interpreted as a quantitative version of the Gross-Prasad conjecture [5, 6].

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