Modular cycles on arithmetic quotients of classical domains

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Let $\mathcal{D} = G/K$ be a Hermitian symmetric domain with G a real semisimple Lie group and K a maximal compact subgroup of G. For an arithmetic subgroup Γ of G, which has a finite covolume $\mathrm{vol}(\Gamma\backslash G)$ with respect to the Haar measure of G, the quotient $\Gamma\backslash G$ is a complex algebraic variety, thanks to C. L. Siegel, I. Satake, and W. Baily-A. Borel.

Given a Lie subgroup H of G, and a point x of $\mathcal{D}$, the orbit Hx in $\Gamma\backslash\mathcal{D}$ defines a reasonable topological chain in the quotient $\Gamma\backslash\mathcal{D}$, if the subgroup H is the real point of some algebraic subgroup of G defined with respect to the rational structure specified by Γ. Such a chain, which is sometimes extendable to a cycle on some good compactification of $\Gamma\backslash\mathcal{D}$, is called a (*generalized*) *modular symbol*.

Though there was some prototype investigation even by E. Hecke, a systematic investigation of this subject was initiated by Yu. Manin [Mn72], [Mn73] (and B. Mazur-P. Swinnerton-Dyer [MzSw74]) in early '70's, for the case: $G = SL_2(\mathbf{R})$, Γ a congruence subgroup of $SL_2(\mathbf{Z})$, and $H \cong \mathbf{G}_{m,\mathbf{R}}$. As already shown by a paper of the T. Shintani [Sh75], the periods of modular forms along these modular symbols appears as Fourier coefficients of elliptic modular forms of half-integral weight, and sometime later this was extended to the case of theta correspondence for $SL_2(\mathbf{R}) \times SO(2, q)$ by the author [Od77].
Remark There are other people, Kudla, Rallis-Schiffman, Kudla,Vigneras, who discussed "lifting" of modular forms. However only the author seem to investiagte the "converse", i.e. the adjoint of the lifting at this time. The "periods integrals" appear as Fourier coefficients of elliptic modular forms. Later this was found to be very important to investigate periods of Hilbert modular surfaces (*cf.* [Od82]).

The author has to refer to the important result of J. Millson [Ml76] and the begining of the work of A. Ash [As77] on higher-dimensional modular symbols. Though the author's interest shifted slightly to the direction of quotients of Hermitian symmetric spaces and the Hodge structures of cohomology groups of discrete subgroups, general results on modular symbols expanded in 80's. One good reference on these developement might be found in [LbSc89].

The purpose of this note is to review the recent two results by the author and his coauthors. After some review of fundamental results of the cohomology groups of discrete subgroups in Section 1, we discuss a new type vanshing theorem of specified Hodge components of the Poincaré dual of certain modular symbols in Section 2 ([KO98]). This is a joint work with Toshiyuki Kobayashi.

In Section 3, we discuss the construction of Green current for certain modular divisors, which is a joint work with Masao Tsuzuki ([TsO00]). Some of the results are overlapped with a recent paper of Bruinier [Br00].

1 Cohomology of dsicrete subgroups

In this section, we recall basic facts on cohomology groups of discrete subgroups in real semisimple Lie groups. A good reference is a survey artilce by A. Borel [Brl76]. The book [BrWl80] of Borel-Wallach also has been a very important reference, though this is a bit difficult to penetrate since it is not written as a textbook. The original paper about this theme is Matsushima's [Mt62]. In the case when G/K is Hermitian, there is a paper by the author which is more specialized to the Hodge theoretic aspect of the problem [Od85].

1.1 Shift to the relative Lie algebra cohomology groups

Given an arithmetic discrete subgroup Γ in a semisimple real Lie group G, we consider its Eilenberg-Maclane cohomology group $\mathrm{H}^i(\Gamma, \mathbf{C})$. Or more generally if a finite dimensional rational representation $r : G \to GL(V)$ of G is given, we may regard its as Γ-module by restriction, and we can form cohomology group $\mathrm{H}^i(\Gamma, V)$.

Fix a maximal compact subgroup K of G to get a Riemannnian symmetric space $X = G/K$. For simplicity assume that we have no elements of finite order in Γ, then Γ acts on X from the right side without fixed point, and the quotinet $\Gamma\backslash X$ becomes a manifold. In this case Γ is isomorphic to the fundamental group of this manifold (X is contractible to a point), and the Γ-modules V defines a local system $\tilde{V}$ on this quotient manifold. Then we have an isomorphism of cohomology groups: $\mathrm{H}^*(\Gamma, V) \cong \mathrm{H}^*(\Gamma\backslash X, \tilde{V})$.

Let $\sigma : X \to \Gamma\backslash X$ be the canonical map, by pulling back differential forms with respect to σ we have a monomorphism of de Rham complexes $\sigma^0 : \Omega^*_{\Gamma\backslash X} \to \Omega^*_X$. Moreover on the target complex, Γ acts naturally. Let $(\Omega^*_X)^\Gamma$ be the invariant subcomplex. Then it coincides with the image of σ^0, and by the de Rham theorem, we have an isomorphism of cohomology groups:

$$\mathrm{H}^*(\Gamma, V) = \mathrm{H}^*(\Gamma\backslash X, \tilde{V}) = \mathrm{H}^*(\Omega_X(V)^\Gamma).$$

The last complex in the above isomorphims is identified with the complex of differentail forms on G as follows.

Let π^0 be the pull-back homomorphism of differential forms with respect to the canonical map $\pi : \Gamma\backslash G \to \Gamma\backslash X$. Then a form $\omega \in \Omega^d_X(V)^\Gamma$ defines a differential form ω^0 on G by

$$x \in G \mapsto r(x)^{-1}\pi^0(\omega)(x),$$

and denote by $A^*_0(G, \Gamma, V)$ the image of this homomorphism. Then since σ^0 is a monomorphism, we have an isomorphism of complexes $\Omega^*_X(V)^\Gamma \cong A^*_0(G, \Gamma, V)$.

Here the last complex is identified with the right K-invariant subcomplex of the de Rham complex $\Omega^*_{\Gamma\backslash G}(V) = \Omega^*_{\Gamma\backslash G} \otimes V$, which is defined over $\Gamma\backslash G$ and takes values in V.

Since the tangent space at each point of G/K is identified with the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ with resepect to the Killing form, the module of the i-th cochains becomes $\mathrm{Hom}_K(\wedge^i\mathfrak{p}, C^\infty(\Gamma\backslash G)\otimes V)$. Therefore, the cochain comlex defined in this manner gives the relative Lie algebra cohomology groups. When G is connected we have an isomorphism:

$$\mathrm{H}^*(\Gamma, V) \cong \mathrm{H}^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G), V).$$

1.2 Matsushima isomorphism

Let G be a connected semisimple real Lie group with finite center. Assume that the discrete subgroup Γ is cocompact, i.e. the quotinet $\Gamma\backslash G$ is compact.

Let $L^2(\Gamma\backslash G)$ be the space of L^2-functions on G with respect the Haar measure on G, on which G acts unitarily by the right action. By assumption the space $C^\infty(\Gamma\backslash G)$ is a subspace of this space.

Proposition (1.1) (Gelfand-Graev,Piateskii-Shapiro) If Γ is cocompact, we have the direct sum decomposition of the unitary representation $L^2(\Gamma\backslash G)$ into irreducible components:

$$L^2(\Gamma\backslash G) = \tilde{\oplus}_{\pi\in\hat{G}} m(\pi,\Gamma)M_\pi,$$

with finite multipicities $m(\pi,\Gamma)$. Here $\hat{G}$ is the unitary dual of G, i.e.the unitary equivalence classes of irreducible unitary representations of G, and M_π denotes the representation sapce of π.

By the above proposition, one has a decomposition

$$C^\infty(\Gamma\backslash G) = \tilde{\oplus}_{\pi\in\hat{G}} m(\pi,\Gamma)M_\pi^\infty$$

of topological linear spaces. Here M_π^∞ is the subspace consisting of C^∞-vectors in the representation space M_π.

Theorem (1.2) For a finite dimensional rational G-module V over the complex number field, we have an isomorphism:

$$\mathrm{H}^*(\Gamma, V) = \oplus_{\pi\in\hat{G}} m(\pi,\Gamma)\mathrm{H}^*(\mathfrak{g}, K, M_\pi^\infty\otimes V)$$

$$= \oplus_{\pi\in\hat{G}}\{\mathrm{Hom}_G(M_\pi, L^2(\Gamma\backslash G)\otimes_{\mathbb{C}} \mathrm{H}^*(\mathfrak{g}, K, M_\pi^\infty\otimes V)\}$$

The key point of the proof here is the complete direct sum $\tilde{\oplus}$ is replaced by a simple algebraic direct sum by passing to the cohomology (*cf* Borel [Brl76]).

We can equipp a K-invaraint inner product on V. By using this, we may regard $H^m(\mathfrak{g}, K, M_\pi^\infty\otimes V)$ as a space of a kind of harmonic forms, i.e. the totality of cochains vanishing by the Laplace operator. Thus we have the following.

Proposition (1.3) Let (r, V) be an irreducible G-module of finite dimension. For any $\pi\in\hat{G}$ we have the follwing.

(i) If $\chi_r(C) = \chi_\pi(C)$, there is an isomorphism

$$\mathrm{Hom}_K(\wedge^m, M_\pi\otimes^\infty\otimes V) = C^m(\mathfrak{g}, K, M_\pi^\infty\otimes V) = \mathrm{H}^*(\mathfrak{g}, K, M_\pi^\infty\otimes V)$$

(ii) If $\chi_r(C) \neq \chi_\pi(C)$, there is an isomorphism

$$\mathrm{H}^m(\mathfrak{g}, K, M_\pi^\infty\otimes V) = \{0\}.$$

Here C denotes the Casimir operator.

In particular, when V is the trivial G-module $\mathbb{C}$, the above theorem is no other than the original formula of Betti numbers of $\Gamma\backslash X$ by Matsushima ([Mt62]).

Also there is a variant of this type vanishing theorem shown by D. Wigner. But it is omitted here.

We recall here that the relative Lie algebra cohomology group $\mathrm{H}^m(\mathfrak{g}, K, M_\pi^\infty \otimes V)$ is isomorphic to the continuous cohomology group $\mathrm{H}^m_{ct}(G, M_\pi^\infty \otimes V)$ if G is connected. This is shown by using differential cohomology and van Est spectral sequence.

1.3 "Classical" vanishing theorems

A number of vanishing theorems were found in '60's: Calabi-Vesentini, Weil etc. In their proof, the same type of computation of "curvature forms" is done, which is similar to a proof of Kodaira vanishing theorem.

Firstly, Matsushima's vanishing theorem of the 1-st Betti number of $\Gamma\backslash X$ was also proved by such method ([Mt62]).

This type of vanishing theorem is vastly improved by representation theoretic method. Probably the best resut of this category is the following result by Zuckermann ([Zkrm78]) (see also [BrWl80], Chapter V, §2 − §3 (p.150–155)).

Theorem(1.6) Let G be a simple real algebraic group, (π, H) a nontrivial irreducible unitary representation of G, and (r, V) a finite dimensional representation of G. Then for $k < \mathrm{rank}_{\mathbb{R}} G$,

$$\mathrm{H}^k_{ct}(G, H^\infty \otimes V) = \{0\}.$$

Corollary(1.7) Given a cocompact discrete subgroup Γ of G, for $k < \mathrm{rank}_{\mathbb{R}} G$, the restriction homomorphism

$$\mathrm{H}^k_{ct}(G, V) \to \mathrm{H}^k(\Gamma, V)$$

is an isomorphism.

1.4 Enumeration and construction of unitary cohomological representations

We shortly review the state of arts on the cohomological representations defined below.

Definition(1.8) An irreducible (unitary) representation $(\pi, H_\pi) \in \hat{G}$ is called *cohomological*, if there is a finite dimensional G-module F such that

$$\mathrm{H}^i(\mathfrak{g}, K; H_\pi^\infty \otimes_{\mathbb{C}} F) \neq \{0\}$$

for some $i \in \mathbb{N}$. The set of equivalent classes of cohomological representations is denoted by $\hat{G}_{coh}$.

Enumeration of such cohomological representations was done for the case trivial $F = \mathbf{C}$ by Kumaresan [Km80], and for general case by Vogan-Zuckermann [VgZm84]. This was originally described by susing the cohomological induction functor $\mathcal{A}_\lambda(\mathfrak{g})$ first. And later a global realiztion of this Zuckermann module was obtained by H.-W. Wong [Wng95].

Recently Tosiyuki Kobayashi is developping a theory of branching rule for such cohomological representations when they are restricted to a large reductive subgroup H of G ([Kb93],[Kb94],[Kb98a,b]).

2 A new vanishing theorem

2.1 Formulation of problem

Firstly we recall the construction of generalized modular symbols associated with redutive subgroups of G. Given a double coset space $V = \Gamma\backslash/K$, which is compact. Let $\iota : H \subset G$ be a reductive subgroup of G, such that
(a) $H \cap K$ is maximally compact in H;
(b) $H \cap \Gamma$ is cocompact discrete subgroup of H.
Then we have a natural map of double cosets:

$$\tilde{\iota} : (H \cap \Gamma)\backslash H/(H \cap K) \to V.$$

Set $d = \dim_{\mathbf{R}} W = \dim_{\mathbf{R}} H/(H \cap K)$, $N = \dim_{\mathbf{R}} V = \dim_{\mathbf{R}} G/K$. Then the fundamental class $[W] \in \mathrm{H}_d(W, \mathbf{C})$ mapped naturally by ι to

$$\iota_*([W]) \in \mathrm{H}_d(V, \mathbf{C}) \cong_P \mathrm{H}^{N-d}(V, \mathbf{C}) \cong \mathrm{H}^{N-d}(\Gamma, \mathbf{C})$$

$$\cong \oplus_{\pi \in \hat{G}, \chi_\infty(pi) = \chi_\infty(1)} \{\mathrm{Hom}_G(H_\pi, L^2(\Gamma\backslash G)) \otimes \mathrm{H}^{N-d}(\mathfrak{g}, K; H_\pi^\infty)\}.$$

Here the first isomorphism denoted by P is the Poincaré duality. According to the last decomposition in the above formula, we have a decomposition

$$P \cdot \iota_*([W]) = \sum_{\pi \in \hat{G}, \chi_\infty(\pi) = \chi_\infty(1)} \mathcal{M}^{(\pi)}(W),$$

where $\mathcal{M}^{(\pi)}(W)$ denotes the π-th component of the Poincaré dual of $[W]$.
Problem (2.1) Describe $\mathcal{M}^{(\pi)}(W)$, using the special values of automorphic L-functions etc, $\cdots$

Dually speaking, it is to consider the restriction map:

$$\mathrm{H}^d(V, \mathbf{C}) = \oplus_{\pi \in \hat{G}} \{\mathrm{Hom}_G(H_\pi, L^2(\Gamma\backslash G) \oplus_{\mathbf{C}} \mathrm{H}^d(\mathfrak{g}, K; H_\pi)\}.$$

$$\to \mathrm{H}^d(W, \mathbf{C}) = \mathbf{C}.$$

This is done by invetigation of the period integrals $\int_W \omega$ for elements ω in the π-component

$$\mathrm{H}^d(V, \mathbf{C})^{(\pi)} = \{\mathrm{Hom}_G(H_\pi, L^2(\Gamma\backslash G) \oplus_{\mathbf{C}} \mathrm{H}^d(\mathfrak{g}, K; H_\pi)\}.$$

of $\mathrm{H}^d(V, \mathbf{C})$.

A nice situation is when the retriction $\pi|H$ of $\pi \in \hat{G}_{coh}$ is *admissble*, i.e.
(a) $\pi|H$ decomposes discretely as H-modules;
(b) $\pi|H = \tilde{\oplus}_{\sigma \in \hat{H}} m(\pi, \sigma)\sigma$ has finite multiplicity $m(\pi, \sigma) < \infty$.

For cohomological representations $\pi = \mathcal{A}_\lambda(\mathfrak{q})$, Koybayashi obtained certain sufficient condition for the admissibility of $\pi|H$, which is described geometrically.

As an application, under the same condition as the above criterion of admissibility, the restriction to

$$\mathrm{H}_d(W, \mathbf{C}) \times \mathrm{H}^d(V, \mathbf{C})^{(\pi)}$$

of the natural pairing:

$$\mathrm{H}_d(V, \mathbf{C}) \times \mathrm{H}^d_{dR}(V, \mathbf{C}) \to \mathbf{C}$$

vanishes. Here $d = \dim W$.

2.2 Example

Let $G = SO(2n, 2) \times$ *compact factor* such that there is an algebraic group $\mathbf{G}$ over $\mathbf{Q}$ with $G = \mathbf{G}(\mathbf{R})$. Then we may form an algebraic subgroup $\mathbf{H}$ of $\mathbf{G}$ such that $H = \mathbf{H}(\mathbf{R})$ is isomorphic to $SO(2n, 1) \times$ *compact factor* and there is a maximal compact subgroup K of G with $H \cap K$ being maximal compact in H.

Set $X = G/K$ and $X_H = H/(H \cap K)$. Then X_H is a totally real submanifold of X. This means that for some holomorphic local coordinates $(z_1, z_2, \cdots, z_{2n})$ at each point of X_H, it is defined locally by the equalities $\mathrm{Im}(z_1) = 0, \cdots, \mathrm{Im}(z_{2n}) = 0$.

Now for a cocompact arithmetic discrete subgroup Γ of G, we can define a generalized modular symbol:

$$\iota : W = (H \cap \Gamma)\backslash X_H \to V = \Gamma\backslash X.$$

In this case the fundamental class $[W] \in \mathrm{H}_{2n}(W, \mathbf{C})$ mapped to

$$\mathrm{H}_{2n}(V, \mathbf{C}) \cong \mathrm{H}^{2n}_{dR}(V, \mathbf{C}) = \oplus_{p+q=2n} \mathrm{H}^{p,q}$$

by ι_*. Here the last isomorphism is the Hodge decomposition. Hence we have the Hodge decomposition of $\mathcal{M}(W)$:

$$\mathcal{M}(W) = \sum_{p+q=2n} \mathcal{M}^{(p,q)}(W).$$

Its (n, n)-type component $\mathcal{M}^{(n,n)}(W)$ has further decoposition:

$$\mathcal{M}^{(n,n)}(W) = \sum_{\pi \in \hat{G}, \mathrm{H}^{n,n}(\mathfrak{g}, K; H^\infty_\pi) \neq \{0\}} \mathcal{M}^{(\pi)}(W).$$

Here $\mathrm{H}^{n,n}(\mathfrak{g}, K; \mathrm{H}^\infty_\pi)$ is the (n, n)-type component of $\mathrm{H}^{2n}(\mathfrak{g}, K; H^\infty_\pi)$. In this case we have the following.

Proposition (2.2) If $\pi \neq \infty$, $\mathcal{M}^{(\pi)})(W) = 0$ for π satisfying $\mathrm{H}^{n,n}(\mathfrak{g}, K; H^\infty_\pi) \neq \{0\}$. This means that $\mathcal{M}(n, n)(W)$ is a consatnt multiple of a universal cohomology class $\eta \in \mathrm{H}^{(n,n)}_{dR}(V, \mathbf{C})$, which is one of two natural generators of

$$\mathrm{H}^{n,n}(\mathfrak{g}, K; \infty) = \mathbf{C} \wedge^n \kappa \oplus \mathbf{C}\eta,$$

where κ is the Kaehler class. Moreover the constant is $\mathrm{vol}(W)/\mathrm{vol}(V)$.

Remark In place of the pair $(SO(2n, 2), SO(2n, 1))$, we can consider the pair $(SU(2n, 2), Sp(n, 1))$ for example.

3 Construction of Green functions for modular divisors on arithmetic quotients of bounded symmetric domains

This part is a joint work with Masao Tsuzuki.

3.1 Logarithmic Green functions

Given a (smooth, for simplicity) submanifold Y of codimension d in a smooth quasi-projective complex algebraic variety X of dimension n, a current δ_Y, i.e. a differential forms with coefficients in distributions, of type (d, d) on X is defined by associating the values of the integral

$$\int_Y i^*(\omega)$$

for every $(n-d, n-d)$-type C^∞ form ω on X. Here $i : Y \subset X$ is the inclusion map. This is a closed form.

Now, if there is a current g of type $(d-1, d-1)$ on X such that

$$dd^c g + \delta_Y$$

is a smooth differential form of type (d, d) on X, then g is called a Green current for Y. Here $d^c = (\partial - \bar{\partial})/2\pi\sqrt{-1}$ if we write $d = \partial + \bar{\partial}$ as a sum of holomorphic part and antiholomorphic part. The form g is not unique.

In the intersection theory of Arakelov type for higher dimensional cases, Gillet-Soulé [GlSl92] defined Green current of logarithmic type. When the codimension $d > 1$, this is defined by using the resolution of singularities of Hironaka, but when $d = 1$, i.e. when Y is a divisor, it is done more directly.

If one defines a Hermitian metric $\| \ \|$ on the holomorphic line bundle$L = O_X(Y)$ by putting $\|f\| = e^{-g}|f|$ for local section f of L, then the Chern form of L with respect to this metric is given by $dd^c g + \delta_Y$. Conversely, if $(L, \| \ \|)$ a holomorphic line bundle with Hermitian metric, then its Green current is known by the following theorem of Poincaré-Lelong.

Proposition (3.0) For a meromorphic section s of L, $-\log\|s\|^2$ is a locally integrable function X and gives a logarithmic Green function for for $Y = \mathrm{div}(s)$. Moreover the right hand side of

$$dd^c(-\log\|s\|^2) + \delta_Y = c_1(L, \| \ \|)$$

is the Chern form for the metric line bundle $(L, \| \ \|)$.

Existence of logarithmic Green currents are guaranteed in general, but littel is known about their concrete construction.

From now on, when X and Y are arithmetic quotients and Y is a divisor of X, we show one way to construct the logarithmic Green current of Y.

3.2 The secondary spherical functions for affine symmetric pairs (G, H) of rank 1

For our construction, we need a pair (G, H) of real reductive Lie groups satisfying the follwoing "axioms":

(i) G, a connected real semisimple (algebraic) Lie group such that the quotient G/K by a maximal compact subgroup K is Hermitian symmetric ($K = G^\theta$ with θ a Cartan involution);

(ii) H, a reductive subgroup of G, such that there exists an involution $\sigma : G \to G$ satisfying $\theta\sigma = \sigma\theta$ and $(G^\sigma)_0 \subset H \subset G^\sigma$. Moreover $H\backslash G$ is a semisimple symmetric space of real rank 1.

Then for the Lie algebras $\mathfrak{g} = \mathrm{Lie}(G),\ \mathfrak{h} = \mathrm{Lie}(H)$ we have

$$\mathfrak{g} = \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\theta} \oplus \mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{\theta} \oplus \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{-\theta} \oplus \mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\theta}$$

and a maximal abelian subalgebra $\mathfrak{a}$ in the last factor $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\theta}$ is of dimension 1.

By the classification table of symmetric spaces, we have two cases:

$$(\mathrm{U-type}) : \mathfrak{g} = \mathfrak{s}u(n,1), \mathfrak{h} = \mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(n-1,1))$$

$$(\mathrm{O-type}) : \mathfrak{g} = \mathfrak{s}o(n,2), \mathfrak{h} = \mathfrak{s}o(n-1,2).$$

We choose a generator $Y_0 \in \mathfrak{a}$ such that $\lambda(Y_0) = 1$ for the short root λ in the root system $\Psi = \Psi(\mathfrak{a}, \mathfrak{g})$. Set $U = \exp(\mathfrak{g}_{\lambda} + \mathfrak{g}_{2\lambda})$ and $2\rho_0 = \mathrm{tr}(\mathrm{ad}(Y_0)|_{\mathrm{Lie}(U)})$. Also we normalize the Casimir operator Ω such that it corresponds to the bilinear form

$$X, Y = \frac{1}{B(Y_0, Y_0)} B(X, Y) \quad (X, Y \in \mathfrak{g}).$$

Set $A = \exp(\mathfrak{a}) = \{a_t = \exp(tY_0) | t \in \mathbf{R}\}$, then $G = HAK$.

Now we can consider a left H-invaraint and right K-invariant spherical function $\phi_s^{(1)}(g) \in C^{\infty}(H\backslash G/K)$ satisfying

$$\phi_s^{(1)}(g) * \Omega = (s^2 - \rho_0^2)\phi_s^{(1)}(g) \quad (s \in \mathbf{C}).$$

This function generates class one principal series representation of G in $C^{\infty}(H\backslash G)$ by right translation under G, which has H-invariant. This *ordinary* spherical function is not necessarily good one to define Poincaré series. In place of this, we consider the *secondary* spherical functions $\phi^{(2)}(g) \in C^{\infty}(G - H \cdot K)$ which has logarithmic singularities along $H \cdot K$.

Proposition (3.1) Let $s \in \mathbf{C}, \mathrm{Re}(s) > \rho_0$. Then there exists the unique function satisfying the conditions (a)-(d) below:

(a) $\phi_s^{(2)}$ is C^{∞} on $G - H \cdot K$, and left H-invaraint and right K-invariant;

(b) $\phi_s^{(2)}$ satisfies the diferential equation:

$$\phi_s^{(2)} * \Omega = (s^2 - \rho_0^2)\phi_s^{(2)} \text{ on } G - HK;$$

(c) For sufficiently small $\delta > 0$,

$$\phi_s^{(2)}(a_t) - \log(t) \text{ is bounded in } (0, \delta);$$

(d) $\phi_s^{(2)}(a_t)$ is rapidly decreasing for $t \to +\infty$.

For our later purpose, it is better to introduce a vector-valued spherical function ψ_s. Recall the Cartan decomposition and its complexification:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-.$$

Here the subpace $\mathfrak{p}_{\pm}$ is the $\pm i$-eigenspaces with respect to the given complex structure in the complexification $\mathfrak{p} \otimes \mathbf{C}$.

For a function $F \in C^{\infty}(G/K)$ we can define the gradient $\nabla F = \nabla_+ F + \nabla_- F \in C^{\infty}(G) \otimes \mathfrak{p}_{\mathbf{C}}$. Then we have $\nabla_- \nabla_+ F : G \to \mathfrak{p}_+ \otimes \mathfrak{p}_-$, which is K-equivariant under the right K-action on G and the tensor product of the adjoint representation $\mathrm{Ad}_{\mathfrak{p}_{\pm}}$. The K-module $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ is a direct sum of the trivial representation and the other irreducible representation V_{11}. We denote by pr the projection to V_{11} from $\mathfrak{p}_+ \otimes \mathfrak{p}_-$

Definition (3.2) $\phi_s = pr \cdot \nabla_- \nabla_+ \phi_s^{(2)}$.

3.3 Poincaré series

We can now define Poincaré series.

Definition (3.3) For $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \rho_0$, set

$$G_s(z) = \sum_{\gamma \in (\Gamma \cap H)\backslash \Gamma} \phi_s^{(2)}(l, \gamma(z)) \quad (g \in G)$$

and

$$\Psi_s(z) = \sum_{\gamma \in (\Gamma \cap H)\backslash \Gamma} \psi_s^{(2)}(l, \gamma(z)) \quad (g \in G).$$

Both converges s currents for $\mathrm{Re}(s) > \rho_0$ real analytic except for on the image of

$$(\Gamma \cap H)\backslash H/(K \cap H) \to \Gamma\backslash G/K.$$

We can show
· a criterion for $G_s \in L^p(\Gamma\backslash G)$ in particulra $G_s \in L^2(\Gamma\backslash G)$,
· the analytic continuation, when $G_s \in L^2(\Gamma\backslash G)$, (this is a kind of functional equation for G_s and G_{-s}.
· $G_s(g)$ has a simple pole at $s = \rho_0$ with residue

$$\frac{\mathrm{vol}((H \cap \Gamma)\backslash H)}{\mathrm{vol}(\Gamma\backslash G)} \frac{1}{2\rho_0}.$$

3.4 $\partial\bar{\partial}$-formula

Let $d = \partial + \bar{\partial}$ be the decompostion of the exterior derivative on G/K into the holomorphic part and the antiholomorphic part.

Theorem (3.4) We have

$$\text{(a)} \quad 4\sqrt{-1}\partial\bar{\partial}G_s + \tilde{\delta}_{D_0} = -\frac{c(\mathfrak{g})^2}{2n}(s^2 - \rho_0^2)G_s \wedge \omega_{\Gamma\backslash G/K} - \sqrt{-1}\Psi_s$$

$$\text{(b)} \quad \Delta\Psi_s = c(\mathfrak{g})^2(s^2 - \rho_2^2)(\frac{1}{2}\Psi_s - \frac{\sqrt{-1}}{2}\tilde{\delta}_{D_0} + \frac{\sqrt{-1}}{2n}\delta_{D_0} \wedge \omega_{\Gamma\backslash G/K}$$

Here Ψ_s is the Poincaré series of (1,1)-form, $\tilde{\delta}_{D_0}$ the current associated with the divisor

$$(H \cap \Gamma)\backslash H/(H \cap K) \to \Gamma\backslash G/K,$$

$c(\mathfrak{g})$ a constant gievn by $c(\mathfrak{g}) = 1$ (U-type), $= 2,$ (O-type), and $\omega_{\Gamma\backslash G/K}$ the Kaehler form on $\Gamma\backslash G/K$.

Theorem (3.5)
(i) The $(1,1)$-type current Ψ_s is holomorphic at $s = \rho_0$, and Ψ_{ρ_0} is harmonic C^∞-form of $(1,1)$-type on $\Gamma\backslash G/K$.
(ii) Put $\mathcal{G} = \lim_{s\to\rho_0}(G_s - \frac{1}{s-\rho_0}\mathrm{Res}_{s-\rho_0}G_s)$. Then

$$4\sqrt{-1}\partial\bar{\partial}\mathcal{G} + \tilde{\delta}_{D_0} = -\kappa\omega_{\Gamma\backslash G/K} - \sqrt{-1}\Psi_{\rho_0}.$$

(*cf.* [TsO01,Chapter 7, Theoem (7.6.1)] and its application).

Here is the meaning of the last formula. There is the subspace of square-integrable forms $\mathrm{H}^2_{(2)}(\Gamma\backslash G/K, \mathbf{C})$ in $\mathrm{H}^2(\Gamma, \mathbf{C}) = \mathrm{H}^2(\Gamma\backslash G/K, \mathbf{C})$. There are two irreducible unitary representation of G which contribute to the (1,1)-type Hodge component of $\mathrm{H}^2_{(2)}(\Gamma\backslash G/K, \mathbf{C})$: one is the trivial representation$\mathbf{C}$ and the other is a certain infinite dimensional representations, which we denote by $\pi_{1,1}$. Both representations π have $\mathrm{H}^{1,1}(\mathfrak{g}, K; H_{1,1}) \cong \mathbf{C}$. Hence

$$\mathrm{H}^{1,1} = \mathrm{Hom}_G(1, L^2(\Gamma\backslash G)) \oplus \mathrm{Hom}_G(\pi_{1,1}, L^2(\Gamma\backslash G)).$$

The first factor of the right hand side is generated by the Kaehler form κ. The right hand side of the statement (ii) of the last theorem is compatible with this decomposition.

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