On Ikeda's lifting theorem

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1. Introduction

In 1999, T. Ikeda [2] gave a proof of a conjecture of Duke-Imamoglu, according to which to each normalized Hecke eigenform f of even integral weight $2k$ on $SL_2(\mathbf{Z})$ there is associated a Hecke eigenform F of weight $k+n$ of genus $2n$ (where n is any natural number with $n \equiv k \pmod 2$) such that the standard zeta function of F is $\zeta(s)$ times the product of shifted Hecke L-functions of f. If $n=1$, this lift is equal to the Saito-Kurokawa lift.

In fact, Ikeda gives a formula for the Fourier coefficients of F which is highly non-linear and in which squarefree Fourier coefficients of half-integral weight modular forms and values of local singular series polynomials enter.

In this note we shall give an explicit linear version of Ikeda's lifting theorem as a linear map from half-integral weight modular forms to Siegel modular forms where no singular series polynomials occur anymore. When specialized to $n=1$ (i.e. genus 2), one gets back (as one can show) a classical formula due to Eichler-Zagier for the Maass lift as a linear map on half-integral weight modular forms [1, p.74].

More details and proofs can be found elsewhere. Below, we use standard notation.

2. Background

Conjecture *(Duke-Imamoglu, 1996). Let $f \in S_{2k}$ be a normalized Hecke eigenform with L-function $L(f,s)$. Let $n \in \mathbf{N}$ with $n \equiv k \pmod 2$. Then there is a Hecke eigenform $F \in S_{k+n}(\Gamma_{2n})$ (where $\Gamma_{2n} := Sp_{2n}(\mathbf{Z})$) such that*

$$L_{st}(F;s) = \zeta(s) \prod_{j=1}^{2n} L(f, s+k+n-j).$$

Theorem *(Ikeda [2]). The conjecture is true. Moreover, the Fourier coefficients $A(T)$ ($T>0$ half-integral) of F are given by*

$$A(T) = c(|D_{T,0}|) f_T^{k-1/2} \prod_{p|D_T} \tilde{F}(T;\alpha_p)$$

where the notation is as follows:

– $D_T = (-1)^n \det(2T)$ is the discriminant of T ($\equiv 0,1 \pmod 4$), $D_T = D_{T,0} f_T^2$ with $D_{T,0}$ a fundamental discriminant and $f_T \in \mathbf{N}$;

– $c(|D_{T,0}|) = |D_{T,0}|$-th Fourier coefficient of a Hecke eigenform $g \in S^+_{k+1/2}$ corresponding to f under the Shimura correspondence;

– $F_p(T;p^{-s}) := \frac{b_p(T;s)}{\gamma_p(T;p^{-s})}$ where $b_p(T;s)$ is the local p-singular series polynomial in p^{-s} attached to T, $\gamma_p(T;X) := (1-X)(1-(\frac{D_{T,0}}{p})p^nX)^{-1}\prod_{j=1}^n(1-p^{2j}X^2)$; then $F_p(T;p^{-s})$ is a polynomial in p^{-s}, and we put $\tilde{F}_p(T;X) := X^{-v_p(f_T)}F_p(T;p^{-n-1/2}X)$ where v_p denotes the usual p-adic order (note that $\tilde{F}_p(T;X)$ is a Laurent polynomial which by work of Katsurada [3] is invariant w.r.t. $X \mapsto X^{-1}$);

– $a(p)$ is the p-th Fourier coefficient of f,

$$1 - a(p)X + p^{2k-1}X^2 = (1-p^{k-1/2}\alpha_pX)(1-p^{k-1/2}\alpha_p^{-1}X).$$

2. Results

Theorem. *Let g be an arbitrary element in $S^+_{k+1/2}$ with Fourier coefficients $c(m)$ ($m \equiv 0,1 \pmod 4$) and for $T > 0$ half-integral define*

$$A_g(T) := \sum_{a|f_T} a^{k-1/2}\Big(\sum_{d^2|a}\ \sum_{G\in\mathcal{D}(T),|\det G|=d}\rho_{T[G^{-1}]}(a/d^2)\Big)\,c(|D_{T,0}|f_T/a^2)$$

where the notation (roughly speaking) is as follows:

– $\mathcal{D}(T) := GL_{2n}(\mathbf{Z})\backslash\{G \in M_{2n}(\mathbf{Z}) \cap GL_{2n}(\mathbf{Q}) \mid T[G^{-1}]$ half-integral$\}$ (this is a finite set);

– $\rho_{T[G^{-1}]}(t)$ $(t \in \mathbf{N}, G \in \mathcal{D}(T))$ is a certain multiplicative function such that
i) $\rho_{T[G^{-1}]}(t)t^{k-1/2}$ is an integer,
ii) $\rho_{T[G^{-1}]}(p^\nu)$ only depends on some simple local p-invariants of $T[G^{-1}]$ and essentially is equal to an elementary symmetric function in powers of p.

Then for g a Hecke eigenform the Fourier coefficients of the Ikeda lift F are given by $A_g(T)$.

Corollary. *The map*

$$g \mapsto \sum_{T>0} A_g(T)e^{2\pi i \operatorname{tr}(TZ)} \qquad (Z \in \mathcal{H}_{2n})$$

is a linear map from $S^+_{k+1/2}$ to $S_{k+n}(\Gamma_{2n})$ which on Hecke eigenforms coincides with the Ikeda lift.

Remark. One can show (not obvious): if $n = 1$, then

$$A_g(T) = \sum_{a|\operatorname{cont} T} a^k c(|D_T|/a^2)$$

where cont T is the content of T (greatest common divisor of the coefficients of T). This is the formula of Eichler-Zagier quoted above.

The idea of proof of the Theorem can be described shortly as follows. One combines the following three ingredients:

i) A formula of Böcherer-Kitaoka which expresses the product $\prod_{p|D_T} \tilde{F}_p(T; X_p)$ as a finite sum over $G \in \mathcal{D}(T)$ of "simple" polynomials (however, with no obvious and compatible functional equations);

ii) Katsurada's functional equation (see above) plus a certain symmetrization argument;

iii) the multiplicative structure of the Fourier coefficients $c(|D_0|m^2)$ (D_0 fixed) of a Hecke eigenform $g \in S^+_{k+1/2}$.

We remark further that with the same method one can also obtain a closed formula for the positive definite Fourier coefficients of the Siegel-Eisenstein series of genus $2n$, in terms of the generalized Cohen class number formula (specialized to the case $n = 1$, one again gets back an identity given in [1]). This formula is quite different from the formulas given in [3].

References

[1] M. Eichler and D. Zagier: The theory of Jacobi forms. Progress in Math., vol. 55, Birkhäuser, Boston 1985

[2] T. Ikeda: On the lifting of elliptic modular forms to Siegel cusp forms of degree $2n$. Preprint 1999

[3] H. Katsurada: An explicit form for Siegel series. Amer. J. Math. 121 (1999), 415-452

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