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Supplementary material to "Noncommutative Lebesgue decomposition and contiguity with applications in quantum statistics"

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This supplementary material is devoted to proofs of Remark 3.4, Theorem 4.4, Theorem 4.5, Lemma 5.6, Theorem 6.1, Theorem 7.1, Theorem 7.2, and Theorem 7.6 of [1].

Proof of Remark 3.4. Recall that σ is decomposed as $\sigma = E^*\tilde{\sigma}E$, where

$$E = \begin{pmatrix} I & 0 & 0 \\ 0 & I & \sigma_0^{-1}\alpha \\ 0 & 0 & I \end{pmatrix}, \qquad \tilde{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \beta - \alpha^*\sigma_0^{-1}\alpha \end{pmatrix}.$$

Then there is a unitary operator U that satisfies

$$\sqrt{\tilde{\sigma}}\, E = U\sqrt{\sigma},$$

and the operator R, modulo the singular part R_2, is given by

$$\begin{aligned} E^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_0 \# \rho_0^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} E &= E^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\sigma_0}\left(\sqrt{\sqrt{\sigma_0}\rho_0\sqrt{\sigma_0}}\right)^{-1}\sqrt{\sigma_0} & 0 \\ 0 & 0 & 0 \end{pmatrix} E \\ &= E^*\sqrt{\tilde{\sigma}}\left(\sqrt{\sqrt{\tilde{\sigma}}\rho\sqrt{\tilde{\sigma}}}\right)^+ \sqrt{\tilde{\sigma}}\, E \\ &= E^*\sqrt{\tilde{\sigma}}\left(\sqrt{\sqrt{\tilde{\sigma}}E\rho E^*\sqrt{\tilde{\sigma}}}\right)^+ \sqrt{\tilde{\sigma}}\, E \\ &= \sqrt{\sigma}\, U^*\left(\sqrt{U\sqrt{\sigma}\rho\sqrt{\sigma}U^*}\right)^+ U\sqrt{\sigma} \\ &= \sqrt{\sigma}\, U^*\left(U\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\, U^*\right)^+ U\sqrt{\sigma} \\ &= \sqrt{\sigma}\left(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right)^+ \sqrt{\sigma}. \end{aligned}$$

This proves the claim (3.14). □

Proof of Theorem 4.4. We first prove the 'if' part. Due to Remark 3.4, for each $n \in \mathbb{N}\cup\{\infty\}$, the operator

$$R^{(n)} := \sqrt{\sigma^{(n)}}\, Q^{(n)+}\sqrt{\sigma^{(n)}}$$

is a version of the square-root likelihood ratio $\mathcal{R}\left(\sigma^{(n)} \middle| \rho^{(n)}\right)$, where

$$Q^{(n)} := \sqrt{\sqrt{\sigma^{(n)}}\rho^{(n)}\sqrt{\sigma^{(n)}}}\,.$$

Let the spectral (Schatten) decomposition of $Q^{(n)}$ be

$$Q^{(n)} = \sum_{i=1}^{\dim \mathcal{H}} q_i^{(n)} E_i^{(n)}, \qquad (\mathrm{rank}\, E_i^{(n)} = 1)$$

where the eigenvalues are arranged in the increasing order. Take an arbitrary positive number λ that is smaller than the minimum positive eigenvalue of $Q^{(\infty)}$. Then there is an $N \in \mathbb{N}$ and an index d, $(1 \le d \le \dim \mathcal{H})$, such that for all $n \ge N$,

$$q_1^{(n)} \le q_2^{(n)} \le \cdots \le q_{d-1}^{(n)} < \lambda < q_d^{(n)} \le \cdots \le q_{\dim \mathcal{H}}^{(n)}$$

and, if $d \ge 2$, then $q_{d-1}^{(n)} \to 0$ as $n \to \infty$. Consequently, for $n \ge N$,

$$\mathbb{1}_\lambda(Q^{(n)}) = \sum_{i=1}^{d-1} E_i^{(n)} \underset{n\to\infty}{\longrightarrow} \sum_{i=1}^{d-1} E_i^{(\infty)} = \mathbb{1}_\lambda(Q^{(\infty)}) = \mathbb{1}_0(Q^{(\infty)}).$$

Let us introduce

$$O^{(n)} := \sqrt{\sigma^{(n)}}\, \mathbb{1}_\lambda(Q^{(n)}) Q^{(n)+} \sqrt{\sigma^{(n)}}.$$

Then it is shown that $O^{(n)} = o_{L^2}(\rho^{(n)})$. In fact,

$$\begin{aligned}
\mathrm{Tr}\, \rho^{(n)} O^{(n)2} &= \mathrm{Tr}\, \sigma^{(n)} \mathbb{1}_\lambda(Q^{(n)}) Q^{(n)+} Q^{(n)2} Q^{(n)+} \\
&\le \mathrm{Tr}\, \sigma^{(n)} \mathbb{1}_\lambda(Q^{(n)}) \\
&\to \mathrm{Tr}\, \sigma^{(\infty)} \mathbb{1}_0(Q^{(\infty)}) \\
&= \mathrm{Tr}\, \sigma^{(\infty)\perp} \\
&= 0.
\end{aligned}$$

Here, the inequality follows from

$$Q^{(n)+} Q^{(n)2} Q^{(n)+} = \sum_{i : q_i^{(n)} > 0} E_i^{(n)} = I - \mathbb{1}_0(Q^{(n)}),$$

the second last equality from

$$\begin{aligned}
\sigma^{(\infty)ac} &= R^{(\infty)} \rho^{(\infty)} R^{(\infty)} \\
&= \sqrt{\sigma^{(\infty)}}\, Q^{(\infty)+} Q^{(\infty)2} Q^{(\infty)+} \sqrt{\sigma^{(\infty)}} \\
&= \sqrt{\sigma^{(\infty)}} (I - \mathbb{1}_0(Q^{(\infty)})) \sqrt{\sigma^{(\infty)}},
\end{aligned}$$

and the last equality from $\sigma^{(\infty)} \ll \rho^{(\infty)}$.

We next introduce

$$\overline{R}^{(n)} := R^{(n)} - O^{(n)} = \sqrt{\sigma^{(n)}} \left(I - \mathbb{1}_\lambda(Q^{(n)}) \right) Q^{(n)+} \sqrt{\sigma^{(n)}}.$$

Then $\overline{R}^{(n)}$ is positive. Moreover, it is shown that $\mathrm{Tr}\, \rho^{(n)} \overline{R}^{(n)2} \to 1$ as $n \to \infty$. In fact,

$$\left(I - \mathbb{1}_\lambda(Q^{(n)}) \right) Q^{(n)+} = \left(\sum_{i : q_i^{(n)} > \lambda} E_i^{(n)} \right) \left(\sum_{i : q_i^{(n)} > 0} \frac{1}{q_i^{(n)}} E_i^{(n)} \right) = \sum_{i : q_i^{(n)} > \lambda} \frac{1}{q_i^{(n)}} E_i^{(n)}, \tag{S.1}$$

which converges to

$$\left(I - \mathbb{1}_\lambda(Q^{(\infty)}) \right) Q^{(\infty)+} = \sum_{i : q_i^{(\infty)} > \lambda} \frac{1}{q_i^{(\infty)}} E_i^{(\infty)}.$$

In addition, since

$$\mathbb{1}_\lambda(Q^{(\infty)})Q^{(\infty)+} = \left(\sum_{i:q_i^{(\infty)}=0} E_i^{(\infty)}\right)\left(\sum_{i:q_i^{(\infty)}>0} \frac{1}{q_i^{(\infty)}} E_i^{(\infty)}\right) = 0,$$

we have

$$\left(I - \mathbb{1}_\lambda(Q^{(n)})\right) Q^{(n)+} \longrightarrow Q^{(\infty)+}. \tag{S.2}$$

Thus

$$\overline{R}^{(n)} \longrightarrow \sqrt{\sigma^{(\infty)}}\, Q^{(\infty)+} \sqrt{\sigma^{(\infty)}} = R^{(\infty)},$$

so that

$$\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} \overline{R}^{(n)^2} = \operatorname{Tr} \rho^{(\infty)} R^{(\infty)^2} = \operatorname{Tr} \sigma^{(\infty)} = 1.$$

Here, the second equality follows from $\sigma^{(\infty)} \ll \rho^{(\infty)}$. This identity is combined with $O^{(n)} = o_{L^2}(\rho^{(n)})$ to conclude that $\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} R^{(n)^2} = 1$. Furthermore, due to (S.1), the family $\overline{R}^{(n)}$ is uniformly bounded, in that

$$\overline{R}^{(n)} \le \frac{1}{\lambda} \sigma^{(n)} \le \frac{1}{\lambda}.$$

Thus, the sequence $\overline{R}^{(n)^2}$ is uniformly integrable under $\rho^{(n)}$. This proves $\sigma^{(n)} \lhd \rho^{(n)}$.

We next prove the 'only if' part. Let $R^{(n)}$ be a version of the square-root likelihood ratio $\mathcal{R}\left(\sigma^{(n)} \mid \rho^{(n)}\right)$. Due to assumption, there is an L^2-infinitesimal sequence $O^{(n)}$ of observables such that $\sigma^{(n)} \lhd_{O^{(n)}} \rho^{(n)}$. Let

$$\overline{R}^{(n)} = \sum_{i=1}^{\dim \mathcal{H}} r_i^{(n)} E_i^{(n)}, \qquad (\operatorname{rank} E_i^{(n)} = 1)$$

be the spectral (Schatten) decomposition of $\overline{R}^{(n)} = R^{(n)} + O^{(n)}$, where the eigenvalues are arranged in the increasing order, so that

$$r_1^{(n)} \le r_2^{(n)} \le \cdots \le r_{\dim \mathcal{H}}^{(n)}.$$

Let us choose the index d, $(1 \le d \le \dim \mathcal{H})$, that satisfies

$$\sup \left\{ r_d^{(n)} \,\middle|\, n \in \mathbb{N} \right\} < \infty \qquad \text{and} \qquad \sup \left\{ r_{d+1}^{(n)} \,\middle|\, n \in \mathbb{N} \right\} = \infty,$$

and let us define

$$A^{(n)} := \sum_{i=1}^{d} r_i^{(n)} E_i^{(n)} \qquad \text{and} \qquad B^{(n)} := \sum_{i=d+1}^{\dim \mathcal{H}} r_i^{(n)} E_i^{(n)}.$$

Then $A^{(n)}$ is the uniformly bounded part of $\overline{R}^{(n)}$, and $\overline{R}^{(n)} = A^{(n)} + B^{(n)}$.

Take a convergent subsequence $A^{(n_k)}$ of $A^{(n)}$, so that

$$A_{(\infty)} := \lim_{k\to\infty} A^{(n_k)}.$$

Then for any M that is greater than $M_0 := \sup \left\{ r_d^{(n)} \,\middle|\, n \in \mathbb{N} \right\}$,

$$\lim_{k\to\infty} \overline{R}^{(n_k)} \mathbb{1}_M(\overline{R}^{(n_k)}) = A_{(\infty)}.$$

It then follows from the assumption $\sigma^{(n)} \lhd_{O^{(n)}} \rho^{(n)}$ that

$$\operatorname{Tr} \rho^{(\infty)} A_{(\infty)}^2 = \lim_{M\to\infty} \lim_{k\to\infty} \operatorname{Tr} \rho^{(n_k)} \overline{R}^{(n_k)^2} \mathbb{1}_M(\overline{R}^{(n_k)}) = 1. \tag{S.3}$$

Furthermore, since

$$\operatorname{Tr} \rho^{(n)} \overline{R}^{(n)^2} = \operatorname{Tr} \rho^{(n)} (A^{(n)} + B^{(n)})^2 = \operatorname{Tr} \rho^{(n)} A^{(n)^2} + \operatorname{Tr} \rho^{(n)} B^{(n)^2},$$

we see that $B^{(n_k)} = o_{L^2}(\rho^{(n_k)})$, and so is $C^{(n_k)} := R^{(n_k)} - A^{(n_k)} = B^{(n_k)} - O^{(n_k)}$. As a consequence, for any unit vector $x \in \mathcal{H}$,

$$\begin{aligned}&\left\langle x \,\middle|\, R^{(n_k)}\rho^{(n_k)}R^{(n_k)}x\right\rangle\\ &\quad= \left\langle x \,\middle|\, A^{(n_k)}\rho^{(n_k)}A^{(n_k)}x\right\rangle + 2\,\mathrm{Re}\left\langle x \,\middle|\, A^{(n_k)}\rho^{(n_k)}C^{(n_k)}x\right\rangle + \left\langle x \,\middle|\, C^{(n_k)}\rho^{(n_k)}C^{(n_k)}x\right\rangle\\ &\quad\longrightarrow \left\langle x \,\middle|\, A_{(\infty)}\rho^{(\infty)}A_{(\infty)}x\right\rangle\end{aligned}$$

as $k \to \infty$. In fact

$$\left|\left\langle x \,\middle|\, C^{(n_k)}\rho^{(n_k)}C^{(n_k)}x\right\rangle\right| \le \mathrm{Tr}\, C^{(n_k)}\rho^{(n_k)}C^{(n_k)} \longrightarrow 0$$

and, due to the Schwartz inequality,

$$\left|\left\langle x \,\middle|\, A^{(n_k)}\rho^{(n_k)}C^{(n_k)}x\right\rangle\right|^2 \le \left\langle x \,\middle|\, A^{(n_k)}\rho^{(n_k)}A^{(n_k)}x\right\rangle \left\langle x \,\middle|\, C^{(n_k)}\rho^{(n_k)}C^{(n_k)}x\right\rangle \longrightarrow 0.$$

It then follows from the inequality

$$\sigma^{(n_k)} \ge R^{(n_k)}\rho^{(n_k)}R^{(n_k)}$$

that

$$0 \le \left\langle x \,\middle|\, \left(\sigma^{(n_k)} - R^{(n_k)}\rho^{(n_k)}R^{(n_k)}\right)x\right\rangle \underset{k\to\infty}{\longrightarrow} \left\langle x \,\middle|\, \left(\sigma^{(\infty)} - A_{(\infty)}\rho^{(\infty)}A_{(\infty)}\right)x\right\rangle.$$

Since $x \in \mathcal{H}$ is arbitrary, we have

$$\sigma^{(\infty)} \ge A_{(\infty)}\rho^{(\infty)}A_{(\infty)}.$$

Combining this inequality with (S.3), we conclude that

$$\sigma^{(\infty)} = A_{(\infty)}\rho^{(\infty)}A_{(\infty)}.$$

This implies that $\sigma^{(\infty)} \ll \rho^{(\infty)}$. □

Proof of Theorem 4.5. We first prove the 'if' part. Let

$$\overline{R}^{(n)} = R^{(n)} = \sqrt{\sigma^{(n)}}\sqrt{\sqrt{\sigma^{(n)}}\rho^{(n)}\sqrt{\sigma^{(n)}}}^{+}\sqrt{\sigma^{(n)}}.$$

Due to assumption, there is an $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $n \ge N$ implies $\mathrm{Tr}\,\rho^{(n)}\sigma^{(n)} > \varepsilon$. Since $\rho^{(n)}$ is pure, the operator $\sqrt{\sigma^{(n)}}\rho^{(n)}\sqrt{\sigma^{(n)}}$ is rank-one, and its positive eigenvalue is greater than ε. Thus

$$\overline{R}^{(n)} \le \frac{1}{\sqrt{\varepsilon}}\sigma^{(n)} \le \frac{1}{\sqrt{\varepsilon}}$$

for all $n \ge N$. This implies that $\overline{R}^{(n)}$ is uniformly bounded, so that $\overline{R}^{(n)^2}$ is uniformly integrable.

We next prove the 'only if' part. Due to assumption, there is an L^2-infinitesimal sequence $O^{(n)}$ of observables such that $\sigma^{(n)} \lhd_{O^{(n)}} \rho^{(n)}$. Let

$$\overline{R}^{(n)} = \sum_i r_i^{(n)} E_i^{(n)}$$

be the spectral decomposition of $\overline{R}^{(n)} = R^{(n)} + O^{(n)}$, and let $\rho^{(n)} = \left|\psi^{(n)}\right\rangle\left\langle\psi^{(n)}\right|$ for some unit vector $\psi^{(n)} \in \mathcal{H}^{(n)}$. Since $\lim_{n\to\infty} \mathrm{Tr}\,\rho^{(n)}R^{(n)^2} = 1$ is equivalent to $\lim_{n\to\infty} \mathrm{Tr}\,\rho^{(n)}\overline{R}^{(n)^2} = 1$, we have

$$\lim_{n\to\infty}\sum_i r_i^{(n)^2} p_i^{(n)} = 1,$$

where $p_i^{(n)} := \left\langle \psi^{(n)} \,\middle|\, E_i^{(n)}\psi^{(n)}\right\rangle$. Further, since $\overline{R}^{(n)^2}$ is uniformly integrable, for any $\varepsilon > 0$, there exists an $M > 0$ such that

$$\limsup_{n\to\infty} \sum_{i:\, r_i^{(n)} > M} r_i^{(n)^2} p_i^{(n)} < \varepsilon.$$

It then follows that

$$\begin{aligned}
\liminf_{n\to\infty} \sqrt{\operatorname{Tr}\rho^{(n)}\sigma^{(n)}} &\geq \liminf_{n\to\infty} \sqrt{\operatorname{Tr}\rho^{(n)}R^{(n)}\rho^{(n)}R^{(n)}} \\
&= \liminf_{n\to\infty} \left\langle \psi^{(n)} \middle| R^{(n)} \middle| \psi^{(n)} \right\rangle \\
&= \liminf_{n\to\infty} \left\langle \psi^{(n)} \middle| \overline{R}^{(n)} \middle| \psi^{(n)} \right\rangle \\
&= \liminf_{n\to\infty} \sum_i r_i^{(n)} p_i^{(n)} \\
&\geq \liminf_{n\to\infty} \sum_{i:\, r_i^{(n)} \leq M} r_i^{(n)} p_i^{(n)} \\
&\geq \liminf_{n\to\infty} \sum_{i:\, r_i^{(n)} \leq M} \frac{{r_i^{(n)}}^2}{M} p_i^{(n)} \\
&= \frac{1}{M}\left(1 - \limsup_{n\to\infty} \sum_{i:\, r_i^{(n)} > M} {r_i^{(n)}}^2\; p_i^{(n)}\right) \\
&> \frac{1}{M}(1-\varepsilon).
\end{aligned}$$

This completes the proof. □

Proof of Lemma 5.6. We shall prove the following series of equalities for any $\{\xi_t\}_{t=1}^r \subset \mathbb{R}^d$ and $\eta_1, \eta_2 \in \mathbb{R}$:

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{Tr}\rho^{(n)} e^{\sqrt{-1}\eta_1\left(Z^{(n)}+O^{(n)}\right)} \left\{\prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}}\right\} e^{\sqrt{-1}\eta_2\left(Z^{(n)}+O^{(n)}\right)} \\
&\quad= \lim_{n\to\infty} \operatorname{Tr}\rho^{(n)} e^{\sqrt{-1}\eta_1\left(Z^{(n)}+O^{(n)}\right)} \left\{\prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}}\right\} e^{\sqrt{-1}\eta_2 Z^{(n)}} \\
&\quad= \lim_{n\to\infty} \operatorname{Tr}\rho^{(n)} e^{\sqrt{-1}\eta_1 Z^{(n)}} \left\{\prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}}\right\} e^{\sqrt{-1}\eta_2 Z^{(n)}}.
\end{aligned}$$

The first equality follows from the Schwartz inequality and (5.2):

$$\begin{aligned}
&\left|\operatorname{Tr}\rho^{(n)} e^{\sqrt{-1}\eta_1\left(Z^{(n)}+O^{(n)}\right)} \left\{\prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}}\right\} \left\{e^{\sqrt{-1}\eta_2\left(Z^{(n)}+O^{(n)}\right)} - e^{\sqrt{-1}\eta_2 Z^{(n)}}\right\}\right|^2 \\
&\leq \operatorname{Tr}\rho^{(n)} \left\{e^{\sqrt{-1}\eta_2\left(Z^{(n)}+O^{(n)}\right)} - e^{\sqrt{-1}\eta_2 Z^{(n)}}\right\}^* \left\{e^{\sqrt{-1}\eta_2\left(Z^{(n)}+O^{(n)}\right)} - e^{\sqrt{-1}\eta_2 Z^{(n)}}\right\} \\
&= 2 - 2\operatorname{Re}\operatorname{Tr}\rho^{(n)} e^{-\sqrt{-1}\eta_2\left(Z^{(n)}+O^{(n)}\right)} e^{\sqrt{-1}\eta_2 Z^{(n)}} \\
&\longrightarrow 2 - 2\operatorname{Re}\operatorname{Tr}\rho^{(n)} e^{-\sqrt{-1}\eta_2 Z^{(n)}} e^{\sqrt{-1}\eta_2 Z^{(n)}} = 0.
\end{aligned}$$

The proof of the second equality is similar. □

Proof of Theorem 6.1. We first prove that ψ is a well-defined normal state. Let $\overline{R}^{(n)} := R^{(n)} + O^{(n)}$. It then follows from assumption (ii) and the sandwiched version of the quantum Lévy-Cramér theorem (Lemma 5.3) that

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{Tr}\rho^{(n)} \mathbb{1}_M\left(\overline{R}^{(n)}\right) \overline{R}^{(n)} \left\{\prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}}\right\} \overline{R}^{(n)} \mathbb{1}_M\left(\overline{R}^{(n)}\right) \qquad \text{(S.4)} \\
&\quad= \phi\left(\mathbb{1}_M\left(R^{(\infty)}\right) R^{(\infty)} \left\{\prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(\infty)}}\right\} R^{(\infty)} \mathbb{1}_M\left(R^{(\infty)}\right)\right),
\end{aligned}$$

where M is taken to be a non-atomic point of the probability measure μ having the characteristic function $\varphi_\mu(\eta) := \phi(e^{\sqrt{-1}\eta R^{(\infty)}})$. Setting $\xi_t = 0$ for all t, taking the limit $M \to \infty$, and recalling the uniform integrability of $\overline{R}^{(n)^2}$ as well as the identity $\lim_{n\to\infty} \mathrm{Tr}\, \rho^{(n)} \overline{R}^{(n)^2} = 1$, we have

$$\lim_{M\to\infty} \phi\left(\mathbb{1}_M(R^{(\infty)}) R^{(\infty)^2}\right) = 1. \tag{S.5}$$

Let ρ be the density operator that represents the state ϕ. For notational simplicity, we set $R := R^{(\infty)}$ and $R_M := \mathbb{1}_M(R)R$. Then, for any $A \in \mathcal{B}(\mathcal{H}^{(\infty)})$,

$$\phi(R_M A R_M) = \mathrm{Tr}\, \rho R_M A R_M = (R_M\sqrt{\rho}, AR_M\sqrt{\rho})_{\mathrm{HS}},$$

where $(B, C)_{\mathrm{HS}} := \mathrm{Tr}\, B^* C$ is the Hilbert-Schmidt inner product. To verify the well-definedness of ψ, it suffices to prove that $\phi(RAR)$ exists and

$$\phi(RAR) = \lim_{M\to\infty} \phi(R_M A R_M)$$

for any $A \in \mathcal{B}(\mathcal{H}^{(\infty)})$. To put it differently, it suffices to prove that $\left\| R\sqrt{\rho} \right\|_{\mathrm{HS}} = 1$, and that $\left\| R_M\sqrt{\rho} - R\sqrt{\rho} \right\|_{\mathrm{HS}} \to 0$ as $M \to \infty$, where $\|\cdot\|_{\mathrm{HS}} := \sqrt{(\,\cdot\,,\,\cdot\,)_{\mathrm{HS}}}$. Let

$$R = \int_0^\infty \lambda\, dE_\lambda$$

be the spectral decomposition of R, and let $d\nu(\lambda) := \phi(dE_\lambda)$ be the induced probability measure on $\mathbb{R}$. It then follows from (S.5) that

$$\|R\sqrt{\rho}\|_{\mathrm{HS}}^2 = \mathrm{Tr}\, \rho R^2 = \int_0^\infty \lambda^2\, d\nu(\lambda) = \lim_{M\to\infty} \int_0^M \lambda^2\, d\nu(\lambda) = \lim_{M\to\infty} \phi(R_M^2) = 1,$$

and that

$$\|R_M\sqrt{\rho} - R\sqrt{\rho}\|_{\mathrm{HS}}^2 = \mathrm{Tr}\, \rho R^2 - \mathrm{Tr}\, \rho R_M^2 = 1 - \phi(R_M^2) \longrightarrow 0$$

as $M \to \infty$.

We next show that for any $\varepsilon > 0$ there is an $M > 0$ that satisfies

$$\begin{aligned} \sup_n \Bigg| \mathrm{Tr}\, \rho^{(n)} \overline{R}^{(n)} \left\{ \prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} \overline{R}^{(n)} \\ - \mathrm{Tr}\, \rho^{(n)} \mathbb{1}_M\left(\overline{R}^{(n)}\right) \overline{R}^{(n)} \left\{ \prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} \overline{R}^{(n)} \mathbb{1}_M\left(\overline{R}^{(n)}\right) \Bigg| < \varepsilon. \end{aligned} \tag{S.6}$$

In fact,

$$\begin{aligned} (\mathrm{LHS}) \quad \le \quad & \sup_n \left| \mathrm{Tr}\, \rho^{(n)} \overline{R}^{(n)} \left\{ \prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} \left\{ \overline{R}^{(n)} - \overline{R}^{(n)} \mathbb{1}_M\left(\overline{R}^{(n)}\right) \right\} \right| \\ & + \sup_n \left| \mathrm{Tr}\, \rho^{(n)} \left\{ \overline{R}^{(n)} - \mathbb{1}_M\left(\overline{R}^{(n)}\right) \overline{R}^{(n)} \right\} \left\{ \prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} \overline{R}^{(n)} \mathbb{1}_M\left(\overline{R}^{(n)}\right) \right|, \end{aligned}$$

and by using the uniform integrability of $\overline{R}^{(n)^2}$, we see that

$$(\text{first term in RHS}) \le \sup_n \sqrt{\mathrm{Tr}\, \rho^{(n)} \overline{R}^{(n)^2}} \sqrt{\mathrm{Tr}\, \rho^{(n)} \left(I - \mathbb{1}_M(\overline{R}^{(n)})\right) \overline{R}^{(n)^2}} < \frac{\varepsilon}{2},$$

and

$$(\text{second term in RHS}) \le \sup_n \sqrt{\mathrm{Tr}\, \rho^{(n)} \left(I - \mathbb{1}_M(\overline{R}^{(n)})\right) \overline{R}^{(n)^2}} \sqrt{\mathrm{Tr}\, \rho^{(n)} \mathbb{1}_M(\overline{R}^{(n)}) \overline{R}^{(n)^2}} < \frac{\varepsilon}{2}.$$

An important consequence of (S.6) is the following identity

$$\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} \overline{R}^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} \overline{R}^{(n)} = \psi \left(\left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(\infty)}} \right\} \right), \tag{S.7}$$

which follows by taking the limit $M \to \infty$ in (S.4).

We next observe that

$$\begin{aligned} \lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} \overline{R}^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} \overline{R}^{(n)} &= \lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} R^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} \overline{R}^{(n)} \\ &= \lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} R^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} R^{(n)}. \end{aligned} \tag{S.8}$$

In fact, the first equality follows from

$$\left| \operatorname{Tr} \rho^{(n)} O^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} \overline{R}^{(n)} \right| \le \sqrt{\operatorname{Tr} \rho^{(n)} O^{(n)2}} \sqrt{\operatorname{Tr} \rho^{(n)} \overline{R}^{(n)^2}} \longrightarrow 0,$$

and the second from

$$\left| \operatorname{Tr} \rho^{(n)} R^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} O^{(n)} \right| \le \sqrt{\operatorname{Tr} \rho^{(n)} R^{(n)2}} \sqrt{\operatorname{Tr} \rho^{(n)} O^{(n)2}} \longrightarrow 0.$$

We further observe that

$$\lim_{n\to\infty} \operatorname{Tr} \sigma^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} = \lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} R^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} R^{(n)}. \tag{S.9}$$

In fact,

$$\begin{aligned} \left| \operatorname{Tr} \sigma^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} - \operatorname{Tr} \rho^{(n)} R^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} R^{(n)} \right| &\le \operatorname{Tr} \left| \sigma^{(n)} - R^{(n)} \rho^{(n)} R^{(n)} \right| \\ &= 1 - \operatorname{Tr} \rho^{(n)} R^{(n)^2} \longrightarrow 0. \end{aligned}$$

Combining (S.9), (S.8), and (S.7), we have

$$\lim_{n\to\infty} \operatorname{Tr} \sigma^{(n)} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} = \psi \left(\prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(\infty)}} \right). \tag{S.10}$$

This completes the proof. □

Proof of Theorem 7.1. Let

$$R^{(n)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_0^{(n)} & R_1^{(n)} \\ 0 & R_1^{(n)*} & R_2^{(n)} \end{pmatrix}$$

be a version of the square-root likelihood ratio $\mathcal{R}\left(\sigma^{(n)} \middle| \rho^{(n)}\right)$ that satisfies

$$R^{(n)} \rho^{(n)} R^{(n)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_0^{(n)} \rho_0^{(n)} R_0^{(n)} & R_0^{(n)} \rho_0^{(n)} R_1^{(n)} \\ 0 & R_1^{(n)*} \rho_0^{(n)} R_0^{(n)} & R_1^{(n)*} \rho_0^{(n)} R_1^{(n)} \end{pmatrix} \le \sigma^{(n)} \tag{S.11}$$

and

$$\left(\sigma^{(n)} - R^{(n)} \rho^{(n)} R^{(n)} \right) \perp \rho^{(n)}. \tag{S.12}$$

Since $R_1^{(n)^*}\rho_0^{(n)}R_1^{(n)} \le \sigma_2^{(n)}$ and $\lim_{n\to\infty}\operatorname{Tr}\sigma_2^{(n)}=0$, we see that

$$\lim_{n\to\infty}\operatorname{Tr}\rho_0^{(n)}R_1^{(n)}R_1^{(n)^*}=0. \tag{S.13}$$

Further, let

$$\tilde\sigma_0^{(n)}:=\frac{\sigma_0^{(n)}}{\operatorname{Tr}\sigma_0^{(n)}},\qquad \tilde\rho_0^{(n)}:=\frac{\rho_0^{(n)}}{\operatorname{Tr}\rho_0^{(n)}},\qquad \tilde R_0^{(n)}:=\frac{1}{\kappa^{(n)}}R_0^{(n)}$$

where

$$\kappa^{(n)}=\sqrt{\frac{\operatorname{Tr}\sigma_0^{(n)}}{\operatorname{Tr}\rho_0^{(n)}}}.$$

Then it follows from (S.11) and (S.12) that $\tilde R_0^{(n)}\tilde\rho_0^{(n)}\tilde R_0^{(n)}\le\tilde\sigma_0^{(n)}$ and $\left(\tilde\sigma_0^{(n)}-\tilde R_0^{(n)}\tilde\rho_0^{(n)}\tilde R_0^{(n)}\right)\perp\tilde\rho_0^{(n)}$. This implies that $\tilde R_0^{(n)}$ is a version of the square-root likelihood ratio $\mathcal{R}\left(\tilde\sigma_0^{(n)}\middle|\tilde\rho_0^{(n)}\right)$.

The assumption $\tilde\sigma_0^{(n)}\lhd\tilde\rho_0^{(n)}$ ensures the existence of a sequence $O_0^{(n)}=o_{L^2}(\tilde\rho_0^{(n)})$ such that $\tilde\sigma_0^{(n)}\lhd_{O_0^{(n)}}\tilde\rho_0^{(n)}$. Let $\overline R_0^{(n)}:=\tilde R_0^{(n)}+O_0^{(n)}$, and let

$$\overline R^{(n)}=\begin{pmatrix}0&0&0\\0&\kappa^{(n)}\overline R_0^{(n)}&0\\0&0&0\end{pmatrix}.$$

Then we see that

$$O^{(n)}:=\overline R^{(n)}-R^{(n)}=\begin{pmatrix}0&0&0\\0&\kappa^{(n)}O_0^{(n)}&-R_1^{(n)}\\0&-R_1^{(n)^*}&-R_2^{(n)}\end{pmatrix}$$

is L^2-infinitesimal with respect to $\rho^{(n)}$. In fact, due to (S.13),

$$\lim_{n\to\infty}\operatorname{Tr}\rho^{(n)}O^{(n)^2}=\lim_{n\to\infty}\operatorname{Tr}\rho_0^{(n)}\left\{\kappa^{(n)^2}O_0^{(n)^2}+R_1^{(n)}R_1^{(n)^*}\right\}=0.$$

Furthermore,

$$\lim_{n\to\infty}\operatorname{Tr}\rho^{(n)}\overline R^{(n)^2}=\lim_{n\to\infty}\kappa^{(n)^2}\operatorname{Tr}\rho_0^{(n)}\overline R_0^{(n)^2}=\lim_{n\to\infty}(\operatorname{Tr}\sigma_0^{(n)})\operatorname{Tr}\tilde\rho_0^{(n)}\overline R_0^{(n)^2}=1,$$

and

$$\begin{aligned}\lim_{M\to\infty}\liminf_{n\to\infty}\operatorname{Tr}\rho^{(n)}\overline R^{(n)^2}\mathbb{1}_M(\overline R^{(n)}) &= \lim_{M\to\infty}\liminf_{n\to\infty}\kappa^{(n)^2}\operatorname{Tr}\rho_0^{(n)}\overline R_0^{(n)^2}\mathbb{1}_M(\kappa^{(n)}\overline R_0^{(n)})\\ &= \lim_{M\to\infty}\liminf_{n\to\infty}(\operatorname{Tr}\sigma_0^{(n)})\operatorname{Tr}\tilde\rho_0^{(n)}\overline R_0^{(n)^2}\mathbb{1}_{M/\kappa^{(n)}}(\overline R_0^{(n)})\\ &\ge \lim_{M\to\infty}\liminf_{n\to\infty}(\operatorname{Tr}\sigma_0^{(n)})\operatorname{Tr}\tilde\rho_0^{(n)}\overline R_0^{(n)^2}\mathbb{1}_{\lambda M}(\overline R_0^{(n)})=1,\end{aligned}$$

where

$$\lambda:=\liminf_{n\to\infty}\frac{1}{\kappa^{(n)}}=\liminf_{n\to\infty}\sqrt{\operatorname{Tr}\rho_0^{(n)}}>0.$$

Thus $\sigma^{(n)}\lhd_{O^{(n)}}\rho^{(n)}$. □

Proof of Theorem 7.2. We first prove the 'only if' part. Due to assumption, there is an L^2-infinitesimal sequence $O^{(n)}$ of observables satisfying the condition that for any $\varepsilon>0$, there is an $M>0$ such that

$$\liminf_{n\to\infty}\operatorname{Tr}\rho^{(n)}\mathbb{1}_M(\overline R^{(n)})\overline R^{(n)^2}>1-\varepsilon,$$

where $\overline{R}^{(n)} := R^{(n)} + O^{(n)}$ with $R^{(n)} := \bigotimes_{i=1}^{n} R_i$. It then follows that

$$\begin{aligned}\prod_{i=1}^{\infty} \operatorname{Tr} \rho_i R_i &= \lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} R^{(n)} \\ &= \lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} \overline{R}^{(n)} \\ &\geq \liminf_{n\to\infty} \operatorname{Tr} \rho^{(n)} \overline{R}^{(n)} \mathbb{1}_M(\overline{R}^{(n)}) \\ &\geq \liminf_{n\to\infty} \operatorname{Tr} \rho^{(n)} \frac{\overline{R}^{(n)^2}}{M} \mathbb{1}_M(\overline{R}^{(n)}) \\ &> \frac{1}{M}(1-\varepsilon).\end{aligned}$$

Further, the equivalence of (7.1) and (7.2) is well known, (see [4, Section 14.12], for example).

We next prove the 'if' part. Since $\sigma^{(n)} \ll \rho^{(n)}$, we have $\operatorname{Tr} \rho^{(n)} R^{(n)^2} = 1$ for all n. It then suffices to prove that $R^{(n)^2}$ is uniformly integrable under $\rho^{(n)}$. For each $i \in \mathbb{N}$, let

$$R_i = \sum_{x\in\mathcal{X}_i} r_i(x) \left|\psi_i(x)\right\rangle \left\langle\psi_i(x)\right|$$

be a Schatten decomposition of R_i, where $\mathcal{X}_i = \{1, \ldots, \dim \mathcal{H}_i\}$ is a standard reference set that put labels on the eigenvalues $r_i(x)$ and eigenvectors $\psi_i(x)$. Note that the totality $\{\psi_i(x)\}_{x\in\mathcal{X}_i}$ of eigenvectors forms an orthonormal basis of $\mathcal{H}_i$. Let

$$p_i(x) := \langle \psi_i(x) \,|\rho_i \psi_i(x)\rangle \,, \qquad q_i(x) := \langle \psi_i(x) \,|\sigma_i \psi_i(x)\rangle \,.$$

Then $P_i := (p_i(x))_{x\in\mathcal{X}_i}$ and $Q_i := (q_i(x))_{x\in\mathcal{X}_i}$ are regarded as classical probability distributions on $\mathcal{X}_i$. Due to the identity $\sigma_i = R_i \rho_i R_i$, we have

$$q_i(x) = p_i(x) r_i(x)^2, \qquad (\forall x \in \mathcal{X}_i),$$

which implies that $Q_i \ll P_i$ for all $i \in \mathbb{N}$. Now, since

$$\operatorname{Tr} \rho_i R_i = \sum_{x\in\mathcal{X}_i} p_i(x) r_i(x) = \sum_{x\in\mathcal{X}_i} \sqrt{p_i(x) q_i(x)},$$

assumption (7.1) is equivalent to

$$\prod_{i=1}^{\infty} \left(\sum_{x\in\mathcal{X}_i} \sqrt{p_i(x) q_i(x)} \right) > 0.$$

This is nothing but the celebrated Kakutani criterion for the infinite product measure $\prod_i Q_i$ to be absolutely continuous to $\prod_i P_i$, (cf. [3, 4]). As a consequence, the classical likelihood ratio process

$$L^{(n)}(X_1, \ldots, X_n) := \prod_{i=1}^{n} \frac{q_i(X_i)}{p_i(X_i)}$$

is uniformly integrable under $\prod_i P_i$, (cf. [4, Section 14.17]). The uniform integrability of $R^{(n)^2}$ under $\rho^{(n)}$ now follows immediately from the identity

$$\operatorname{Tr} \rho^{(n)} \mathbb{1}_M(R^{(n)}) R^{(n)^2} = E_{P^{(n)}} \left[\mathbb{1}_{M^2}(L^{(n)}) L^{(n)} \right],$$

where $P^{(n)} := \prod_{i=1}^{n} P_i$. □

Proof of Theorem 7.6. Since the symmetric logarithmic derivative L_i at θ_0 satisfies $\mathrm{Tr}\,\rho_{\theta_0} L_i = 0$ for all $i \in \{1, \dots, d\}$, the property (i) in Definition 7.4 is an immediate consequence of an i.i.d. version of the quantum central limit theorem [2, 5].

In order to prove (ii) in Definition 7.4, we first calculate the square-root likelihood ratio $\mathcal{R}\left(\rho_\theta^{\otimes n} \middle| \rho_{\theta_0}^{\otimes n}\right)$ between $\rho_\theta^{\otimes n}$ and $\rho_{\theta_0}^{\otimes n}$. Let $\rho_\theta = \rho_\theta^{ac} + \rho_\theta^{\perp}$ be the Lebesgue decomposition with respect to ρ_{θ_0}. Then

$$\rho_\theta^{\otimes n} \ge (\rho_\theta^{ac})^{\otimes n} = (R_\theta \rho_{\theta_0} R_\theta)^{\otimes n} = R_\theta^{\otimes n}\, \rho_{\theta_0}^{\otimes n}\, R_\theta^{\otimes n}, \tag{S.14}$$

where $R_\theta = \mathcal{R}\left(\rho_\theta | \rho_{\theta_0}\right)$. On the other hand,

$$\mathrm{Tr}\,\rho_{\theta_0}\rho_\theta = \mathrm{Tr}\,\rho_{\theta_0}\rho_\theta^{ac} + \mathrm{Tr}\,\rho_{\theta_0}\rho_\theta^{\perp} = \mathrm{Tr}\,\rho_{\theta_0}\rho_\theta^{ac} = \mathrm{Tr}\,\rho_{\theta_0}\left(R_\theta \rho_{\theta_0} R_\theta\right).$$

Therefore,

$$\mathrm{Tr}\,\rho_{\theta_0}^{\otimes n}\left[\rho_\theta^{\otimes n} - (R_\theta \rho_{\theta_0} R_\theta)^{\otimes n}\right] = (\mathrm{Tr}\,\rho_{\theta_0}\rho_\theta)^n - \left(\mathrm{Tr}\,\rho_{\theta_0}\left(R_\theta \rho_{\theta_0} R_\theta\right)\right)^n = 0.$$

Due to Lemma 2.1, this implies that

$$\rho_{\theta_0}^{\otimes n} \perp \left[\rho_\theta^{\otimes n} - (R_\theta \rho_{\theta_0} R_\theta)^{\otimes n}\right]. \tag{S.15}$$

From (S.14) and (S.15), we have the quantum Lebesgue decomposition

$$\rho_\theta^{\otimes n} = (\rho_\theta^{\otimes n})^{ac} + (\rho_\theta^{\otimes n})^{\perp}$$

with respect to $\rho_{\theta_0}^{\otimes n}$, where

$$(\rho_\theta^{\otimes n})^{ac} = R_\theta^{\otimes n}\, \rho_{\theta_0}^{\otimes n}\, R_\theta^{\otimes n} \qquad \text{and} \qquad (\rho_\theta^{\otimes n})^{\perp} = \rho_\theta^{\otimes n} - R_\theta^{\otimes n}\, \rho_{\theta_0}^{\otimes n}\, R_\theta^{\otimes n}.$$

Consequently, $R_\theta^{\otimes n}$ gives a version of the square-root likelihood ratio $\mathcal{R}\left(\rho_\theta^{\otimes n} \middle| \rho_{\theta_0}^{\otimes n}\right)$.

Let us proceed to the proof of (ii) in Definition 7.4. Since R_h is differentiable at $h = 0$ and $R_0 = I$, it is expanded as

$$R_h = I + \frac{1}{2} A_i h^i + o(\|h\|).$$

Due to assumption (7.7),

$$\rho_{\theta_0 + h} = R_h \rho_{\theta_0} R_h + o(\|h\|^2) = \rho_{\theta_0} + \frac{1}{2}\left(A_i \rho_{\theta_0} + \rho_{\theta_0} A_i\right) h^i + o(\|h\|).$$

As a consequence, the selfadjoint operator A_i is also a version of the ith SLD at θ_0. To evaluate the higher order term of R_h, let

$$B(h) := R_h - I - \frac{1}{2} A_i h^i.$$

Then

$$\begin{aligned}
\mathrm{Tr}\,\rho_{\theta_0} R_h^2 &= \mathrm{Tr}\,\rho_{\theta_0}\left(I + \frac{1}{2} A_i h^i + B(h)\right)^2 \\
&= \mathrm{Tr}\,\rho_{\theta_0}\left(I + \frac{1}{4} A_i A_j h^i h^j + 2B(h) + A_i h^i + B(h)^2 + \frac{1}{2} A_i h^i B(h) + \frac{1}{2} B(h) A_i h^i\right) \\
&= 1 + \frac{1}{4} J_{ji} h^i h^j + 2\mathrm{Tr}\,\rho_{\theta_0} B(h) + o(\|h\|^2).
\end{aligned}$$

This relation and assumption (7.7) lead to

$$\mathrm{Tr}\,\rho_{\theta_0} B(h) = -\frac{1}{8} J_{ji} h^i h^j + o(\|h\|^2). \tag{S.16}$$

In order to prove (ii), it suffices to show that

$$\begin{aligned} O_h^{(n)} &:= \exp\left[\frac{1}{2}\left(h^i\Delta_i^{(n)} - \frac{1}{2}J_{ji}h^ih^j\right)\right] - (R_{h/\sqrt{n}})^{\otimes n} \\ &= e^{-\frac{1}{4}J_{ji}h^ih^j}\left\{e^{\frac{1}{2\sqrt{n}}h^iL_i}\right\}^{\otimes n} - (R_{h/\sqrt{n}})^{\otimes n} \end{aligned}$$

is L^2-infinitesimal under $\rho_{\theta_0}^{\otimes n}$, setting the D-infinitesimal residual term $o_D\left(h^i\Delta_i^{(n)}, \rho_{\theta_0}^{(n)}\right)$ in (ii) to be zero for all n. In fact,

$$\begin{aligned} \mathrm{Tr}\,\rho_{\theta_0}^{\otimes n}O_h^{(n)^2} &= e^{-\frac{1}{2}J_{ji}h^ih^j}\left\{\mathrm{Tr}\,\rho_{\theta_0}e^{\frac{1}{\sqrt{n}}h^iL_i}\right\}^n + \left\{\mathrm{Tr}\,\rho_{\theta_0}R_{h/\sqrt{n}}^2\right\}^n \\ &\quad -2\,e^{-\frac{1}{4}J_{ji}h^ih^j}\,\mathrm{Re}\left\{\mathrm{Tr}\,\rho_{\theta_0}e^{\frac{1}{2\sqrt{n}}h^iL_i}R_{h/\sqrt{n}}\right\}^n. \end{aligned} \tag{S.17}$$

The first term in the right-hand side of (S.17) is evaluated as follows:

$$\begin{aligned} e^{-\frac{1}{2}J_{ji}h^ih^j}\left\{\mathrm{Tr}\,\rho_{\theta_0}e^{\frac{1}{\sqrt{n}}h^iL_i}\right\}^n &= e^{-\frac{1}{2}J_{ji}h^ih^j}\left\{\mathrm{Tr}\,\rho_{\theta_0}\left(I + \frac{1}{\sqrt{n}}h^iL_i + \frac{1}{2n}L_iL_jh^ih^j + o\left(\frac{1}{n}\right)\right)\right\}^n \\ &= e^{-\frac{1}{2}J_{ji}h^ih^j}\left(1 + \frac{1}{2n}J_{ji}h^ih^j + o\left(\frac{1}{n}\right)\right)^n \longrightarrow 1. \end{aligned}$$

The second term is evaluated from (7.7) as

$$\left\{\mathrm{Tr}\,\rho_{\theta_0}R_{h/\sqrt{n}}^2\right\}^n = \left(1 - o\left(\frac{1}{n}\right)\right)^n \longrightarrow 1.$$

Finally, the third term is evaluated from (S.16) as

$$\begin{aligned} &e^{-\frac{1}{4}J_{ji}h^ih^j}\left\{\mathrm{Tr}\,\rho_{\theta_0}e^{\frac{h^i}{2\sqrt{n}}L_i}R_{h/\sqrt{n}}\right\}^n \\ &= e^{-\frac{1}{4}J_{ji}h^ih^j}\left\{\mathrm{Tr}\,\rho_{\theta_0}\left(I + \frac{h^i}{2\sqrt{n}}L_i + \frac{1}{8n}L_iL_jh^ih^j + o\left(\frac{1}{n}\right)\right)\left(I + \frac{h^k}{2\sqrt{n}}A_k + B\left(\frac{h}{\sqrt{n}}\right)\right)\right\}^n \\ &= e^{-\frac{1}{4}J_{ji}h^ih^j}\left\{1 + \frac{1}{4n}J_{ki}h^ih^k + o\left(\frac{1}{n}\right)\right\}^n \longrightarrow 1. \end{aligned}$$

This proves (ii).

Having established that $\{\rho_\theta^{\otimes n}\}_n$ is q-LAN at θ_0, the property (7.8) is now an immediate consequence of Corollary 7.5 as well as the quantum central limit theorem

$$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \overset{\rho_{\theta_0}^{\otimes n}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau* & J \end{pmatrix}\right). \tag{S.18}$$

This completes the proof. □

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