PULLBACKS OF SAITO-KUROKAWA LIFTS

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In this note, we announce that Ikeda's conjecture [12] holds for $r = 1$ and $n = 0$.

1. Statement of the main theorem

Let κ be an odd positive integer. Let

$$f(\tau) = \sum_{N>0} a_f(N) q^N \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$$

be a normalized Hecke eigenform and

$$h(\tau) = \sum_{\substack{N>0 \\ -N \equiv 0,1 \bmod 4}} c_h(N) q^N \in S^+_{\kappa+1/2}(\Gamma_0(4))$$

a Hecke eigenform associated to f by the Shimura correspondence. Let

$$F(Z) = \sum_{B>0} A(B) e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)} \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$$

be the Saito-Kurokawa lift of h, where

$$A\left(\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}\right) = \sum_{d|(n,r,m)} d^\kappa c_h\left(\frac{4nm - r^2}{d^2}\right).$$

For each normalized Hecke eigenform

$$g(\tau) = \sum_{N>0} a_g(N) q^N \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z})),$$

we consider the period integral $\langle F|_{\mathfrak{h}_1 \times \mathfrak{h}_1}, g \times g \rangle$ given by

$$\langle F|_{\mathfrak{h}_1 \times \mathfrak{h}_1}, g \times g \rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}_1} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}_1} F\left(\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}\right) \overline{g(\tau_1) g(\tau_2)} y_1^{\kappa-1} y_2^{\kappa-1} \, d\tau_1 \, d\tau_2.$$

Define the Petersson norms of f, g, h by

$$\langle f, f\rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}_1} |f(\tau)|^2 y^{2\kappa-2}\, d\tau,$$
$$\langle g, g\rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}_1} |g(\tau)|^2 y^{\kappa-1}\, d\tau,$$
$$\langle h, h\rangle = \frac{1}{6}\int_{\Gamma_0(4)\backslash\mathfrak{h}_1} |h(\tau)|^2 y^{\kappa-3/2}\, d\tau,$$

respectively.

For each prime p, let $\{\alpha_p, \alpha_p^{-1}\}$ and $\{\beta_p, \beta_p^{-1}\}$ denote the Satake parameters of g and f at p, respectively. Then

$$1 - a_g(p)X + p^\kappa X^2 = (1 - p^{\kappa/2}\alpha_p X)(1 - p^{\kappa/2}\alpha_p^{-1} X),$$
$$1 - a_f(p)X + p^{2\kappa-1} X^2 = (1 - p^{\kappa-1/2}\beta_p X)(1 - p^{\kappa-1/2}\beta_p^{-1} X).$$

We put

$$A_p = p^\kappa \begin{pmatrix} \alpha_p^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha_p^{-2} \end{pmatrix}, \quad B_p = p^{\kappa-1/2} \begin{pmatrix} \beta_p & 0 \\ 0 & \beta_p^{-1} \end{pmatrix}.$$

Define the L-function $L(s, \mathrm{Sym}^2(g) \otimes f)$ by an Euler product

$$L(s, \mathrm{Sym}^2(g) \otimes f) = \prod_p \det(\mathbf{1}_6 - A_p \otimes B_p \cdot p^{-s})^{-1}$$

for $\mathrm{Re}(s) \gg 0$. Let $\Lambda(s, \mathrm{Sym}^2(g) \otimes f)$ be the completed L-function given by

$$\Lambda(s, \mathrm{Sym}^2(g) \otimes f) = \Gamma_\mathbb{C}(s)\Gamma_\mathbb{C}(s-\kappa)\Gamma_\mathbb{C}(s-2\kappa+1) L(s, \mathrm{Sym}^2(g) \otimes f),$$

where $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. It satisfies the functional equation

$$\Lambda(4\kappa - s, \mathrm{Sym}^2(g) \otimes f) = \Lambda(s, \mathrm{Sym}^2(g) \otimes f).$$

Our main result is as follows.

Theorem 1.1.

$$\Lambda(2\kappa, \mathrm{Sym}^2(g) \otimes f) = 2^{\kappa+1} \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{|\langle F|_{\mathfrak{h}_1\times\mathfrak{h}_1}, g \times g\rangle|^2}{\langle g, g\rangle^2}.$$

Theorem 1.1 has an application to Deligne's conjecture [3].

Corollary 1.2. *For* $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\left(\frac{\Lambda(2\kappa, \mathrm{Sym}^2(g) \otimes f)}{\langle g, g\rangle^2 c^+(f)}\right)^\sigma = \frac{\Lambda(2\kappa, \mathrm{Sym}^2(g^\sigma) \otimes f^\sigma)}{\langle g^\sigma, g^\sigma\rangle^2 c^+(f^\sigma)}.$$

Here $c^+(f)$ *is the period of* f *as in* [19].

Proof. The assertion follows from Theorem 1.1 and the Kohnen-Zagier formula [15]

$$\Lambda(\kappa, f, \chi_{-\mathbf{D}}) = 2^{-\kappa+1}\mathbf{D}^{1/2}|c_h(\mathbf{D})|^2 \frac{\langle f, f\rangle}{\langle h, h\rangle},$$

where $-\mathbf{D} < 0$ is a fundamental discriminant. □

Remark 1.3. It seems that Corollary 1.2 does not follow from the algebraicity of central critical values of triple product L-functions. Notice that

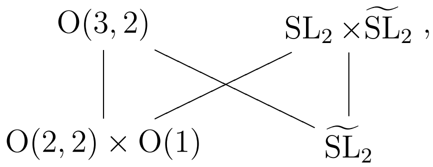
$$\Lambda(2\kappa, g \otimes g \otimes f) = \Lambda(2\kappa, \mathrm{Sym}^2(g) \otimes f)\Lambda(\kappa, f) = 0.$$

2. Proof of Theorem 1.1

Since F is a cusp form, the usual unfolding method does not work. Instead, we use seesaws in the sense of Kudla [16].

We may assume that $c_h(N) \in \mathbb{R}$ for all $N \in \mathbb{N}$. Let $\mathbf{f}$ and $\mathbf{g}$ denote the automorphic forms on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to f and g, respectively. Let $\mathbf{h}$ and Θ denote the automorphic forms on $\widetilde{\mathrm{SL}}_2(\mathbb{A}_\mathbb{Q})$ associated to h and θ, respectively. Here $\theta(\tau) = \sum_{N\in\mathbb{Z}} q^{N^2}$ is the theta function. Let π be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by $\mathbf{g}$.

Proposition 2.1. *For the seesaw*



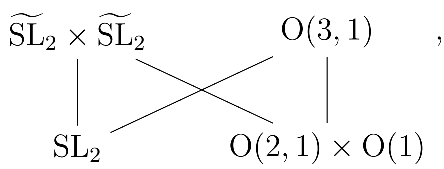
the identity

$$\langle F|_{\mathfrak{h}_1\times\mathfrak{h}_1}, g \times g\rangle = 2^{\kappa+2}\xi(2)\langle g, g\rangle\langle \mathbf{h}\Theta, \mathbf{g}^\sharp\rangle$$

holds. Here $\mathbf{g}^\sharp \in \pi$ *and* $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Fix a fundamental discriminant $-\mathbf{D} < 0$ with $-\mathbf{D} \equiv 1 \bmod 8$ such that $\Lambda(\kappa, f, \chi_{-\mathbf{D}}) \neq 0$ (and hence $c_h(\mathbf{D}) \neq 0$). Such a discriminant exists by [21], [2]. Let $\pi_\mathcal{K}$ be the base change of π to the imaginary quadratic field $\mathcal{K} = \mathbb{Q}(\sqrt{-\mathbf{D}})$.

Proposition 2.2. *For the seesaw*



the identity

$$\langle \mathbf{h}\Theta, \mathbf{g}^\sharp \rangle = (\sqrt{-1})^\kappa \mathbf{D}^{-1/2} c_h(\mathbf{D})^{-1} \langle f, f \rangle^{-1} \langle h, h \rangle \mathcal{I}(\mathbf{g}^\sharp_{\mathcal{K}}, \mathbf{f})$$

holds. Here $\mathbf{g}^\sharp_{\mathcal{K}} \in \pi_{\mathcal{K}}$ *and*

$$\mathcal{I}(\mathbf{g}^\sharp_{\mathcal{K}}, \mathbf{f}) = \int_{\mathbb{A}_{\mathbb{Q}}^\times \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})} \mathbf{g}^\sharp_{\mathcal{K}}(h) \mathbf{f}(h) \, dh.$$

The following proposition follows from the regularized Siegel-Weil formula by Kudla and Rallis [17] and the integral representation of triple product L-functions by Garrett [5], Piatetski-Shapiro and Rallis [18].

Proposition 2.3. *For the seesaw*

$$\begin{array}{ccc} \mathrm{Sp}_3 & & \mathrm{R}_{\mathcal{K}/\mathbb{Q}} \, \mathrm{O}(2,2) \times \mathrm{O}(2,2) \\ | & \times & | \\ \mathrm{R}_{\mathcal{K}/\mathbb{Q}} \, \mathrm{SL}_2 \times \mathrm{SL}_2 & & \mathrm{O}(2,2) \end{array},$$

the identity

$$\Lambda(2\kappa, \mathrm{Sym}^2(g) \otimes f) \Lambda(\kappa, f, \chi_{-\mathbf{D}}) = -2^{2\kappa+6} \mathbf{D}^{-1/2} \xi(2)^2 \mathcal{I}(\mathbf{g}^\sharp_{\mathcal{K}}, \mathbf{f})^2$$

holds.

Now Theorem 1.1 follows from Propositions 2.1–2.3 and the Kohnen-Zagier formula [15].

3. The Gross-Prasad conjecture

In this section, we interpret our result in terms of the Gross-Prasad conjecture [6], [7], which has been refined in a joint work with Tamotsu Ikeda [11].

Let $H_1 = \mathrm{SO}(n+1)$ and $H_0 = \mathrm{SO}(n)$ be special orthogonal groups over a number field k with embedding $\iota : H_0 \hookrightarrow H_1$. Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible cuspidal automorphic representation of $H_i(\mathbb{A}_k)$. We assume that

$$\mathrm{Hom}_{H_0(k_v)}(\pi_{1,v}, \pi_{0,v}) \neq 0$$

for all v.

Conjecture 3.1 (Gross-Prasad)**.** *Assume that* π_1 *and* π_0 *are tempered. Then the period integral*

$$\langle F_1|_{H_0}, F_0 \rangle = \int_{H_0(k) \backslash H_0(\mathbb{A}_k)} F_1(\iota(h_0)) \overline{F_0(h_0)} \, dh_0$$

does not vanish for some $F_1 \in \pi_1$ and some $F_0 \in \pi_0$ if and only if

$$L\left(\frac{1}{2}, \pi_1 \times \pi_0\right) \neq 0.$$

To relate our result to the Gross-Prasad conjecture, we would like to remove the assumption that π_1 and π_0 are tempered, and give an explicit formula for the period integral in terms of special values. We put

$$\mathcal{P}_{\pi_1,\pi_0}(s) = \frac{L(s, \pi_1 \times \pi_0)}{L\left(s+\frac{1}{2}, \pi_1, \mathrm{Ad}\right) L\left(s+\frac{1}{2}, \pi_0, \mathrm{Ad}\right)},$$

where $\mathrm{Ad} : {}^L H_i \to \mathrm{GL}(\mathrm{Lie}({}^L H_i))$ is the adjoint representation. In [11], we conjectured that the identity

$$\frac{|\langle F_1|_{H_0}, F_0\rangle|^2}{\langle F_1, F_1\rangle\langle F_0, F_0\rangle} = \mathcal{P}_{\pi_1,\pi_0}\left(\frac{1}{2}\right)$$

holds up to an elementary constant, and gave an example for $n = 5$ with non-tempered π_1, π_0. Also, this conjectural identity is compatible with the results of Waldspurger [20] for $n = 2$, Harris and Kudla [8], [9] for $n = 3$, Böcherer, Furusawa, and Schulze-Pillot [1] for $n = 4$.

We now discuss the case $n = 4$. Let π_1 (resp. π_0) be the irreducible cuspidal automorphic representation of $\mathrm{SO}(3,2)(\mathbb{A}_\mathbb{Q}) \simeq \mathrm{PGSp}_2(\mathbb{A}_\mathbb{Q})$ (resp. $\mathrm{SO}(2,2)(\mathbb{A}_\mathbb{Q}) \simeq [\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})]_0/\mathbb{A}_\mathbb{Q}^\times$) generated by F (resp. $g \times g$). Let π (resp. σ) be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by g (resp. f). Then

$$L(s, \pi_1) = L(s, \sigma)\zeta\left(s+\frac{1}{2}\right)\zeta\left(s-\frac{1}{2}\right),$$
$$L(s, \pi_0) = L(s, \pi \times \pi).$$

By Theorem 1.1 and the result of Kohnen and Skoruppa [14],

$$\frac{|\langle F|_{\mathfrak{h}_1\times\mathfrak{h}_1}, g \times g\rangle|^2}{\langle F, F\rangle\langle g, g\rangle^2} = \mathcal{P}_{\pi_1,\pi_0}\left(\frac{1}{2}\right).$$

This identity might hold even if F is not a Saito-Kurokawa lift. Using Dokchitser's computer program [4] and Katsurada's formula [13] for $\langle F, F\rangle$, one might check it numerically.

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