MOTIVES AND SIEGEL MODULAR FORMS

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Introduction

This report is an informal resumé of [Y2], though I add a little new material in §3. The main theme is the notion of the fundamental periods of a motive and it's interplay with automorphic forms. I would like to develop more comprehensive theory on a future occasion.

§1. Critical values

We list major historical events concerning critical values of zeta functions.

$$1 - 1/3 + 1/5 - 1/7 + \cdots = \frac{\pi}{4}, \qquad \text{Leibnitz, around 1670.}$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \text{Euler, 1735, discovered experimantally.}$$

$$\zeta(2n)/\pi^{2n} \in \mathbf{Q}, \quad 1 \leqq n \in \mathbf{Z}, \qquad \text{Euler, 1742.}$$

$$\sum_{z} z^{-4n}/\varpi^{4n} \in \mathbf{Q}, \quad 1 \leqq n \in \mathbf{Z}, \qquad \text{Hurwitz, 1899,}$$

where z extends over all nonzero Gaussian integers and $\varpi = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}}$.

$$\frac{L(n,\Delta)}{(2\pi i)^n c^{\pm}(\Delta)} \in \mathbf{Q}, \qquad 1 \leqq n \leqq 11, \quad \pm 1 = (-1)^n, \qquad \text{Shimura, 1959,}$$

where

$$\Delta(z) = \sum_{n=1}^{\infty} a_n q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \qquad q = \exp(2\pi i z)$$

is the cusp form of weight 12 with respect to $SL(2,\mathbf{Z})$ and $c^{\pm}(\Delta) \in \mathbf{R}^{\times}$. Similarly for a Hecke eigenform $f \in S_k(\Gamma_0(N),\psi)$ and $\sigma \in \mathrm{Aut}(\mathbf{C})$,

$$\left(\frac{L(n,f)}{(2\pi i)^n c^{\pm}(f)}\right)^{\sigma} = \frac{L(n,f^{\sigma})}{(2\pi i)^n c^{\pm}(f^{\sigma})},$$
$$1 \leqq n \leqq k-1, \quad \pm 1 = (-1)^n, \qquad \text{Shimura, 1977,}$$

where $c^{\pm}(f^{\sigma}) \in \mathbf{C}^{\times}$ ([Sh1]).

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By these results, it was expected that the critical values of zeta functions are related to periods of integrals. Here the notion of critical values, which is generally accepted now, can be defined as follows. Suppose that a zeta function $Z(s)$ multiplied by its gamma factor $G(s)$ satisfies a functional equation of standard type under the symmetry $s \to v - s$. Then $Z(n)$, $n \in \mathbf{Z}$ is a critical value of $Z(s)$ if both of $G(n)$ and $G(v-n)$ are finite.

In 1979, Deligne ([D]) published a general conjecture which gives a prediction on critical values of the L-function of a motive. For a nice concise exposition of the theory of motives, we refer the reader to a paper of Jannsen [J]; for more comprehensive information, see [JKS].

Let M be a motive over $\mathbf{Q}$ with coefficients in an algebraic number field E. Put $R = E \otimes_{\mathbf{Q}} \mathbf{C}$. We have $E \subset R$ canonically and $R \cong \mathbf{C}^{J_E}$, J_E being the set of all isomorphisms of E into $\mathbf{C}$. Let λ be a finite place of E. From the λ-adic realization $H_\lambda(M)$, we have the λ-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $H_\lambda(M)$. By this represention, we can define the L-function $L(M, s)$ in the usual manner, which takes values in R.

Next let $H_B(M)$ be the Betti realization of M. We put $d = d(M) = \dim_E H_B(M)$. We call d *the rank* of M. Let F_∞ denote the complex conjugation; F_∞ acts E-linearly on $H_B(M)$. We have

$$H_B(M) = H_B^+(M) \oplus H_B^-(M) \tag{1.1}$$

where $H_B^\pm(M)$ denotes the eigenspace of $H_B(M)$ with the eigenvalue ± 1. We put $d^\pm = d^\pm(M) = \dim_E H_B^\pm(M)$. Furthermore $H_B(M)$ has the Hodge decomposition:

$$H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_{p,q \in \mathbf{Z}} H^{pq}(M), \tag{1.2}$$

where $H^{pq}(M)$ is a free R-module. If $H^{pq}(M) = \{0\}$ whenever $p + q \neq w$, M is called *of pure weight* w.

Remark. The Hodge decomposition determines the gamma factor of the conjectural functional equation of $L(M, s)$. Conversely, the gamma factor of the functional equation of $L(M, s)$ determines the Hodge decomposition if M is of pure weight.

We shall assume hereafter that M is of pure weight. We can define Deligne's periods $c^\pm(M) \in R^\times$ (see below). Deligne's conjecture states that

$$L(M, n)/(1 \otimes 2\pi i)^{d^\pm(M)n} c^\pm(M) \in E$$

for critical values $L(M, n)$. Here $\pm 1 = (-1)^n$.

§2. Fundamental periods of a motive

It is important to know the change of Deligne's periods for various algebraic operations for motives; for example $M \otimes N$, the tensor product of two motives. For this purpose, we are going to introduce fundamental periods.

Let $H_{DR}(M)$ be the de Rham realization of M which is a d-dimensional vector space over E. There exists a spectral sequence

$$E_1^{p,q} = H^q(M, \Omega^p) \Longrightarrow H_{\mathrm{DR}}^{p+q}(M). \tag{2.1}$$

The Hodge filtration $\{F^p\}$ is the filtration on $H^{p+q}_{\mathrm{DR}}(M)$ given by this spectral sequence. Since $H^w_{\mathrm{DR}}(M) = H_{\mathrm{DR}}(M)$, we have

$$F^p(H^w_{\mathrm{DR}}(M))/F^{p+1}(H^w_{\mathrm{DR}}(M)) \cong E_\infty^{p,w-p}, \qquad p \in \mathbf{Z}.$$

Since this spectral sequence degenerates at E_1-terms, we have

$$(2.2) \qquad F^p(H_{\mathrm{DR}}(M))/F^{p+1}(H_{\mathrm{DR}}(M)) \cong E_1^{p,w-p} = H^{w-p}(M,\Omega^p), \qquad p \in \mathbf{Z}.$$

We write $F^p(H_{\mathrm{DR}}(M))$ as $F^p(M)$ or simply as F^p. Let

$$I : H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{\mathbf{Q}} \mathbf{C}$$

be the comparison isomorphism. We have

$$(2.3) \qquad I(\oplus_{p' \geqq p} H^{p' q}(M)) = F^p(M) \otimes_{\mathbf{Q}} \mathbf{C}.$$

Now we are going to define a period matrix of M. Let $\{v_1^+, v_2^+, \ldots, v_{d^+}^+\}$ (resp. $\{v_1^-, v_2^-, \ldots, v_{d^-}^-\}$) be a basis of $H_B^+(M)$ (resp. $H_B^-(M)$) over E. We write the Hodge filtration as

$$(2.4) \qquad H_{DR}(M) = F^{i_1} \supsetneqq F^{i_2} \supsetneqq \cdots \supsetneqq F^{i_m} \supsetneqq F^{i_{m+1}} = \{0\}$$

so that there are no different filtrations between successive members. The choice of numbers i_μ may not be unique for F^{i_μ}. For the sake of simplicity, we assume that i_μ is chosen, for $1 \leqq \mu \leqq m$, so that it is the maximum number. Put

$$s_\mu = \mathrm{rank}(H^{i_\mu, w-i_\mu}(M)), \qquad 1 \leqq \mu \leqq m$$

where rank means the rank as a free R-module. Then we have

$$(2.5) \qquad i_c + i_{m+1-c} = w, \quad 1 \leqq c \leqq m, \qquad s_\mu = s_{m+1-\mu}, \quad 1 \leqq \mu \leqq m.$$

We also have a partion of d:

$$(2.6) \qquad d = s_1 + s_2 + \cdots + s_d, \qquad s_\mu > 0, \quad 1 \leqq \mu \leqq m.$$

By (2.3), we have

$$s_\mu = \dim_E F^{i_\mu} - \dim_E F^{i_{\mu+1}}, \qquad \dim_E F^{i_\mu} = s_\mu + s_{\mu+1} + \cdots + s_m, \quad 1 \leqq \mu \leqq m.$$

We take a basis $\{w_1, w_2, \ldots, w_d\}$ of $H_{DR}(M)$ over E so that $\{w_{s_1+s_2+\ldots+s_{\mu-1}+1}, \ldots, w_d\}$ is a basis of F^{i_μ} for $1 \leqq \mu \leqq m$. Writing

$$(2.7) \qquad I(v_j^\pm) = \sum_{i=1}^d x_{ij}^\pm w_i, \qquad x_{ij}^\pm \in R, \quad 1 \leqq j \leqq d^\pm,$$

we obtain a matrix $X^{\pm} = (x_{ij}^{\pm}) \in M(d, d^{\pm}, R)$. Let P_M be the lower parabolic subgroup of $GL(d)$ which corresponds to the partition (2.6). Then the coset of $X^{\pm}$ in

$$P_M(E)\backslash M(d, d^{\pm}, R)/GL(d^{\pm}, E)$$

does not depend on the choices of basis. We put $X = (X^+\, X^-) \in M(d, d, R)$ and call it *a period matrix of* M. The coset of X in

$$P_M(E)\backslash M(d, d, R)/(GL(d^+, E) \times GL(d^-, E))$$

is well defined. Here $GL(d^+, E) \times GL(d^-, E)$ is embedded in $GL(d)$ as diagonal blocks.

Thus we are interested in a polynomial function on $M(d, d)$ rational over $\mathbf{Q}$, which satisfies

$$(*) \qquad f(px\gamma) = \lambda_1(p)\lambda_2(\gamma)f(x) \qquad \text{for all } \ p \in P_M,\ \gamma \in GL(d^+) \times GL(d^-).$$

Here λ_1 and λ_2 are characters of P_M and $GL(d^+) \times GL(d^-)$ respectively given by

$$\lambda_1\left(\begin{pmatrix} p_{11} & 0 & \dots & 0 \\ * & p_{22} & \dots & 0 \\ * & * & \ddots & \vdots \\ * & * & * & p_{mm} \end{pmatrix}\right) = \det(p_{11})^{a_1}\det(p_{22})^{a_2}\cdots\det(p_{mm})^{a_m},$$

where $p_{ii} \in GL(s_i)$,

$$\lambda_2\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = (\det a)^{k^+}(\det b)^{k^-}, \qquad a \in GL(d^+),\ b \in GL(d^-).$$

We call f to be *of the type* (λ_1, λ_2) or *of the type* $\{(a_1, a_2, \dots, a_m); (k^+, k^-)\}$. All of such an f generate a graded algebra over $\mathbf{Q}$.

Theorem 1. *The graded algebra of all f satisfying $(*)$ is isomorphic to a polynomial ring with explicitly given generators. Each graded piece is at most one dimensional.*

Let $f(x) = \det(x)$, $x \in M(d, d)$. Then $f(x)$ is of the type $\{(1, 1, \dots, 1); (1, 1)\}$ and $f(X)$ is Deligne's period $\delta(M)$. Assume that $s_1 + s_2 + \cdots + s_{p^+} = d^+$ for some p^+. Let $f^+(x)$ be the determinant of the upper left $d^+ \times d^+$-submatrix of $x \in M(d, d)$. Then $f^+(x)$ is of the type $\{(\overbrace{1, 1, \dots, 1}^{p^+}, 0, \dots, 0); (1, 0)\}$ and $f^+(X)$ is Deligne's period $c^+(M)$. Similarly if $s_1 + s_2 + \cdots + s_{p^-} = d^-$ for some p^-, let $f^-(x)$ be the determinant of the upper right $d^- \times d^-$-submatrix of x. Then $f^-(x)$ is of the type $\{(\overbrace{1, 1, \dots, 1}^{p^-}, 0, \dots, 0); (0, 1)\}$ and $f^-(X)$ is Deligne's period $c^-(M)$. Either one of the above conditions is equivalent to that $F^{\mp}(M)$, hence also $c^{\pm}(M)$ can be defined (cf. [D], §1, [Y1], §2). We have $F^{\mp}(M) = F^{i_{p^{\pm}+1}}(M)$; $F^{\pm}(M)$ can be defined if M has a critical value.

Let $\mathcal{P} = \mathcal{P}(M)$ denote the set of integers p such that $s_1 + s_2 + \cdots + s_p < \min(d^+, d^-)$. Put $q = m - p$. Then $p < q$ and $s_1 + s_2 + \cdots + s_q = d - (s_1 + \cdots + s_p)$ by (2.5).

Theorem 2. *For every $p \in \mathcal{P}$, there exists a non-zero polynomial f_p of the type* $\{(\overbrace{2,\dots,2}^{p},\overbrace{1,\dots,1}^{m-2p},\overbrace{0,\dots,0}^{p});(1,1)\}$. *Every polynomial satisfying $(*)$ can be written uniquely as a monomial of* $\det(x)$, *$f^+(x)$, $f^-(x)$, $f_p(x)$, $p \in \mathcal{P}$.*

We put $c_p(M) = f_p(X)$. We call $\delta(M)$, $c^\pm(M)$, $c_p(M)$, $p \in \mathcal{P}$ *the fundamental periods* of M. By Theorem 2, any period invariant of M can be written as a monomial of the fundamental periods.

Deligne's conjecture states that $L(M,0)/c^+(M) \in E$ if 0 is critical for M. Other period invariants are hidden for the relation with $L(M,0)$ but will manifest themselves when we make various algebraic operations on M (tensor products with other motives, exterior powers, etc.). Deligne showed $c^\pm(M) \in R^\times$. We can prove that other period invariants are also invertible elements of R. Hereafter we understand the equality between period invariants $\bmod E^\times$.

Let us explain a general principle concerning the variation of fundamental periods of motives under algebraic operations, taking the case $M = M_1 \otimes M_2 \otimes \cdots \otimes M_n$ as an example. Here all motives are defined over $\mathbf{Q}$, and with coefficients in E. Let X_i be a period matrix of M_i for $1 \leqq i \leqq n$ and X be a period matrix of M. We see that:

(i) Every entries of X can be written as a polynomial of entries of X_i with coefficients in $\mathbf{Q}$.

(ii) If we replace X_i by a matrix in $P_{M_i}(E)X_i(GL(d^+(M_i),E) \times GL(d^-(M_i),E))$, then X is replaced by a matrix in $P_M(E)X(GL(d^+(M),E) \times GL(d^-(M),E))$.

We see that (i) and (ii) hold even when we regard X_i as a variable matrix. Now let $p \in \mathcal{P}(M)$ and let $c_p(M) = f_p(X)$ be a fundamental period of M, where f_p is a polynomial on $M(d(M),d(M))$. By (i), we have $c_p(M) = f_p(g(X_1,\dots,X_n))$ with a polynomial g. We regard X_i as a variable matrix on $M(d(M_i),d(M_i))$. By (ii), we see that $f_p(g(X_1,\dots,X_n))$ is, as a polynomial of X_i, a polynomial f_i of some admissible type. By the one dimensionality (Theorem 1), we conclude that $c_p(M) = cf_1(X_1)\cdots f_n(X_n)$ with $c \in \mathbf{Q}^\times$. The same argument can be used also for $c^\pm(M)$, $\delta(M)$. Therefore every fundamental period of M is a monomial of fundamental periods of M_i, $1 \leqq i \leqq n$.

§3. Siegel modular forms

We are interested in the nature of the critical values of L-functions attached to Siegel modular forms and related period invariants.

Let Γ be a congruence subgroup of $Sp(m,\mathbf{Z})$. Let $S_k^{(m)}(\Gamma)$ denote the space of Siegel modular cusp forms of weight k with respect to Γ. We normalize the Petersson norm so that

$$\langle f,f\rangle = \mathrm{vol}(\Gamma\backslash\mathfrak{H}_m)^{-1}\int_{\Gamma\backslash\mathfrak{H}_m}|f(z)|^2(\det y)^{k-m-1}dxdy,$$

where $\mathfrak{H}_m$ denotes the Siegel upper half space of degree m and $z = x+iy$ with real symmetric matrices x and y. We take a non-zero Hecke eigenform $f \in S_k^{(m)}(\Gamma)$. Then we can define two L-functions $L_{\mathrm{st}}(s,f)$ and $L_{\mathrm{sp}}(s,f)$ called standard and spinor respectively.

For simplicity, we assume $\Gamma = Sp(m, \mathbf{Z})$. Put $w = mk - m(m+1)/2$. For a prime number p, let $\alpha_0, \alpha_1, \ldots, \alpha_m$ be the Satake parameters attached to the eigenform f. Then $\alpha_0^2\alpha_1 \cdots \alpha_m = p^w$. The Euler p-factors of $L_{\mathrm{st}}(s, f)$ and $L_{\mathrm{sp}}(s, f)$ are

$$[(1-p^{-s})\prod_{i=1}^{m}(1-\alpha_i p^{-s})(1-\alpha_i^{-1}p^{-s})]^{-1} \quad \text{and}$$

$$[[(1-\alpha_0 p^{-s})\prod_{r=1}^{m}\prod_{1 \leqq i_1 < \ldots < i_r \leqq m}(1-\alpha_0\alpha_{i_1}\cdots\alpha_{i_r}p^{-s})]^{-1}$$

respectively. We assume that the Fourier coefficients of f are contained in a totally real algebraic number field E. We assume the existence of motives $M_{\mathrm{st}}(f)$ and $M_{\mathrm{sp}}(f)$, which are defined over $\mathbf{Q}$ with coefficients in E and which satisfy

$$L(M_{\mathrm{st}}(f), s) = (L_{\mathrm{st}}(s, f^{\sigma}))_{\sigma \in J_E},$$

$$L(M_{\mathrm{sp}}(f), s) = (L_{\mathrm{sp}}(s, f^{\sigma}))_{\sigma \in J_E}.$$

Considering the degrees of the Euler products, we have

$$\operatorname{rank} M_{\mathrm{st}}(f) = 2m+1, \qquad \operatorname{rank} M_{\mathrm{st}}(f) = 2^m.$$

Conjecture. *If one of two motives $M_{\mathrm{st}}(f)$ and $M_{\mathrm{sp}}(f)$ is not of pure weight, then the associated automorphic representation to f is not tempered. Furthermore f can be obtained as a lifting from lower degree forms.*

Hereafter we assume that $M_{\mathrm{st}}(f)$ and $M_{\mathrm{sp}}(f)$ are of pure weight, expecting that the problems can be reduced to lower degree cases otherwise. (This is in fact the case if $m = 2$.) Then the functional equations of the L-functions ([A], [Boc2]; still conjectural for $L_{\mathrm{sp}}(s, f)$ for $m > 2$) forces the Hodge types of motives. We have to assume that

$$\wedge^{2m+1} M_{\mathrm{st}}(f) \cong T(0),$$

$$\begin{aligned} H_B(M_{\mathrm{st}}(f)) \otimes_{\mathbf{Q}} \mathbf{C} =& H^{0,0}(M_{\mathrm{st}}(f)) \\ & \oplus_{i=1}^{m}(H^{-k+i,k-i}(M_{\mathrm{st}}(f)) \oplus H^{k-i,-k+i}(M_{\mathrm{st}}(f))). \end{aligned}$$

We further assume that F_∞ acts on $H^{0,0}(M_{\mathrm{st}}(f))$ (the first factor of the right hand side) by $(-1)^m$. For $M_{\mathrm{sp}}(f)$, we assume:

$$\wedge^{2^m} M_{\mathrm{sp}}(f) \cong T(2^{m-1}(mk - \frac{m(m+1)}{2})),$$

$$H_B(M_{\mathrm{sp}}(f)) \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_{p,q} H^{p,q}(M_{\mathrm{sp}}(f)),$$

where (p, q) extends over all pairs such that

$$\begin{aligned} &p = (k-i_1) + (k-i_2) + \cdots + (k-i_r), \quad q = (k-j_1) + (k-j_2) + \cdots + (k-j_s), \\ &r + s = m, \qquad 1 \leqq i_1 < \ldots < i_r \leqq m, \qquad 1 \leqq j_1 < \ldots < j_s \leqq m, \\ &\{i_1, \ldots, i_r\} \cup \{j_1, \ldots, j_s\} = \{1, 2, \ldots, m\}, \end{aligned}$$

including the cases $r = 0$ or $s = 0$. We further assume that if $w = mk - m(m+1)/2$ is even, then the eigenvalues $+1$ and -1 of F_∞ on $H^{pp}(M_{\mathrm{sp}}(f))$ occur with the equal multiplicities.

Proposition 3. *Assume Deligne's conjecture. If $k > 2m$, then we have*

$$c^{\pm}(M_{\mathrm{st}}(f) = \pi^{mk}(\langle f^{\sigma}, f^{\sigma}\rangle)_{\sigma\in J_E}.$$

This proposition can be proved by comparing critical values $L_{\mathrm{st}}(n, f)$ predicted by Deligne's conjecture with results of Böcherer [Boc1], Mizumoto [M] and Shimura [Sh5].

We are interested how many independent periods exist for $M_{\mathrm{st}}(f)$ and for $M_{\mathrm{sp}}(f)$. Let $J_E = \{\sigma_1, \sigma_2, \dots, \sigma_l\}$, $l = [E : \mathbf{Q}]$ and write $x \in R \cong \mathbf{C}^{J_E}$ as $x = (x^{(1)}, x^{(2)}, \dots, x^{(l)})$, $x^{(i)} \in \mathbf{C}$ so that $x^{(i)} = x^{\sigma_i}$ for $x \in E$.

Theorem 4. *Let the notation be the same as above. We assume that two motives over $\mathbf{Q}$ having the same L-function are isomorphic (Tate's conjecture). Then there exist $p_1, p_2, \dots, p_r \in \mathbf{C}^{\times}$, $1 \leqq r \leqq m+1$ such that for any fundamental period $c \in R^{\times}$ of $M_{\mathrm{st}}(f)$ or $M_{\mathrm{sp}}(f)$, we have $c^{(1)} = \alpha\pi^{A}p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ with $\alpha \in \overline{\mathbf{Q}}^{\times}$ and non-negative integers A, a_i, $1 \leqq i \leqq r$.*

We have the relations of the L-functions:

$$L(M_{\mathrm{sp}}(f) \otimes M_{\mathrm{sp}}(f), s) = \prod_{j=0}^{m} L(\wedge^j M_{\mathrm{st}}(f), s - w), \tag{3.1}$$

$$L(\wedge^2 M_{\mathrm{sp}}(f), s) = \prod_{j=0}^{m-1} L(\wedge^j M_{\mathrm{st}}(f), s - w)^{\epsilon_j}, \tag{3.2}$$

$$L(\mathrm{Sym}^2 M_{\mathrm{sp}}(f), s) = \prod_{j=0}^{m} L(\wedge^j M_{\mathrm{st}}(f), s - w)^{\delta_j}, \tag{3.3}$$

where $\epsilon_j = 1$ if $j \equiv m-1$ or $m-2 \mod 4$ and $\epsilon_j = 0$ otherwise, $\delta_m = 1$, $\epsilon_j + \delta_j = 1$ for $0 \leqq j \leqq m-1$. By the method of §2, we can compare the fundamental periods of $M_{\mathrm{sp}}(f)$ with those of $M_{\mathrm{st}}(f)$ using (3.2) and (3.3). In this way, Theorem 4 can be proved.

By this theory, we can get some more insight for the zeta functions of symplectic Shimura varieties. Let us describe the result in the simplest case. It is generally believed, but proved very little when $m > 2$, that the zeta function of $\Gamma\backslash\mathfrak{H}_m$ can be written using the spinor L-functions of automorphic forms:

$$\zeta(s, \Gamma\backslash\mathfrak{H}_m) \doteqdot \prod_f L_{\mathrm{sp}}(s, f).$$

Put $M = M_{\mathrm{sp}}(f)$. The weight of M is w and the deepest filtration of $H_{\mathrm{DR}}(M)$ is F^w, which is one dimensional. By (2.2), we have

$$F^w(H_{\mathrm{DR}}(M)) \cong H^0(M, \Omega^w).$$

Let

$$\langle\ ,\ \rangle : M \otimes M \longrightarrow U$$

be a polarization of M, where U denotes a motive of rank 1. Take $0 \neq \omega \in F^w(H_{\mathrm{DR}}(M))$. Then we can show that

$$\langle \omega, F_\infty\omega\rangle = c_1(M)\delta(M)^{-1}.$$

The left hand side can be interpreted as the norm of the differential form ω. We can prove the following proposition which is consistent with this picture.

Proposition 5. *With the same assumptions as in Theorem 4, we have*

$$(c_1(M)\delta(M)^{-1})^2 = (\pi^{mk}\langle f^\sigma, f^\sigma\rangle)^2_{\sigma\in J_E}$$

when $k > 2m$.

Remark. It is possible to refine Theorem 4 replacing mod $\overline{\mathbf{Q}}^\times$ by mod $E^\times$. In Proposition 5, it is probable that we can drop the square factor from both sides.

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