Quantum Fisher Metric and Pure State Estimation

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Abstract

A statistical parameter estimation theory for quantum pure state models is presented. We first investigate a general framework of the pure state estimation theory and derive quantum counterparts of the Fisher metric. Then we formulate a one parameter estimation theory, based on the symmetric logarithmic derivatives, and clarify the differences between pure state models and strictly positive models.

1 Introduction

A quantum statistical model is a family of density operators ρ_θ defined on a certain separable Hilbert space $\mathcal{H}$ with finite-dimensional real parameters $\theta = (\theta^i)_{i=1}^n$ which are to be estimated statistically. In order to avoid singularities, the conventional quantum estimation theory [1][2] has been often restricted to models that are composed of strictly positive density operators. It was Helstrom [3] who successfully introduced the symmetrized logarithmic derivative for the one parameter estimation theory as a quantum counterpart of the logarithmic derivative in the classical estimation theory. The right logarithmic derivative is another successful counterpart introduced by Yuen and Lax [4] in the expectation parameter estimation theory for quantum gaussian models, which provided a theoretical background of optical communication theory. Quantum information theorists have also kept away from degenerated states, such as pure states, for mathematical convenience [5]. Indeed, the von Neumann entropy cannot distinguish the pure states, and the relative entropies diverge.

In this paper, however, we try to construct an estimation theory for pure state models, and clarify the differences between the pure state case and the strictly positive state case. In Sec. 2, we prove some crucial lemmas which will provide fudamentals of the pure state

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estimation theory. In Sec. 3, we study the quantum counterpart of the logarithmic derivative and the Fisher information which played important roles in the classical estimation theory. The quantum statistical significance of the Fubini–Study metric is also clarified. In Sec. 4, we provide one parameter pure state estimation theory based on the symmetric logarithmic derivative. This is rather analogous to the conventional quantum estimation theory, but reveals the essential difference between the pure state models and the strictly positive models. Some examples are also given in Sec. 5.

2 Preliminaries

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\psi|\varphi\rangle$ for every $\psi, \varphi \in \mathcal{H}$. Further, let $\mathcal{L}$ and $\mathcal{L}_{sa}$ are, respectively, the set of all the (bounded) linear operators and all the self-adjoint operators on $\mathcal{H}$. Given a possibly degenerated density operator ρ, we define sesquilinear forms on $\mathcal{L}$:

$$(A, B)_\rho = \mathrm{Tr}\,\rho B A^*, \tag{1}$$

$$\langle A, B\rangle_\rho = \frac{1}{2}\mathrm{Tr}\,\rho(BA^* + A^* B), \tag{2}$$

where $A, B \in \mathcal{L}$. These are *pre-inner product* on $\mathcal{L}$, i.e., possessing all properties of inner product except that $(K, K)_\rho$ and $\langle K, K\rangle_\rho$ may be equal to zero for a nonzero $K \in \mathcal{L}$. Note that the Schwarz inequality also holds for pre-inner product. The forms $(\cdot, \cdot)_\rho$ and $\langle\cdot, \cdot\rangle_\rho$ become inner products if and only if $\rho > 0$. If $\mathrm{rank}\,\rho = 1$ or equivalently $\rho^2 = \rho$, ρ is called *pure*. The following lemmas are fundamental.

Lemma 1 *Suppose ρ is pure. Then the following 3 conditions for linear operators $K \in \mathcal{L}$ are equivalent.*

(i) $(K, K)_\rho = 0$,

(ii) $\rho K = 0$,

(iii) $\mathrm{Tr}\,\rho K = 0$ *and* $\rho K + K^*\rho = 0$.

Proof Let us express as $\rho = |\psi\rangle\langle\psi|$ where $|\psi\rangle$ is a normalized vector in $\mathcal{H}$. Then the following equivalent sequence

$$(K, K)_\rho = 0 \iff \langle\psi|KK^*|\psi\rangle = 0 \iff \langle\psi|K = 0 \iff |\psi\rangle\langle\psi|K = 0,$$

yield (i)⇔(ii). Further, (ii)⇒(iii) is trivial. (iii)⇒(ii) is shown as follows. Operating $\langle\psi|$ from the left to the assumption

$$|\psi\rangle\langle\psi|K + K^*|\psi\rangle\langle\psi| = 0,$$

and invoking another assumption $\operatorname{Tr} \rho K = 0 \Leftrightarrow \langle\psi|K^*|\psi\rangle = 0$, we have

$$0 = \langle\psi|\psi\rangle\langle\psi|K + \langle\psi|K^*|\psi\rangle\langle\psi| = \langle\psi|K.$$

Therefore $|\psi\rangle\langle\psi|K = 0$. ■

Lemma 2 *Suppose ρ is pure. Then the following 3 conditions for linear operators $K \in \mathcal{L}$ are equivalent.*

(i) $\langle K, K\rangle_\rho = 0$,

(ii) $\rho K = \rho K^* = 0$,

(iii) $\operatorname{Tr} \rho K = 0$, $\rho K + K^*\rho = 0$, *and* $\rho K^* + K\rho = 0$.

Proof (i)⇔(ii) is shown as follows:

$$\begin{aligned} \langle K, K\rangle_\rho = 0 \quad &\Longleftrightarrow \quad \langle\psi|KK^*|\psi\rangle + \langle\psi|K^*K|\psi\rangle = 0 \\ &\Longleftrightarrow \quad \langle\psi|K = 0, \quad \langle\psi|K^* = 0 \\ &\Longleftrightarrow \quad \rho K = \rho K^* = 0. \end{aligned}$$

(ii)⇔(iii) is a sraightforward consequence of Lemma 1. ■

Lemma 3 *Suppose ρ is pure. Then the following 3 conditions for self–adjoint operators $K \in \mathcal{L}_{sa}$ are equivalent.*

(i) $\langle K, K\rangle_\rho = 0$,

(ii) $\rho K = 0$,

(iii) $\rho K + K\rho = 0$.

Proof Straightforward by setting $K = K^*$ in Lemma 2. ■

Note that in either lemmas, equivalence of the conditions (i) and (ii) holds for any ρ, whereas the condition (iii) is characteristic of pure states. These lemmas are, therefore, effectively employed in the pure state estimation theory. Denote by $\mathcal{K}(\rho)$ the set of linear operators $K \in \mathcal{L}$ satisfying $(K, K)_\rho = 0$, which are called the kernel of the pre-inner product $(\cdot, \cdot)_\rho$. Also denote by $\mathcal{K}_{sa}(\rho)$ the set of self-adjoint operators $K \in \mathcal{L}_{sa}$ satisfying $\langle K, K\rangle_\rho = 0$, which are called the kernel of the pre-inner product $\langle\cdot, \cdot\rangle_\rho$.

3 Quantum Fisher metric

suppose we are given a n parameter pure state model:

$$\mathcal{S} = \{\rho_\theta \,;\; \rho_\theta^* = \rho_\theta,\ \mathrm{Tr}\,\rho_\theta = 1,\ \rho_\theta^2 = \rho_\theta,\ \theta \in \Theta \subset \mathbf{R}^n\}. \tag{3}$$

We define a family of quantum analogues of the logarithmic derivative by

$$\frac{\partial \rho_\theta}{\partial \theta^j} = \frac{1}{2}[\rho_\theta L_{\theta,j} + L_{\theta,j}^* \rho_\theta], \quad \mathrm{Tr}\,\rho_\theta L_{\theta,j} = 0. \tag{4}$$

For instance,

$$\frac{\partial \rho_\theta}{\partial \theta^j} = \frac{1}{2}[\rho_\theta L_{\theta,j}^S + L_{\theta,j}^S \rho_\theta], \quad L_{\theta,j}^S = L_{\theta,j}^{S*} \tag{5}$$

defines the *symmetric logarithmic derivative* (SLD) $L_{\theta,j}^S$ introduced by Helstrom [3]. Furthermore, since every pure state model is written in the form $\rho_\theta = U_\theta \rho_0 U_\theta^*$, where U_θ is unitary, we have another useful logarithmic derivative

$$\frac{\partial \rho_\theta}{\partial \theta^j} = \frac{1}{2}[\rho_\theta L_{\theta,j}^A - L_{\theta,j}^A \rho_\theta], \quad \mathrm{Tr}\,\rho_\theta L_{\theta,j}^A = 0, \quad L_{\theta,j}^A = -L_{\theta,j}^{A*}, \tag{6}$$

which may be called the *anti-symmetric logarithmic derivative* (ALD). Indeed, the ALD is closely related to the local generator $A_{\theta,j} = -i(\partial U_\theta/\partial\theta^j)U_\theta^*$ of the unitary U_θ such as $L_{\theta,j}^A = -2iA_{\theta,j}$. Thus, (4) defines a certain family of logarithmic derivatives [6]. Denote by $\mathcal{T}(\rho_\theta)$ all the logarithmic derivatives which satisfy (4).

Lemma 4 *Suppose ρ_θ is pure and an arbitrary linear operator $A \in \mathcal{L}$ is given. Then all the quantities $(A, L_{\theta,j})_{\rho_\theta}$ with respect to the common θ^j are identical for every logarithmic derivatives $L_{\theta,j} \in \mathcal{T}(\rho_\theta)$.*

Proof Take any logarithmic derivatives $L_{\theta,j}$ and $L'_{\theta,j}$ which correspond to the same θ^j, and denote $K = L_{\theta,j} - L'_{\theta,j}$. Then, from (4), K satisfies the condition (iii) of Lemma 1. Therefore $(K,K)_{\rho_\theta} = 0$ holds. This and the Schwarz inequality

$$|(A,K)_{\rho_\theta}|^2 \le (A,A)_{\rho_\theta}(K,K)_{\rho_\theta},$$

lead us to $(A,K)_{\rho_\theta} = 0$ for all $A \in \mathcal{L}$. ■

From Lemma 4, we can define uniquely the complex Fisher information matrix J_θ for the family of logarithmic derivatives (4) whose (j,k) entry is $(L_{\theta,j}, L_{\theta,k})_{\rho_\theta}$.

The SLD is also not uniquely determined for pure state models. Denote by $\mathcal{T}^S(\rho_\theta)$ all the SLD's which satisfy (5).

Lemma 5 *Suppose ρ_θ is pure and an arbitrary self-adjoint operator $A \in \mathcal{L}_{sa}$ is given. Then all the quantities $\langle A, L^S_{\theta,j}\rangle_{\rho_\theta}$ with respect to the common θ^j are identical for every SLD $L^S_{\theta,j} \in \mathcal{T}^S(\rho_\theta)$.*

Proof By using Lemma 3, it is proved in the same way as Lemma 4. ■

From Lemma 5, we can define uniquely the real Fisher information matrix J^S_θ for the family of SLD (5) whose (j,k) entry is $\langle L^S_{\theta,j}, L^S_{\theta,k}\rangle_{\rho_\theta}$, which is called the *SLD–Fisher information matrix*. The above results are summarized by the following theorem.

Theorem 1 *Suppose ρ_θ is pure. Then the complex Fisher information matrix $J_\theta = [(L_{\theta,j}, L_{\theta,k})_{\rho_\theta}]$ and the SLD–Fisher information matrix $J^S_\theta = \left[\langle L^S_{\theta,j}, L^S_{\theta,k}\rangle_{\rho_\theta}\right]$ are uniquely determined on the quotient spaces $\mathcal{T}(\rho_\theta)/\mathcal{K}(\rho_\theta)$ and $\mathcal{T}^S(\rho_\theta)/\mathcal{K}_{sa}(\rho_\theta)$, respectively. They are related by $J^S_\theta = \mathrm{Re}\, J_\theta$. The (j,k) entry of J^S_θ becomes*

$$(J^S_\theta)_{jk} = 2\mathrm{Tr}\,(\partial_j\rho_\theta)(\partial_k\rho_\theta), \tag{7}$$

where $\partial_j = \partial/\partial\theta^j$. This metric is identical, up to a constant factor, to the Fubini–Study metric.

Proof We only need to prove (7). Differentiating $\rho_\theta = \rho_\theta^2$,

$$\partial_j\rho_\theta = (\partial_j\rho_\theta)\rho_\theta + \rho_\theta(\partial_j\rho_\theta). \tag{8}$$

This relation indicates that $2\partial_j\rho_\theta$ is a representative of the SLD. Then

$$(J^S_\theta)_{jk} = \langle 2\partial_j\rho_\theta, 2\partial_k\rho_\theta\rangle_{\rho_\theta} = 2\mathrm{Tr}\,\rho_\theta[(\partial_j\rho_\theta)(\partial_k\rho_\theta) + (\partial_k\rho_\theta)(\partial_j\rho_\theta)]. \tag{9}$$

Further, multiplying ρ_θ to (8), we have

$$\rho_\theta(\partial_j\rho_\theta)\rho_\theta = 0. \tag{10}$$

Therefore, by using (8) and (10),

$$\begin{aligned}(\partial_j\rho_\theta)(\partial_k\rho_\theta) &= [(\partial_j\rho_\theta)\rho_\theta + \rho_\theta(\partial_j\rho_\theta)][(\partial_k\rho_\theta)\rho_\theta + \rho_\theta(\partial_k\rho_\theta)] \\ &= (\partial_j\rho_\theta)\rho_\theta(\partial_k\rho_\theta) + \rho_\theta(\partial_j\rho_\theta)(\partial_k\rho_\theta)\rho_\theta.\end{aligned}$$

This, along with (9), leads to the relation (7). Denoting $\rho_\theta = |\psi\rangle\langle\psi|$

$$\mathrm{Tr}\,(\partial_j\rho_\theta)(\partial_k\rho_\theta) = 2[\mathrm{Re}\,\langle\partial_j\psi|\partial_k\psi\rangle + \langle\psi|\partial_j\psi\rangle\langle\psi|\partial_k\psi\rangle],$$

which is identical to the Fubini–Study metric [8][9]. ■

The Fubini–Study metric is known as a gauge invariant metric on a projective Hilbert space [10]. Theorem 1 gives another meaning of the Fubini–Study metric, i.e., the statistical distance. Wootters [11] also investigated from a statistical viewpoint the distance between two rays, and obtained $d(\psi, \varphi) = \cos^{-1} |\langle\psi|\varphi\rangle|$. This is identical, up to a constant factor, to the geodesic distance as measured by the Fubini–Study metric [12]. Theorem 1, together with the following Theorem 2, reveals a deeper connection between them.

4 Paremeter estimation of pure states

In this section, we give a parameter estimation theory of pure state models based on the SLD. Given a n parameter pure state model (3). In order to handle simultaneous probability distributions of possibly mutually non-commuting observables, an extended framework of measurement theory is needed [1, p. 53] [2, p. 50]. An *estimator* for θ is identified to a generalized measurement which takes values on Θ. The expectation vector with respect to the measurement M at the state ρ_θ is defined as

$$E_\theta[M] = \int \hat{\theta} P_\theta^M(d\hat{\theta}).$$

The measurement M is called *unbiased* if $E_\theta[M] = \theta$ holds for all $\theta \in \Theta$, i.e.,

$$\int \hat{\theta}^j P_\theta^M(d\hat{\theta}) = \theta^j, \quad (j = 1, \cdots, n). \tag{11}$$

Differentiation yields

$$\int \hat{\theta}^j \frac{\partial}{\partial\theta^k} P_\theta^M(d\hat{\theta}) = \delta_k^j, \quad (j, k = 1, \cdots, n). \tag{12}$$

If (11) and (12) hold at a certain θ, M is called *locally unbiased* at θ. Obviously, M is unbiased iff M is locally unbiased at every $\theta \in \Theta$. Letting M be a locally unbiased measurement at θ, we define the covariance matrix $V_\theta[M] = [v_\theta^{jk}] \in \mathbf{R}^{n\times n}$ with respect to M at the state ρ_θ by

$$v_\theta^{jk} = \int (\hat{\theta}^j - \theta^j)(\hat{\theta}^k - \theta^k) P_\theta^M(d\hat{\theta}). \tag{13}$$

A lower bound for $V_\theta[M]$ is given by the following theorem, which is a quantum version of Cramér–Rao theorem.

Theorem 2 *Given a pure state model ρ_θ, the following inequality holds for any locally unbiased measurement M:*

$$V_\theta[M] \geq \left(J_\theta^S\right)^{-1}. \tag{14}$$

Proof It is proved almost in the same way as the strictly positive case [2, p. 274], except that $\langle \cdot, \cdot \rangle_{\rho_\theta}$ is a pre-inner product now. ■

When the model is one dimensional, the measurement M is identified with a certain self-adjoint operator T, and the inequalities in the theorem become scalar, i.e.,

$$V_\theta[T] \geq \frac{1}{J_\theta^S}. \tag{15}$$

In this case, the lower bound $1/\mathrm{Tr}\,\rho_\theta(L_\theta^S)^2$ can be attained by the unbiased estimators

$$T = \theta I + \frac{2}{J_\theta^S}\frac{d\rho_\theta}{d\theta} + K_\theta, \quad {}^\forall K_\theta \in \mathcal{K}_{sa}(\rho_\theta), \tag{16}$$

where I is the identity. Since $d\rho_\theta/d\theta$ and K_θ do not commute in general, the measurement which attains the lower bound (15) is not determined uniquely. This fact provides significant features in the pure state estimation theory.

On the other hand, when the dimension $n \geq 2$, the matrix equality in (14) cannot be attained in general, because of the impossibility of the exact simultaneous measurement of non-commuting observables (in the von–Neumann's sense). We must, therefore, abandon the strategy of finding the measurement that minimizes the covariance matrix itself. Rather, we often adopt another strategy as follows: Given a positive definite real matrix $G = [g_{jk}] \in \mathbf{R}^{n\times n}$, find the measurement M that minimizes the quantity

$$\mathrm{tr}\; GV_\theta[M] = \sum_{jk} g_{jk} v_\theta^{jk}. \tag{17}$$

If there is a constant C such that tr $GV_\theta[M] \geq C$ holds for all M, C is called a Cramér–Rao type bound or simply a CR bound, which may depend on both G and θ. For instance, it is shown that the folowing quantity is a CR bound [7].

$$C^S = \mathrm{tr}\, G(J_\theta^S)^{-1}.$$

This bound is, however, not always the most informative one unless $n = 1$. For instance, it is shown that the CR bound based on the right logarithmic derivative is the most informative one for coherent models [13][14]. Anyhow, there have been few results that derived the most informative CR bounds, as is the strictly positive case. The construction of the general quantum parameter estimation theory for $n \geq 2$ is left to future study.

5 Examples

Here we give examples of one parameter pure state estimation.

5.1 Time–energy uncertainty relation

Let us consider a model of the form

$$\rho_\theta = e^{i\theta\mathcal{H}/\hbar}\rho_0\, e^{-i\theta\mathcal{H}/\hbar}.$$

Here, $\mathcal{H}$ is the time independent Hamiltonian of the system, $\hbar$ the Planck's constant, and θ the time parameter.

According to the one parameter estimation theory for strictly positive models [6],

$$V[T] \geq \frac{1}{J^S_\theta} \geq \frac{1}{J(L_\theta)} \tag{18}$$

holds, where L_θ is any logarithmic derivative which satisfy (4) and $J(L_\theta) = (L_\theta, L_\theta)_{\rho_\theta}$. The first equality is attained when and only when $T = \theta I + L^S_\theta / J^S_\theta$, and the second equality holds iff $L_\theta = L^S_\theta$.

Now, $L^A_\theta = -2i\mathcal{H}/\hbar$ is an ALD for the model and the corresponding Cramér–Rao inequality becomes

$$V_\theta[T] \geq \frac{\hbar^2}{4V_\theta[\mathcal{H}]}, \tag{19}$$

where T is an arbitrary unbiased estimator T for the time parameter θ. This inequality is nothing but a time-energy uncertainty relation. If $\rho_0 > 0$, then this lower bound cannot be attained for any T since L^A_θ is not an SLD, whereas Theorem 2 asserts that, if ρ_0 is pure, the equality in (19) is locally attainable. This is a significant difference between the strictly positive models and the pure state models. Since the ALD $L^A_\theta = -2i\mathcal{H}/\hbar$ and the SLD–Fisher information for the pure state models $J^S_\theta = 2\mathrm{Tr}\,(d\rho_\theta/d\theta)^2$ are both obtainable directly from the Liouville–von Neumann equation, this result is not specific to the case where the Hamiltonian is time independent, but is quite general.

5.2 Efficient estimator

An unbiased estimator T is called *efficient* if the equality in (15) holds for all $\theta \in \Theta$. Nagaoka [6] has proved that a one parameter model ρ_θ has an efficient estimator when and only when the model takes the form

$$\rho_\theta = e^{\frac{1}{2}[\beta(\theta)T-\gamma(\theta)]}\;\rho_0\; e^{\frac{1}{2}[\beta(\theta)T-\gamma(\theta)]}, \tag{20}$$

where $\beta(\theta)$, $\gamma(\theta)$ are real functions.

Let us consider a model of the form

$$\rho_\theta = e^{if(\theta)A}\rho_0\, e^{-if(\theta)A},$$

where $f(\theta)$ is a real monotonic odd function and $A \in \mathcal{L}_{sa}$. If $\rho_0 > 0$, then it is shown that there exists an efficient estimator for θ only when A is a canonical observable [15]. On the other hand, if ρ_0 is pure, then there may exist an efficient estimator even if A is not canonical, because of the uncertainty $K_\theta \in \mathcal{K}_{sa}(\rho_\theta)$ in (16). For instance, the spin 1/2 model

$$f(\theta) = \frac{1}{2}\left(\frac{\pi}{2} - \cos^{-1}\theta\right), \quad A = \sigma_y, \quad \rho_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

has an efficient estimator σ_z for the parameter θ. Indeed, this model admits another form

$$\rho_\theta = \sqrt{1-\theta^2}\ \exp\left[\frac{1}{4}\log\frac{1+\theta}{1-\theta}\sigma_z\right]\rho_0\ \exp\left[\frac{1}{4}\log\frac{1+\theta}{1-\theta}\sigma_z\right]. \tag{21}$$

This is not a paradox since, in the pure state model, the estimator which attains the Cramér–Rao bound (15) is adjustable for every points ρ_θ up to the uncertainty of the kernel $\mathcal{K}_{sa}(\rho_\theta)$, see also [15].

6 Conclusions

A quantum estimation theory of the pure state models was presented. We first investigated a general framework of the pure state estimation theory and derived quantum counterpart of the Fisher metric. The statistical significance of the Fubini–Study metric was also stressed. We then formulated the one parameter pure state estimation theory based on the symmetric logarithmic derivative and disclosed the characteristics of the pure states. Some examples were also given in order to demonstrate the one parameter pure state estimation, and clarified the difference between the pure state models and the strictly positive models. The construction of the general quantum multi-parameter estimation theory is, however, left to future study, as is the strictly positive model case.

References

[1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[2] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982) (in Russian, 1980).

[3] C. W. Helstrom, "Minimum Mean-Square Error Estimation in Quantum Statistics," Phys. Lett., **25A**, 101-102 (1967).

[4] H. P. H. Yuen and M. Lax, "Multiple-Parameter Quantum Estimation and Measurement of Non-Selfadjoint Observables," IEEE Trans., **IT-19**, 740-750 (1973).

[5] D. Petz, "Entropy in Quantum Probability I," Quantum Probability and Related Topics Vol. VII, pp. 275–297 (World Scientific, 1992).

[6] H. Nagaoka, "On Fisher information of quantum statistical models," SITA '87, 241-246 (1987) (in Japanese).

[7] H. Nagaoka, "A New Approach to Cramér-Rao Bounds for Quantum State Estimation," IEICE Technical Report, **IT89-42**,9-14(1989).

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, II* (John Wiley, New York, 1969).

[9] S. Abe, "Quantized geometry associated with uncertainty and correlation," Phys. Rev. **A48**, 4102–4106 (1993), and the references cited therein.

[10] J. P. Provost and G. Vallee, "Riemannian structure on Manifolds of quantum states," Comm. Math. Phys. **76**, 289–301 (1980).

[11] W. K. Wootters, "Statistical Distance and Hilbert space," Phys. Rev. **D23**, 357–362 (1981).

[12] J. Anandan and Y. Aharonov, "Geometry of quantum evolution," Phys. Rev. Lett. **65**, 1697–1700 (1990).

[13] A. Fujiwara, "Multi-parameter pure state estimation based on the right logarithmic derivative," METR **94-9**, Univ. Tokyo (1994).

[14] A. Fujiwara, "Linear random measurements of two non-commuting observables," METR **94-10**, Univ. Tokyo (1994).

[15] A. Fujiwara, "Information geometry of quantum states based on the symmetric logarithmic derivative," METR **94-11**, Univ. Tokyo (1994).