Estimation of a generalized amplitude damping channel

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Abstract

This paper addresses the problem of finding the optimal strategy for estimating a generalized amplitude damping channel $\Gamma_\eta^{(p)}$ by means of the extension $\mathrm{id} \otimes \Gamma_\eta^{(p)}$. We first evaluate the quantum Fisher information of output states based on the symmetric logarithmic derivative, and specify all the pure state inputs that maximize the quantum Fisher information. We next investigate the ∇^e-autoparallelity of output state manifolds, and characterize the condition for the existence of an efficient estimator. A comparison of these results concludes that, while there is no uniformly optimal input for all p and η, a maximally entangled input is an admissible, relatively optimal one under a non-asymptotic setting.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space that represents the physical system of interest and let $\mathcal{S}(\mathcal{H})$ be the set of density operators on $\mathcal{H}$. It is well known that a dynamical change $\Gamma : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ of the physical system, called a *quantum channel*, is represented by a trace preserving completely positive linear map [1, 2, 3]. Nevertheless, it is a different matter how one can identify the quantum channel that one has in a laboratory. Since almost every quantum protocol assumes a priori knowledge of the behavior of the quantum channel under consideration, there is no doubt that identifying the channel is of fundamental importance in quantum information theory. It is, however, not very long since the quantum channel identification problem was directed proper attention, and the theory of finding an optimal estimation scheme has not been developed so far, with only a few exceptions [4, 5, 6, 7, 8, 9]. The purpose of this paper is to investigate the optimality of an estimation scheme for a (generalized) amplitude damping channel of a two level quantum system, based on noncommutative parameter estimation theory [10, 11] and quantum information geometry [12].

An *amplitude damping* channel $\Gamma_\eta : \mathcal{S}(\mathbb{C}^2) \to \mathcal{S}(\mathbb{C}^2)$, having a one dimensional parameter $\eta \in [0,1]$, is defined by [13, p. 380]

$$\Gamma_\eta(\sigma) = \sum_{i=1}^{2} E_i \, \sigma \, E_i^*,$$

where

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\eta} \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & \sqrt{1-\eta} \\ 0 & 0 \end{bmatrix}.$$

The channel describes the physical process of approach to equilibrium due to coupling with its environment, and the damping parameter η respresents the rate of dissipation, that is, $\eta = e^{-t/T}$ where t is time and T a constant characterizing the speed of the process. This channel induces the following affine map on the Stokes parameter space:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \sqrt{\eta} & & \\ & \sqrt{\eta} & \\ & & \eta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1-\eta \end{bmatrix}.$$

As $\eta \to 0$, the channel transforms every point in the unit ball towards a fixed point at the north pole, the ground state. Thus the environment is regarded as if it were at zero temperature in this model.

A *generalized amplitude damping* channel $\Gamma_\eta^{(p)} : \mathcal{S}(\mathbb{C}^2) \to \mathcal{S}(\mathbb{C}^2)$ describes the effect of dissipation to an environment at finite temperature [13, p. 382]. It is defined by

$$\Gamma_\eta^{(p)}(\sigma) = \sum_{i=1}^{4} E_i \, \sigma \, E_i^*$$

where

$$E_1 = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\eta} \end{bmatrix}, \quad E_2 = \sqrt{p} \begin{bmatrix} 0 & \sqrt{1-\eta} \\ 0 & 0 \end{bmatrix},$$

$$E_3 = \sqrt{1-p}\begin{bmatrix} \sqrt{\eta} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_4 = \sqrt{1-p}\begin{bmatrix} 0 & 0 \\ \sqrt{1-\eta} & 0 \end{bmatrix},$$

and $p \in [0,1]$ is a parameter that represents the temperature of the environment. This channel induces the following affine map on the Stokes parameter space:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \sqrt{\eta} & & \\ & \sqrt{\eta} & \\ & & \eta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ (2p-1)(1-\eta) \end{bmatrix},$$

and the stationary state is

$$\sigma_\infty = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}.$$

When $p = 0$ or 1, the channel is reduced to an amplitude damping channel, and $\Gamma_\eta^{(p)}$ for a generic p is regarded as their mixture:

$$\Gamma_\eta^{(p)} = p\,\Gamma_\eta^{(1)} + (1-p)\,\Gamma_\eta^{(0)}.$$

Since p is an indicator of the temperatute of the environment, it is likely that one can evaluate the true value of p beforehand, independent of the channel. Therefore, the estimation problem of the single parameter η, given $p \in [0,1]$, would be physically feasible. In this paper, we study the optimality of an estimation scheme for the damping parameter η of a generalized amplitude damping channel $\Gamma_\eta^{(p)}$ by means of the extension $\mathrm{id} \otimes \Gamma_\eta^{(p)} : \mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2) \to \mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

Once an input state $\sigma \in \mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is fixed, we have a one dimensional family of output states $\rho_\eta := \mathrm{id} \otimes \Gamma_\eta^{(p)}(\sigma)$, and the parameter estimation for the quantum channel $\mathrm{id} \otimes \Gamma_\eta^{(p)}$ is reduced to that for the quantum state ρ_η. As a consequence, the problem amounts to finding an optimal input state for $\mathrm{id} \otimes \Gamma_\eta^{(p)}$ and an optimal estimator for the corresponding output state. Note that as long as we are concerned only with "local" (or asymptotic) optimality of an estimation scheme, the problem is further reduced to finding an input state that maximizes the symmetric logarithmic derivative (SLD) Fisher information of the output family $\{\rho_\eta\}_\eta$ as shown in [4]. If, however, we switch the subject to "global" optimality under a non-asymptotic setting, such an approach might fail because the locally optimal estimation scheme would, in general, depend on the true value of the parameter η. In this case, we are forced to consider an alternative criterion in discussing the optimality of an estimation scheme. We will show that this is the case with a generalized amplitude damping channel, that is, there is no uniformly optimal input that maximizes the SLD Fisher information for all η unless $p = 0, 1$, or $1/2$. This is in good contrast to the estimation of an $SU(2)$ channel [6] or a generalized Pauli channel [8], in which a maximally entangled input is uniformly optimal in that it simultaneously maximizes the SLD Fisher information for all values of the parameters.

The paper is organized as follows. The main results are stated in Section 2, and are proved in Section 3. The admissibility of an estimation scheme under a non-asymptotic setting is discussed in Section 4. For the reader's convenience, we give a brief account of quantum information geometry in Appendix A.

2 Main results

According to the general prescription presented in [4], let us specify the input state for the channel $\mathrm{id}\otimes\Gamma_\eta^{(p)}$ that maximizes the SLD Fisher information of the output states. Due to the convexity of the SLD Fisher information [4], we can take the input to be a pure state $\sigma = |\psi\rangle\langle\psi|$. Further, due to the covariancy of the channel in the rotation around the z-axis of the Stokes parameter space, we can assume, without loss of generality, that the optimal input takes the form

$$\psi = \psi(\alpha,\phi) := \sqrt{1-\alpha}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\otimes\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} + \sqrt{\alpha}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\otimes\begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}, \tag{1}$$

where $0\le\alpha\le 1$ and $0\le\phi<\pi$. Let us denote the corresponding output state by

$$\rho_\eta = \rho_\eta(\alpha,\phi) := \mathrm{id}\otimes\Gamma_\eta^{(p)}\left(|\psi(\alpha,\phi)\rangle\langle\psi(\alpha,\phi)|\right),$$

and let $J_\eta^{(p)}(\alpha,\phi)$ be the SLD Fisher information about the parameter η. The main results are stated in a series of theorems as follows.

Theorem 1. *(Amplitude damping channel) When $p=1$, the SLD Fisher information takes the maximum if and only if either $(\alpha,\phi)=(0,\pi/2)$ or $(1,0)$. When $p=0$, it takes the maximum if and only if either $(\alpha,\phi)=(0,0)$ or $(1,\pi/2)$. In each optimal case, the output family admits an efficient estimator for the parameter η.*

Theorem 1 has an interesting physical interpretation: the existence of quantum entanglement (i.e., $\alpha\neq 0,1$) strictly deteriorates the accuracy of estimation for an amplitude damping channel. This is in remarkable contrast to the estimation problem of a unitary channel [6] or a Pauli channel [8] in which quantum entanglement actually enhances the accuracy of estimation.

Theorem 2. *(Submodel of Pauli channel) When $p=1/2$, the SLD Fisher information takes the maximum if and only if $\alpha=1/2$. The corresponding output family admits an efficient estimator for the parameter $\sqrt{\eta}$.*

Theorem 2 is a direct consequence of the fact that, when $p=1/2$, the channel forms a ∇^e-autoparallel submodel of a Pauli channel [8].

For other values of p, the optimal input depends on the true value of the parameter η. Let us introduce the function

$$\beta(\eta\,;r) = \begin{cases} \dfrac{r\,(1-\eta)-1+\sqrt{r\,(1-r)}\,(1+\eta)}{(2\,r-1)\,(1-\eta)}, & 0\le\eta\le\eta_r \\ 0, & \eta_r<\eta\le 1 \end{cases}$$

where $1/2<r<1$ and $\eta_r := \sqrt{(1-r)/r}$, see Figure 1. Then we have

Theorem 3. *(Generalized amplitude damping channel) When $1/2<p<1$, the SLD Fisher information takes the maximum if and only if $(\alpha,\phi)=(\beta(\eta;\,p),\pi/2)$ or $(1-\beta(\eta;\,p),0)$. When $0<p<1/2$, it takes the maximum if and only if $(\alpha,\phi)=(\beta(\eta;\,1-p),0)$ or $(1-\beta(\eta;\,1-p),\pi/2)$.*

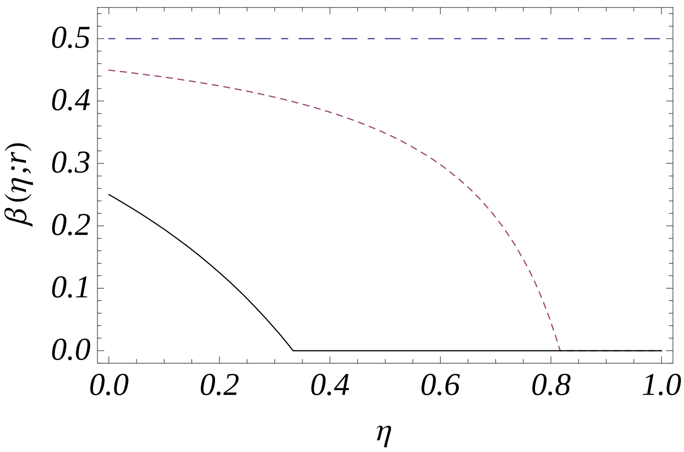

Figure 1: The function $\beta(\eta;\, r)$ that characterizes the optimal degree α of entanglenent. Here the parameter r is taken to be $r = 0.9$ (solid), $r = 0.6$ (dashed), and the limit $r \to 0.5$ (chained).

Theorem 3 implies that, as long as one wishes to achieve the lower bound of the optimal quantum Cramér-Rao inequality [4]

$$V_\eta\,[T] \geq \left(\max_{\alpha,\phi}\; J_\eta^{(p)}(\alpha,\phi) \right)^{-1}$$

for the variance $V_\eta\,[T] := E_\eta\,[(T - \eta I)^2]$ of a locally unbiased estimator T, the estimation scheme for the parameter η is inevitably an adaptive one: one needs to modify successively the degree α of entanglement of the input state according to the temporary estimate of η.

In summary, there is no uniformly optimal input that maximizes the SLD Fisher information for all η unless $p = 0$, 1, or 1/2. In this regard, the next theorem, characterizing the condition for the output family to admit an efficient estimator for all p, would be meaningful in discussing the admissibility of an estimation scheme under a non-asymptotic setting.

Theorem 4. *(Autoparallelity) The family $\{\rho_\eta(\alpha,\phi)\}_{0<\eta<1}$ of output states is ∇^e-autoparallel for all $p \in [0,1]$ if and only if either $\alpha = 1/2$, or $(\alpha,\phi) = (0,0), (0,\pi/2), (1,0), (1,\pi/2)$.*

Note that the output family $\rho_\eta(\alpha,\phi)$ degenerates to a point when $(\alpha,\phi) = (0,0), (1,\pi/2)$ and $p = 1$, or when $(\alpha,\phi) = (0,\pi/2), (1,0)$ and $p = 0$.

3 Proof of Theorems

In this section, we change the variables (p,α,ϕ) into $(q,\delta,c,s) := (2p-1,\ 2\alpha-1,\ \cos 2\phi,\ \sin 2\phi)$. The variable s appears only in Section 3.4. Details of the tedious computation are relegated to the Appendix B. When $|q| < 1$ and $|\delta| < 1$, the SLD of the output family ρ_η is unique, and the SLD Fisher information is given by

$$J_\eta^{(q)}(\delta, c) = \frac{h(\delta, c\,;\,\eta, q)}{4\,\eta\,(1-\eta)\,\left[1 - q^2\,\delta^2 + (1 + 2\,c\,q\,\delta + (c^2 + q^2 - 1)\,\delta^2)\,\eta\right]} \tag{2}$$

where

$$\begin{aligned} h(\delta, c\,;\,\eta, q) = {} & 2 - c^2\,\left(1 - q^2\right)\,\delta^2 - \left(1 + q^2\right)\,\delta^2 + 2\,c\,q\,\delta\,\left(1 - \delta^2\right) \\ & + \left[2 + 6\,c\,q\,\delta + \left(-3 + 2\,q^2\right)\,\delta^2 + 2\,c^3\,q\,\delta^3 + \delta^4 + c^2\,\delta^2\,\left(5 + 2\,q^2 - \delta^2\right)\right]\,\eta \\ & - \left(1 - c^2\right)\,\delta^2\,(q + c\,\delta)^2\,\eta^2. \end{aligned}$$

When $|q| = 1$ or $|\delta| = 1$, on the other hand, the SLD is not unique; however, the SLD Fisher information is well-defined and is identical to the continuous extension of (2) to the boundary. Note that the SLD Fisher information satisfies the following relations.

$$J_\eta^{(q)}(\delta, c) = J_\eta^{(-q)}(-\delta, c) = J_\eta^{(-q)}(\delta, -c) = J_\eta^{(q)}(-\delta, -c). \tag{3}$$

3.1 Proof of Theorem 1

In view of the symmetry (3), we need only prove the case $q = 1$. It is easy to show that the SLD Fisher information

$$J_\eta^{(1)}(\delta, c) = \frac{2\,(1 + \delta\,c) - \delta^2\,(1 - c^2)\,\eta}{4\,\eta\,(1 - \eta)}$$

takes the maximum

$$J_\eta = \frac{1}{\eta\,(1 - \eta)}$$

if and only if $(\delta, c) = (-1, -1)$ or $(1, 1)$. The corresponding output states

$$\rho_\eta|_{(\delta,c)=(-1,-1)} = \begin{bmatrix} 1-\eta & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \rho_\eta|_{(\delta,c)=(1,1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\eta & 0 \\ 0 & 0 & 0 & \eta \end{bmatrix}$$

are both isomorphic to the classical coin flipping in which heads occur with probability η. As a consequence, there is an efficienent estimator for the expectation parameter η. This proves Theorem 1.

3.2 Proof of Theorem 2

When $q = 0$, the channel is reduced to a submodel of the Pauli channel which has been studied in detail in [8]. To be specific, the optimal input is an arbitrary maximally entangled state (i.e., $\delta = 0$), and the corresponding output family is canonically embedded in the 3-dimensional probability simplex $\mathcal{P}^3$ of Pauli channels as follows (see Figure 2).

$$p_\eta = \left(\frac{(1+\sqrt{\eta})^2}{4}, \frac{1-\eta}{4}, \frac{1-\eta}{4}, \frac{(1-\sqrt{\eta})^2}{4}\right).$$

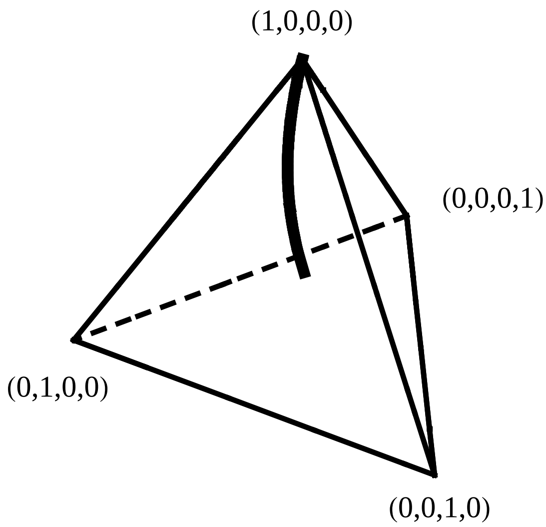

Figure 2: The output family for $q = 0$ and $\delta = 0$ embedded in the probability simplex $\mathcal{P}^3$ of Pauli channels. It is the ∇^e-geodesic connecting the vertex $(1, 0, 0, 0)$ and the center $(1/4, 1/4, 1/4, 1/4)$ of $\mathcal{P}^3$, and has an efficient estimator for the expectation parameter $\sqrt{\eta}$.

Moreover, this turns out to be an exponential family:

$$p_{\eta(\xi)} = \frac{1}{(e^{\xi/2} + e^{-\xi/2})^2}(e^{\xi}, 1, 1, e^{-\xi})$$

having a natural (∇^e-affine) parameter $\xi := 2\,\mathrm{arctanh}\sqrt{\eta}$. We regard the vector $X := (1, 0, 0, -1)$, that is obtained by collecting the coefficients of ξ in the exponents of $(e^{\xi}, 1, 1, e^{-\xi})$, as a random variable that takes values $X(i)$ with probability $p_\eta(i)$, $(i = 1, \ldots, 4)$. According to the theory of exponential family [12, Section 3.5], the random variable X gives the efficient estimator for the expectation parameter

$$E_\eta[X] = \sqrt{\eta}.$$

This proves Theorem 2. We will give an alternative argument in Section 4.

3.3 Proof of Theorem 3

We present the proof in a series of lemmas. The goal is to find the maximum of $J_\eta^{(q)}(\delta, c)$ on the compact square region $[-1, 1] \times [-1, 1]$ for a given $q \neq 0$ and ± 1. Let us consider the extremal conditions:

$$\frac{\partial J_\eta^{(q)}(\delta,c)}{\partial\delta}=0, \qquad \frac{\partial J_\eta^{(q)}(\delta,c)}{\partial c}=0. \tag{4}$$

The next key lemma asserts that the SLD Fisher information $J_\eta^{(q)}(\delta,c)$ takes the maximum either on $\delta=0$ or on the boundary $|\delta|=1$ or $|c|=1$.

Lemma 5. *If $\delta\neq 0$, the equations (4) do not simultaneously hold for any interior point $(\delta,c)\in(-1,1)\times(-1,1)$.*

Proof Consider the quantity

$$K(\delta,c\,;\,\eta,q):=c\left(\frac{\partial J_\eta^{(q)}(\delta,c)}{\partial c}\right)-\delta\left(\frac{\partial J_\eta^{(q)}(\delta,c)}{\partial\delta}\right). \tag{5}$$

If the equations (4) simultaneously hold, then $K(\delta,c\,;\,\eta,q)=0$. It is therefore sufficient to show that $K(\delta,c\,;\,\eta,q)>0$ for all $(\delta,c)\in(-1,1)\times(-1,1)$ unless $\delta=0$. By a direct evaluation

$$K(\delta,c\,;\,\eta,q)=\frac{\delta^2\,f(\delta,c\,;\,\eta,q)}{2\,\eta\,(1-\eta)\,\left[1-q^2\,\delta^2+(1+2\,c\,q\,\delta+(c^2+q^2-1)\,\delta^2)\,\eta\right]^2}$$

where f is a polynomial that has the following decomposition:

$$\begin{aligned} f(\delta,c\,;\,\eta,q)=\eta\left(1-\delta^2+(q+c\,\delta)^2\,\eta\right)^2&+(1-q^2)\Big[\left(1-q^2\right)\eta+\left(1-\delta^4\right)\eta+\left(1-\delta^2\right)^2\eta^2\\ &+(1-\eta)\,\left(1+c\,q\,\delta+c\,q\,\delta\,\eta+c^2\,\delta^2\,\eta\right)^2+(q+c\,\delta)^2\,\eta\,\left(1+\eta^2\right)\Big]. \end{aligned}$$

Since $0<\eta<1$ and $|q|<1$, it is clear from the above decomposition that $f(\delta,c\,;\,\eta,q)$ is strictly positive for all interior points $(\delta,c)\in(-1,1)\times(-1,1)$. This proves the assertion. $\square$

Lemma 6. *If $\delta=0$, the equations (4) have a unique solution $c=0$. The point $(\delta,c)=(0,0)$ is a saddle point of $J_\eta^{(q)}(\delta,c)$.*

Proof By a direct computation, we have

$$\left.\frac{\partial J_\eta^{(q)}(\delta,c)}{\partial\delta}\right|_{\delta=0}=\frac{c\,q}{2\eta\,(1-\eta)}, \qquad \left.\frac{\partial J_\eta^{(q)}(\delta,c)}{\partial c}\right|_{\delta=0}=0.$$

Since $q\neq 0$, the only critical point on $\delta=0$ is $c=0$. The Hessian matrix at the origin $(\delta,c)=(0,0)$ is

$$H(0,0)=\frac{1}{2\eta\,(1-\eta)}\begin{bmatrix} q^2\,(1-\eta) & q\\ q & 0\end{bmatrix},$$

and $\det H(0,0)=-q^2/4\eta^2\,(1-\eta)^2\,<0$. $\square$

Lemma 7. *$J_\eta^{(q)}(\delta, c)$ does not take the maximum on the boundary $|\delta| = 1$ unless $|c| = 1$.*

Proof According to (3), $J_\eta^{(q)}(-1, c) = J_\eta^{(q)}(1, -c)$, so that it suffices to treat the case $\delta = -1$. For $-1 < c < 1$, consider the quantity

$$\hat{K}(c\,;\,\eta, q) := c\left(\frac{\partial J_\eta^{(q)}(-1, c)}{\partial c}\right) + \left[\frac{\partial J_\eta^{(q)}(\delta, c)}{\partial \delta}\right]_{\delta=-1}. \tag{6}$$

Here the second term is understood as the one-sided derivative at the boundary $\delta = -1$. Then

$$\hat{K}(c\,;\,\eta, q) = K(-1, c\,;\,\eta, q) = \frac{f(-1, c\,;\,\eta, q)}{2\,\eta\,(1-\eta)\,\left[\,1 - q^2 + (c-q)^2\,\eta\,\right]^2}.$$

As the function $f(-1, c\,;\,\eta, q)$ is strictly positive for all $c \in (-1, 1)$, so is $\hat{K}(c\,;\,\eta, q)$. Suppose now that $\partial J_\eta^{(q)}(-1, c)/\partial c = 0$ at a certain $c = c_0 \in (-1, 1)$. Then $\left[\partial J_\eta^{(q)}(\delta, c)/\partial \delta\right]_{\delta=-1}$ is strictly positive at $c = c_0$. In other words, $J_\eta^{(q)}(-1, c_0) < J_\eta^{(q)}(-1+\varepsilon, c_0)$ for sufficiently small $\varepsilon > 0$. □

We deduce from Lemmas 5-7 that $J_\eta^{(q)}(\delta, c)$ takes the maximum on the boundary $|c| = 1$. It is now straightforward to obtain the maximum. Due to the symmetry (3), we need only consider the case $c = -1$, in which

$$J_\eta^{(q)}(\delta, -1) = \frac{1 - \delta^2 + (1 - 2q\,\delta + \delta^2)\,\eta}{2\,\eta\,(1-\eta)\,[1 + q\,\delta + (1 - q\,\delta)\,\eta]}.$$

The optimal δ that maximizes $J_\eta^{(q)}(\delta, -1)$ is given by

$$\underset{-1 \le \delta \le 1}{\arg\max}\; J_\eta^{(q)}(\delta, -1) = \begin{cases} \dfrac{(\sqrt{1-q^2}\, - 1)\,(1+\eta)}{q\,(1-\eta)}, & 0 \le \eta \le \eta_q \\[2ex] -1, & \eta_q < \eta \le 1 \end{cases}$$

where $\eta_q := \sqrt{(1-q)/(1+q)}$. This completes the proof.

3.4 Proof of Theorem 4

A quantum statistical model ρ_η is ∇^e-autoparallel if and only if there is a constant operator T and smooth functions $\lambda(\eta)$ and $\mu(\eta)$ such that the SLD L_η satisfies

$$L_\eta = \lambda(\eta) T - \mu(\eta) I. \tag{7}$$

In this case, the operator T is the efficient estimator for an alternative parameter $\zeta := \mu(\eta)/\lambda(\eta)$ of the model. It is important to notice that the condition (7) is invariant under the transformation

$$L_\eta \mapsto \hat{L}_\eta := f(\eta) L_\eta - g(\eta) I,$$

where $f(\eta)$ and $g(\eta)$ are arbitrary smooth functions with $f(\eta) \neq 0$. In fact, $\hat{L}_\eta = \hat{\lambda}(\eta)T - \hat{\mu}(\eta)I$ holds for $\hat{\lambda}(\eta) := f(\eta)\lambda(\eta)$ and $\hat{\mu}(\eta) := f(\eta)\mu(\eta) + g(\eta)$. Specifically, if we let $g(\eta) := f(\eta)(L_\eta)_{11}$, then the $(1,1)$th entry of $\hat{L}_\eta$ vanishes, and we have $\hat{\mu}(\eta) = \hat{\lambda}(\eta)T_{11}$ and

$$\hat{L}_\eta = \hat{\lambda}(\eta)\,\hat{T}, \tag{8}$$

where $\hat{T} := T - T_{11}I$ is a constant operator. Namely, $\hat{L}_\eta$ becomes a constant operator multiplied by a smooth function of η; in particular, the entries of $\hat{L}_\eta$ are linearly dependent functions of η. Conversely, when (8) holds for specific functions $f(\eta)$ and $g(\eta)$, a triad of the original $\lambda(\eta), \mu(\eta)$ and T that satisfies (7) can be retrieved as follows:

$$\lambda(\eta) = \frac{\hat{\lambda}(\eta)}{f(\eta)}, \qquad \mu(\eta) = -\frac{g(\eta)}{f(\eta)}, \qquad T = \hat{T}. \tag{9}$$

Thus (8) gives a necessary and sufficient condition for the model ρ_η to be ∇^e-autoparallel.

Let us proceed to the proof of Theorem 4. We present the proof for $|\delta| < 1$ and $|\delta| = 1$ separately. We first treat the case $|\delta| < 1$. Assume for now that $|q| < 1$: in this case, the model ρ_η is strictly positive and the SLD is uniquely determined in the form $L_\eta = [\ell_{ij}/2\sqrt{\eta}\,D]$ as seen in the Appendix B. Let us set

$$f(\eta) := 2\sqrt{\eta}\,D, \qquad g(\eta) := \ell_{11}.$$

If $\delta \neq 0$ (in addition to $|\delta| < 1$), the $(3,3)$th entry

$$(\hat{L}_\eta)_{33} = 8\,\delta\,\sqrt{\eta}\,\left[1 + c^2 + 2\,c\,q\,\delta + \left(1 - c^2 - \delta^2 + c^2\,\delta^2\right)\,\eta\right]$$

of the transformed SLD $\hat{L}_\eta$ is a nonzero irrational function of η, and the $(1,4)$th and $(2,3)$th entries

$$\begin{aligned}
(\hat{L}_\eta)_{14} &= 4\,(1+c)\,\sqrt{1-\delta^2}\,\left[1 + q\,\delta + \left(1 - \delta^2 + c\,\delta\,(q+\delta)\right)\,\eta + (-1+c)\,\delta\,(q + c\,\delta)\,\eta^2\right] \\
(\hat{L}_\eta)_{23} &= 4\,(-1+c)\,\sqrt{1-\delta^2}\,\left[1 - q\,\delta + \left(1 - \delta^2 + c\,\delta\,(q-\delta)\right)\,\eta + (1+c)\,\delta\,(q + c\,\delta)\,\eta^2\right]
\end{aligned}$$

are polynomials of η that cannot be simultaneously zero. As a consequence, $(\hat{L}_\eta)_{33}$ and at least one of the latter two are linearly independent, and $\hat{L}_\eta$ cannot be of the form (8). If $\delta = 0$, on the other hand, the transformed SLD of the model ρ_η becomes

$$\hat{L}_\eta = 8\,(1+\eta)\,T,$$

where

$$T = \frac{1}{2}\begin{bmatrix} 0 & s & 0 & 1+c \\ s & 0 & -1+c & 0 \\ 0 & -1+c & 0 & -s \\ 1+c & 0 & -s & 0 \end{bmatrix}. \tag{10}$$

This is of the form (8), and the model turns out to be ∇^e-autoparallel for all $q \in (-1, 1)$. We now invoke a continuity argument to conclude that, when $\delta = 0$, the model ρ_η is ∇^e-autoparallel for all $q \in [-1, 1]$.

We next treat the case $\delta = -1$, in which the model ρ_η is factorized as

$$\rho_\eta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \sigma_\eta, \tag{11}$$

where

$$\sigma_\eta = \frac{1}{2} \begin{bmatrix} 1+q+(c-q)\,\eta & s\sqrt{\eta} \\ s\sqrt{\eta} & 1-q-(c-q)\,\eta \end{bmatrix}.$$

The problem amounts to finding the condition for σ_η to be ∇^e-autoparallel for all $q \in [-1,1]$. Since

$$\det \sigma_\eta = \frac{1}{4}(1-\eta)\left[1-q^2+(c-q)^2\,\eta\right],$$

the model σ_η is strictly positive unless $q = c = \pm 1$, and the SLD is uniquely determined as $L_\eta = [\ell_{ij}/8\sqrt{\eta}\,\det\sigma_\eta]$, where

$$\begin{aligned} \ell_{11} &= \sqrt{\eta}\left[(1-q)\,(2c-2q-s^2)-(c-q)\,(2c-2q+s^2)\,\eta\right] \\ \ell_{12} &= s\left[1-q^2+(c-q)^2\,\eta^2\right] \\ \ell_{22} &= -\sqrt{\eta}\left[(1+q)\,(2c-2q+s^2)+(c-q)\,(2c-2q-s^2)\,\eta\right]. \end{aligned}$$

Letting

$$f(\eta) := 8\sqrt{\eta}\,\det\sigma_\eta, \qquad g(\eta) := \ell_{11},$$

the transformed SLD becomes

$$\begin{aligned} (\hat{L}_\eta)_{12} &= s\,\left[1-q^2+(c-q)^2\,\eta^2\right] \\ (\hat{L}_\eta)_{22} &= -2\sqrt{\eta}\left[2c-2q+s^2\,q-s^2\,(c-q)\,\eta\right]. \end{aligned}$$

These elements are linearly dependent for all $q \in (-1,1)$ if and only if $s = 0$. Now by a continuity argument, we conclude that the model σ_η is ∇^e-autoparallel for all $q \in [-1,1]$ if and only if $s = 0$, that is $|c| = 1$. Note that the model σ_η degenerates to a point $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ when $q = c = 1$, and to a point $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ when $q = c = -1$.

Finally, we treat the case $\delta = 1$, in which the model ρ_η is factorized as

$$\rho_\eta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \sigma_\eta,$$

where

$$\sigma_\eta = \frac{1}{2} \begin{bmatrix} 1+q-(c+q)\,\eta & -s\sqrt{\eta} \\ -s\sqrt{\eta} & 1-q+(c+q)\,\eta \end{bmatrix}.$$

A similar argument as above concludes that σ_η is ∇^e-autoparallel for all $q \in [-1,1]$ if and only if $s = 0$. Note that the model σ_η degenerates to a point $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ when $q = -c = 1$, and to a point $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ when $q = -c = -1$. This completes the proof.

4 Discussions

In this paper, we have studied the problem of estimating a generalized amplitude damping channel $\Gamma_\eta^{(p)}$ for a given p through the extension $\mathrm{id} \otimes \Gamma_\eta^{(p)}$. It was shown that there is no uniformly optimal input that simultaneously maximizes the SLD Fisher information for all η unless $p = 0$, 1, or 1/2. Nevertheless, it was also shown that the output family admits an efficient estimator for all p if and only if the input is either a maximally entangled state ($\alpha = 1/2$) or a disentangled state of the type $(\alpha, \phi) = (0,0), (0, \pi/2), (1,0), (1, \pi/2)$. In this section, we discuss the relative merits of these inputs.

When $(\alpha, \phi) = (0,0)$, the SLD of the factorized model σ_η in (11) satisfies

$$L_\eta = \frac{1}{(1-\eta)\,[\eta + p\,(1-\eta)]}\, T - \frac{1}{1-\eta}\, I, \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, the model has an efficient estimator T for the parameter $\zeta = \eta + p\,(1-\eta)$ unless $p = 1$. For example, when $p = 0$, the model σ_η is reduced to the classical coin flipping $\begin{bmatrix} \eta & 0 \\ 0 & 1-\eta \end{bmatrix}$, and T is the efficient estimator for the parameter η, as was seen in Theorem 1. When $p = 1$, however, the model σ_η degenerates to a point $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and has no information about the parameter η. In short, the input $(\alpha, \phi) = (0,0)$ is the best for $p = 0$, whereas it is the worst for $p = 1$. Such a biased nature is convinced also by the fact that the SLD Fisher information

$$J_\eta^{(p)}(0,0) = \frac{1-p}{(1-\eta)\,[\eta + p\,(1-\eta)]}$$

approaches zero monotonously as $p \to 1$. Similar observation also applies to the other disentangled inputs $(\alpha, \phi) = (0, \pi/2), (1,0), (1, \pi/2)$.

When $\alpha = 1/2$, on the other hand, the SLD of the model ρ_η satisfies

$$L_\eta = \frac{1}{(1-\eta)\sqrt{\eta}}\, T - \frac{1}{1-\eta}\, I,$$

where T is given by (10). Therefore, T is the efficient estimator for the parameter $\zeta = \sqrt{\eta}$ irrespective of p. This uniformity is also seen in the SLD Fisher information

$$J_\eta^{(p)}(1/2, \phi) = \frac{1}{2\,\eta\,(1-\eta)},$$

which is independent of p (and ϕ). Such a robustness is in good contrast to the above mentioned biased nature of disentangled inputs, and would be preferable in practical applications, although the SLD Fisher information does not take the maximum at $\alpha = 1/2$ unless $p = 1/2$.

The advantage of the use of maximally entangled inputs can be viewed also from a different angle. The SLD Fisher information $J_\eta^{(p)}(0,0)$ for the disentangled input $(\alpha, \phi) =$

$(0,0)$ is averaged over p as

$$\overline{J}_\eta := \int_0^1 J_\eta^{(p)}(0,0)\, dp = \frac{\eta - \log\eta - 1}{(1-\eta)^3}.$$

The average value $\overline{J}_\eta$ is the same for the other disentangled inputs $(\alpha,\phi)=(0,\pi/2),(1,0),(1,\pi/2)$. Since $J_\eta^{(p)}(1/2,\phi) > \overline{J}_\eta$ for all $\eta\in(0,1)$, we may assert that, among those which admit efficient estimators for all p, maximally entangled inputs are optimal on average.

Appendix A: Quantum Information Geometry

This appendix provides a brief account of quantum information geometry based on the SLD. Let $\mathcal{S}$ be the totality of faithful quantum states on a D-dimensional Hilbert space $\mathcal{H}$. The set $\mathcal{S}$ is naturally regarded as a (D^2-1)-dimensional differentiable manifold, and its dualistic geometrical structure is introduced as follows. We first define a Riemannian metric by

$$g(X,Y) := \frac{1}{2}\mathrm{Tr}\,\rho(L_X L_Y + L_Y L_X) = \mathrm{Tr}\,(X\rho)L_Y,$$

where $X,Y\in T_\rho\mathcal{S}$, and L_X, L_Y are the corresponding SLDs, i.e., the Hermitian operators satisfying

$$X\rho = \frac{1}{2}(\rho L_X + L_X\rho).$$

The metric g is called the *SLD Fisher* metric. We next introduce a pair of affine connections. One is defined by

$$(\nabla^m_X Y)\rho := X(Y\rho),$$

and is called the *mixture* connection. The other is defined by

$$(\nabla^e_X Y)\rho := \frac{1}{2}\{\rho(XL_Y - \mathrm{Tr}\,\rho(XL_Y)) + (XL_Y - \mathrm{Tr}\,\rho(XL_Y))\rho\},$$

and is called the *exponential* connection. These connections are mutually dual with respect to the SLD Fisher metric, in that

$$Xg(Y,Z) = g(\nabla^m_X Y, Z) + g(Y, \nabla^e_X Z).$$

A coordinate system $\xi = (\xi^i)_{1\le i\le D^2-1}$ of $\mathcal{S}$ is called *affine* with respect to a connection ∇ of $\mathcal{S}$ if $\nabla_{\partial_i}\partial_j = 0$ for all i,j, where $\partial_i = \partial/\partial\xi^i$. For example, the components of density matrices $\rho\,(\in\mathcal{S})$, with one diagonal entry removed (since $\mathrm{Tr}\,\rho = 1$), form a ∇^m-affine coordinate system of $\mathcal{S}$. On the other hand, $\mathcal{S}$ does not have a ∇^e-affine coordinate system, since ∇^e-torsion does not vanish because of the noncommutativity of operators.

A submanifold $\mathcal{M}$ of $\mathcal{S}$ is called *autoparallel* with respect to a connection ∇ of $\mathcal{S}$ if $\nabla_X Y\in T_\rho\mathcal{M}$ for all $\rho\in\mathcal{M}$ and $X,Y\in T_\rho\mathcal{M}$. In particular, a one-dimensional ∇-autoparallel submanifold is called a ∇-*geodesic*. When $\mathcal{M}$ is ∇-autoparallel in $\mathcal{S}$, $\mathcal{M}$ has a vanishing embedding curvature with respect to ∇, and one can regard ∇ as a connection of $\mathcal{M}$, just by restricting ∇ onto $\mathcal{M}$. For example, a maximal commutative subset $\mathcal{P}$ of

$\mathcal{S}$ is autoparallel with respect to both ∇^m and ∇^e, so that one can naturally induce a dualistic structure on $\mathcal{P}$ from that of $\mathcal{S}$. In fact, the geometrical structure thus induced on $\mathcal{P}$ is isomorphic to that of the $(D-1)$-dimensional classical probability simplex $\mathcal{P}^{D-1}$. For more information, see [12]. A generalization to manifolds of non-faithful (i.e., degenerate) quantum states is discussed in [14].

Appendix B: Derivation of the SLD Fisher information

In this appendix, we outline the derivation of the SLD Fisher information (2). In what follows, we work with an alternative parametrization $\theta := \sqrt{\eta}$ in order to simplify the computation. The entries of the output state

$$\tilde{\rho}_\theta = \tilde{\rho}_\theta(\alpha, \phi) := \rho_\eta(\alpha, \phi)\big|_{\eta=\theta^2}$$

are

$$\begin{aligned}
(\tilde{\rho}_\theta)_{11} &= \frac{1}{2}(1-\alpha)\left[\,2p\,(1-\theta^2) + \theta^2\,(1+\cos 2\phi)\,\right] \\
(\tilde{\rho}_\theta)_{12} &= \frac{1}{2}(1-\alpha)\,\theta\,\sin 2\phi \\
(\tilde{\rho}_\theta)_{13} &= -\frac{1}{2}\sqrt{\alpha\,(1-\alpha)}\,\theta\,\sin 2\phi \\
(\tilde{\rho}_\theta)_{14} &= \frac{1}{2}\sqrt{\alpha\,(1-\alpha)}\,\theta\,(1+\cos 2\phi) \\
(\tilde{\rho}_\theta)_{22} &= \frac{1}{2}(1-\alpha)\left[\,2-2p\,(1-\theta^2) - \theta^2\,(1+\cos 2\phi)\,\right] \\
(\tilde{\rho}_\theta)_{23} &= -\frac{1}{2}\sqrt{\alpha\,(1-\alpha)}\,\theta\,(1-\cos 2\phi) \\
(\tilde{\rho}_\theta)_{24} &= \frac{1}{2}\sqrt{\alpha\,(1-\alpha)}\,\theta\,\sin 2\phi \\
(\tilde{\rho}_\theta)_{33} &= \frac{1}{2}\alpha\left[\,2p\,(1-\theta^2) + \theta^2\,(1-\cos 2\phi)\,\right] \\
(\tilde{\rho}_\theta)_{34} &= -\frac{1}{2}\sqrt{\alpha\,(1-\alpha)}\,\theta\,\sin 2\phi \\
(\tilde{\rho}_\theta)_{44} &= \frac{1}{2}\alpha\left[\,2-2p\,(1-\theta^2) - \theta^2\,(1-\cos 2\phi)\,\right].
\end{aligned}$$

The SLD of the model $\tilde{\rho}_\theta$ is a selfadjoint operator $\tilde{L}_\theta$ that satisfies the equation

$$\frac{\partial}{\partial\theta}\tilde{\rho}_\theta = \frac{1}{2}(\tilde{\rho}_\theta\tilde{L}_\theta + \tilde{L}_\theta\tilde{\rho}_\theta).$$

Since

$$\det\tilde{\rho}_\theta = p^2\,(1-p)^2\,\alpha^2\,(1-\alpha)^2\,(1-\theta^2)^4,$$

the SLD is uniquely determined if and only if $p \neq 0,\ 1$ and $\alpha \neq 0,\ 1$. In this case

$$\tilde{L}_\theta = \frac{1}{D}\,[\ell_{ij}]_{1\le i,j\le 4}, \qquad \ell_{ij} = \ell_{ji}$$

where

$$D = 2\left(1-\theta^2\right)\Big[-8\left(-\alpha+p\left(-1+2\alpha\right)\right)\left(1-\alpha+p\left(-1+2\alpha\right)\right)+\theta^2\Big\{3-8p(1-2\alpha)^2$$
$$+8p^2(1-2\alpha)^2+4\left(-1+\alpha\right)\alpha+\left(-1+2\alpha\right)\left(\left(-4+8p\right)\,\cos 2\phi+\left(-1+2\,\alpha\right)\,\cos 4\phi\right)\right)\Big\}\Big]\,,$$

and $\ell_{ij}=\ell_{ij}(\theta,p,\alpha,\phi)$ are given by

$$\begin{aligned}
&\ell_{11}(\theta,\,p,\,\alpha,\,\phi)=\ell_{44}(\theta,\,1-p,\,1-\alpha,\,\phi)\\
&\quad=4\,\theta\left(1-\alpha+p\,\left(-1+2\,\alpha\right)\right)\,\left(3-8\,p+2\,\left(-7+8\,p\right)\,\alpha+\left(4-8\,\alpha\right)\,\cos 2\,\phi+\left(1-2\,\alpha\right)\,\cos 4\,\phi\right)\\
&\qquad+\theta^3\,\Big[2\,\Big(-7+18\,p\left(1-2\,\alpha\right)^2-16\,p^2\left(1-2\,\alpha\right)^2+8\,\left(-2+\alpha\right)\,\left(-1+\alpha\right)\,\alpha\Big)\\
&\qquad\quad+\left(-1+2\,\alpha\right)\,\left(17-32\,p+4\,\left(-1+\alpha\right)\,\alpha\right)\,\cos 2\,\phi\\
&\qquad\quad-2\,\left(-1+2\,\alpha\right)\,\Big(-1-2\,\alpha+4\,\alpha^2+p\,\left(-2+4\,\alpha\right)\Big)\,\cos 4\,\phi-\left(-1+2\,\alpha\right)^3\,\cos 6\,\phi\Big]\,,
\end{aligned}$$

$$\begin{aligned}
&\ell_{22}(\theta,\,p,\,\alpha,\,\phi)=\ell_{33}(\theta,\,1-p,\,1-\alpha,\,\phi)\\
&\quad=4\,\theta\left(-\alpha+p\,\left(-1+2\,\alpha\right)\right)\,\left(5-8\,p-2\,\alpha+16\,p\,\alpha+\left(4-8\,\alpha\right)\,\cos 2\,\phi+\left(-1+2\,\alpha\right)\,\cos 4\,\phi\right)\\
&\qquad+\theta^3\,\Big[2\,\Big(-5+14\,p\left(1-2\,\alpha\right)^2-16\,p^2\left(1-2\,\alpha\right)^2+8\left(-1+\alpha\right)^2\alpha\Big)\\
&\qquad\quad-\left(-1+2\,\alpha\right)\,\left(32\,p+\left(-5+2\,\alpha\right)\,\left(3+2\,\alpha\right)\right)\,\cos 2\,\phi\\
&\qquad\quad-2\,\left(-1+2\,\alpha\right)\,\Big(-3+p\,\left(2-4\,\alpha\right)+2\,\alpha+4\,\alpha^2\Big)\,\cos 4\,\phi+\left(-1+2\,\alpha\right)^3\,\cos 6\,\phi\Big]\,,
\end{aligned}$$

$$\begin{aligned}
&\ell_{12}(\theta,\,p,\,\alpha,\,\phi)=-\,\ell_{34}(\theta,\,1-p,\,1-\alpha,\,\phi)\\
&\quad=16\,\left(\alpha-p\,\left(-1+2\,\alpha\right)\right)\,\left(1-\alpha+p\,\left(-1+2\,\alpha\right)\right)\,\sin 2\,\phi\\
&\qquad+16\,\theta^2\left(-1+\alpha\right)\,\alpha\,\,\left(-1-\left(-1+2\,p\right)\,\left(-1+2\,\alpha\right)\,\cos 2\,\phi\right)\,\sin 2\,\phi\\
&\qquad+\theta^4\,\Big[\left(5+16\,\left(-1+p\right)\,p\right)\,\left(1-2\,\alpha\right)^2\,\sin 2\,\phi\\
&\qquad\quad+4\,\left(-1+2\,p\right)\,\left(-1+2\,\alpha\,\left(2+\alpha\,\left(-3+2\,\alpha\right)\right)\right)\,\sin 4\,\phi+\left(1-2\,\alpha\right)^2\,\sin 6\,\phi\Big]\,,
\end{aligned}$$

$$\begin{aligned}
&\ell_{13}(\theta,\,p,\,\alpha,\,\phi)=\ell_{24}(\theta,\,p,\,\alpha,\,\phi)\\
&\quad=8\,\sqrt{\alpha\left(1-\alpha\right)}\,\left(2\,\alpha-1\right)\,\theta\,\left(1-\theta^2\right)\,\left[\left(2p-1\right)\,\left(2\alpha-1\right)+\cos 2\phi\right]\,\sin 2\phi,
\end{aligned}$$

$$\begin{aligned}
&\ell_{14}(\theta,\,p,\,\alpha,\,\phi)=-\,\ell_{23}\left(\theta,\,p,\,1-\alpha,\,\frac{\pi}{2}-\phi\right)\\
&\quad=8\,\sqrt{\alpha\left(1-\alpha\right)}\cos^2\phi\,\big[4-4\,\alpha+4\,p\,\left(-1+2\,\alpha\right)\\
&\qquad+\theta^2\,\left(-8\,\left(-1+\alpha\right)\,\alpha+4\,\left(-1+p+\alpha\right)\,\left(-1+2\,\alpha\right)\,\cos 2\,\phi\right)\\
&\qquad-4\,\theta^4\,\left(-1+2\,\alpha\right)\,\left(-1+2\,p+\left(-1+2\,\alpha\right)\,\cos 2\,\phi\right)\,\sin^2\phi\big]\,.
\end{aligned}$$

The SLD $\tilde{L}_\theta$ of $\tilde{\rho}_\theta$ is related to the SLD L_η of ρ_η by

$$L_\eta=\frac{1}{2\sqrt{\eta}}\,\tilde{L}_{\sqrt{\eta}}$$

and the SLD Fisher information for the parameter η is given by

$$J_\eta^{(p)}(\alpha, \phi) = \mathrm{Tr}\left[\frac{\partial \rho_\eta}{\partial \eta} L_\eta\right] = \frac{1}{4\,\eta}\,\mathrm{Tr}\left[\frac{\partial \tilde{\rho}_\theta}{\partial \theta}\,\tilde{L}_\theta\right]_{\theta=\sqrt{\eta}}.$$

This leads to the formula (2).

When $p = 0, 1$ or $\alpha = 0, 1$, on the other hand, the SLD is not unique; however, the SLD Fisher information is well-defined, and is identical to the continuous extension of (2) to the boundary, since the rank of ρ_θ is invariant for each values of the parameters p and α [14].

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