Petersson Products for L^2-Siegel modular Forms

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Abstract

Using differential operators we extend the Rankin-Selberg-method to non-cuspidal holomorphic Siegel modular forms. For square-integrable Siegel modular forms we get - in some cases - a formula for the Petersson product in terms of a residue of the Rankin-convolution Dirichlet series (in the case of cusp forms, this is a classical result)

Petersson products are well understood for cusp forms; it is desirable to extend them to noncuspidal holomorphic Siegel modular forms. Here we describe a method to extend some properties of Petersson products (well-known for cusp forms) to noncuspidal square-integrable holomorphic Siegel modular forms; a satisfying result is obtained only for the "almost singular" weight $k = \frac{n}{2}$.

To keep our notations simple, most of the time we restrict ourselves to modular forms of integral weight for the full modular group $\Gamma = Sp(n, \mathbb{Z})$; everything can be extended to the case of arbitrary congruence subgroups (also vector-valued modular forms and half-integral weights can be included; details will appear elsewhere [2]).

We use standard notations without further explanation. For a (complex) matrix S of size n we put $\mathbf{e}_n(S) := \exp(trace(S))$.

The main property, which we have in mind, is the relation of Petersson products with Dirichlet series of convolution type ("Rankin convolution"). For two Siegel modular forms

$$\begin{aligned} f(Z) &= \sum_T a(T)\mathbf{e}_n(2\pi i T Z) \\ g(Z) &= \sum_T b(T)\mathbf{e}_n(2\pi i T Z) \end{aligned}$$

we define this Rankin convolution by

$$\mathcal{D}(f, g, s) := \sum_{\{T\}} \frac{a(T)\overline{b(T)}}{\epsilon(T)det(T)^s}.$$

We assume that both f and g are from the same space M_k^n of Siegel modular forms of degree n, weight k. The summation in this Dirichlet series goes over representatives of the $GL(n,\mathbb{Z})$-equivalence classes of positive definite half-integral matrices and $\epsilon(T)$ denotes the number of units of T in $GL(n,\mathbb{Z})$. We remark here that this Dirichlet series always converges in some right half plane and it is identically zero, if the weight k is singular (i.e. $k < \frac{n}{2}$).
If we assume that f and g are cuspidal then $\mathcal{D}(f,g,s)$ has a meromorphic continuation to $\mathbb{C}$ and

$$Res_{s=k}\mathcal{D}(f,g,s) \sim < f,g >, \tag{1}$$

where $<,>$ denotes the Petersson product.
We call this the "Petersson identity"; these properties are immediate consequences of standard properties of (Siegel type) Eisenstein series and the Rankin-Selberg identity, exhibited by

$$\int_{\Gamma\backslash\mathbb{H}_n} f(Z)\overline{g}(Z)E(Z,s)d_kZ = \Gamma_n(s+k-\frac{n+1}{2})\cdot\mathcal{D}(f,g,s+k-\frac{n+1}{2}). \tag{2}$$

The Eisenstein series $E(Z,s)$ is defined by

$$E(Z,s) = \sum_{M\in\Gamma_\infty\backslash\Gamma} det(\Im(M < Z >))^s.$$

To extend (1) beyond cusp forms, one has first to show that the Rankin convolution $\mathcal{D}(f,g,s)$ has a meromorphic continuation also for non cuspidal modular forms; taking this for granted (for the moment), then we can hope that (1) holds for all **square-integrable nonsingular** f,g**.**
We recall that f is nonsingular, if it has a non-zero Fourier coefficient $a(T)$ with T of maximal rank ; this is equivalent to the condition $k \geq \frac{n}{2}$.
We define the space of square-integrable Siegel modular forms by

$$L^2M_k^n := \{f\in M_k^n \mid \int_{\Gamma\backslash\mathbb{H}_n} \mid f\mid^2 d_kZ < \infty\}.$$

There is a characterization of such L^2 modular forms by Satake [11] (and more generally by Weissauer [13]): For a modular form $f(Z) = \sum a(T)e(tr(TZ)$ we define[1]

$$t_f := Min_T\{rank(T) \mid a(T)\neq 0\}.$$

[1] in the case of a congruence subgroup we have to take the Minimum over the Fourier expansions in all cusps

In particular,

$$t_f = \begin{cases} n & \text{if} \quad f \text{ is cuspidal} \\ 0 & \text{if} \quad f \text{ is an Eisenstein series} \end{cases}$$

Then we have the following

Theorem

$$f \in L^2 \iff f \text{ is cuspidal or } \; 2k \leq n + t_f$$

Remarks:

- For small weights ($k \leq \frac{n}{2}$) all modular forms are square-integrable, in particular, all singular modular forms are square-integrable.
- We call the weights k with

  $$\frac{n}{2} + \frac{1}{2} \leq k \leq n - \frac{1}{2}$$

  the **delicate weights** (some modular forms may be square-integrable, some may not).
- In the statement above, we can include half-integral weights and also vector-valued modular forms.

Remark on the singular case: In the singular case the Petersson identity (1) does not make sense at all, because the Dirichlet series $\mathcal{D}(f, g, s)$ is identically zero. All singular modular forms are generated by theta series attached to positive definite quadratic forms. In [1] we gave a formula for the Petersson product of two such singular theta series (at least if the two quadratic forms involved are rationally equivalent), based on some results of J.-S. Li [7]; this formula also involves the values of a (suitably defined) Rankin convolution.

To extend the Rankin-Selberg identity (2) to the case of arbitrary (i.e.not necessarily cuspidal) modular forms, one can use differential operators to eliminate certain singular terms. The use of differential operators for such a purpose is well known for the Mellin transform (see Maaß [8]), but perhaps less familar for Rankin-Selberg convolutions.
It may be helpful for the reader to explain the method of differential operators for degree n=1 first (here also the method of Zagier [14] is available and

it is perhaps interesting to compare the methods; we should also point out that in the work of Mizuno [10] polynomials in the Laplace operator are used to give a variant of Zagier‘s method). We allow any congruence subgroup Γ of $SL(2,\mathbb{Z})$ for the moment and we apply the Laplace operator

$$\Omega := y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

to the Γ-invariant function

$$F := f\cdot \overline{g}\cdot y^k \qquad (f,g\in M_k^1(\Gamma)).$$

Then we have

- $$\Omega(F)-k(k-1)\cdot F$$
  is of rapid decay
- We can define a hermitian form (perhaps indefinite or degenerate) on $M_k^1(\Gamma)$ by
  $$<< f,g >>:= \int (\Omega - k(k-1))F\frac{dxdy}{y^2}$$
- For all L^2-forms (i.e. cusp forms or forms of weight $k<1$)
  $$<< f,g >> = -k(k-1) < f,g >$$
- For arbitrary $f,g\in M_k^1(\Gamma)$ the Rankin-Selberg identity (2) holds for $(\Omega-k(k-1))\,(F)$, in particular
  $$< f,g > \sim Res_{s=k}D(f,g,s)$$
  holds for square-integrable forms.
- Like in [14] one can show that $<<,>>$ defines a non degenerate Hermitian form on M_k^1; it is (negative) definite only for k kongruent 2 modulo 4 (at least for $\Gamma = SL(2,\mathbb{Z})$).

To generalize this approach to higher degree by direct computation seems to be quite complicated; it was done by Maaß [9] for degree 2, using two generalized Laplace operators (without mentioning anything about Petersson products; the consequences of his calculations for Petersson inner products are given in [3] for degree ≤ 2).

To handle the calculus of differential operators for arbitrary degree n we follow a more abstract strategy developed by Deitmar and Krieg in [4] for a somewhat different purpose; it is based on the interplay between the algebra $\mathbb{D}(\mathbb{H}_n)$ of $Sp(n)$- invariant differential operators on $\mathbb{H}_n$ and the algebra $\mathbb{D}(\mathcal{P}_n)$ of $GL(n)$-invariant differential operators on the space $\mathcal{P}_n$ of positive definite symmetric matrices of size n:
We start from the Γ -invariant function

$$F := f(Z)\overline{g(Z)}det(Y)^k \qquad (f, g \in M_k^n(\Gamma))$$

Theorem: *There is an invariant differential operator $\mathcal{D}$ on $\mathbb{H}_n$ and a differential operator D_0 on $\mathcal{P}_n$ such that*

$$\begin{aligned} \mathcal{D}(F)(Z) &= \sum_{S+T>0} a(S)\overline{b(T)} \\ &\quad \mathcal{D}(\mathbf{e}_n(2\pi i(S-T)X)\mathbf{e}_n(-2\pi tr(S+T)Y)det(Y)^k) \end{aligned} \tag{3}$$

$$\int_X D(F)dX = \sum_{T>0} a(T)\overline{b(T)}D_0(\mathbf{e}_n(-4TY)det(Y)^k) \tag{4}$$

and

$$\int_{\Gamma\backslash\mathbb{H}_n} F(Z)E(Z,s)d^*Z = \gamma_n(s)\cdot\sum_T a(T)\overline{b(T)}det(T)^{-s-k+\frac{n+1}{2}}.$$

Here D_0 is given explicitly as

$$D_0 = det(Y)^{n+1-2k}det(\partial Y)det(Y)^{2k+1-n}det(\partial Y),$$

in particular, we can compute explicitly the factor $\gamma_n(s)$. The differential operator D on $\mathbb{H}_n$ is given in a more abstract way: Let $\mathfrak{a}$ be the Lie algebra of the joint maximal torus in $GL(n,\mathbb{R}) \hookrightarrow Sp(n,\mathbb{R})$ and let W_{Sp} and W_{GL} be the Weyl groups. Then the algebras $\mathbb{D}(\mathbb{H}_n)$ and $\mathbb{D}(\mathcal{P}_n)$ are given by the

(respective) Weyl group invariants in the symmetric algebra $S(\mathfrak{a})$, see e.g. [5].
We can now define an inclusion $\iota : \mathbb{D}(\mathbb{H}_n) \hookrightarrow \mathbb{D}(\mathcal{P}_n)$ by the diagram

$$\begin{array}{ccc} \mathbb{D}(\mathbb{H}_n) & \simeq & S(\mathfrak{a})^{W_{Sp}} \\ | & & | \\ \mathbb{D}(\mathcal{P}_n) & \simeq & S(\mathfrak{a})^{W_{GL}} \end{array}$$

We may choose $\mathcal{D}$ such that

$$\iota(\mathcal{D}) = D_0.$$

This differential operator $\mathcal{D}$ depends on k. Then the Rankin-Selberg mechanism works well for $\mathcal{D}(F)$:

$$\int D(F)E(Z,s)d_kZ = \gamma_n(s)\mathcal{D}(f,g,s+k-\frac{n+1}{2})$$

and it also gives the meromorphic continuation of this Dirichlet series (including the case of non cusp forms); as far as we know, this statement is in the literature only for degree 1 and 2 (by [14], [10] and [9]); Deitmar and Krieg [4] implicitly provide almost all the necessary ingredients for the general case.
We define a number $c(n,k)$ and a differential operator $\tilde{\mathcal{D}}$ by

$$\begin{array}{rcl} \mathcal{D}(1) & = & c(n,k) \\ \mathcal{D} & = & c(n,k) + \tilde{\mathcal{D}}. \end{array}$$

A basic fact to be used is

$$\int \tilde{\mathcal{D}}(F)d_kZ = 0 \tag{5}$$

for $F = f(Z)\overline{g(Z)}det(Y)^k$ with f, g in L^2. This important fact can be proved in a way similar to Lemma A 8.3 in [12]; note however that it requires some work to see that the assumptions of that lemma are satisfied (we omit the details here).
The property (5) then implies that $c(n,k)\cdot < f, g >$ can be expressed in terms of $\mathcal{D}(f,g,s)$, provided that $c(n,k) \neq 0$.
Indeed,

$$c(n,k) = k \cdot (k - \frac{1}{2}) \dots (k - \frac{n-1}{2})(k-n) \cdot (k - n + \frac{1}{2}) \dots (k - n + \frac{n-1}{2}).$$

Therefore this constant is zero for singular weights and also for all the delicate weights, but nonzero for $k = \frac{n}{2}$. For this "almost singular" weight we obtain indeed that for *all* modular forms $f, g \in M^n_{\frac{n}{2}}$

$$< f, g > \sim Res_{s=\frac{n}{2}} \mathcal{D}(f, g, s).$$

Final remark: As a consequence of the above (for the case of congruence subgroups), we obtain an orthogonality property for theta series: For a positive definite integral quadratic form of size n we define (for $Z \in \mathbb{H}_n$)

$$\theta^n(S)(Z) := \sum_{M \in \mathbb{Z}^{(n,n)}} \mathbf{e}_n(M^t SMZ).$$

This is a modular form of degree n and weight $\frac{n}{2}$; it is square-integrable and

$$\theta^n(S) \perp \theta^n(T),$$

if the two quadratic forms S and T are not rationally equivalent. This is a simple consequence of the Petersson identity and the trivial observation that the Rankin convolution $\mathcal{D}(\theta^n(S), \theta^n(T), s)$ is identically zero for rationally inequivalent S and T. We do not know any other way of proving such a statement. The scalar product formula of Li [7] seems to work only for rationally equivalent quadratic forms.

Problem: *The map*

$$(f, g) \longmapsto Res_{s=k} \mathcal{D}(f, g, s)$$

defines an Hermitian form on $M^n_k(\Gamma)$*; is it nondegenerate for nonsinguar weights ? What is its signature ? (for n=1 see [14])*

The case of *delicate* weights is more difficult; we hope to include this more general case in [2]

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