TORIC FANO CONTRACTIONS ASSOCIATED TO LONG EXTREMAL RAYS

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Abstract. We show that a toric Fano contraction associated to an extremal ray whose length is greater than the dimension of its fiber is a projective space bundle.

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1. Introduction

Let $X$ be a smooth projective variety defined over an algebraically closed field $k$ of arbitrary characteristic. In his epoch-making paper (see [Mo]), Shigefumi Mori established the following famous cone theorem

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum R_j,$$

where $\overline{NE}(X)$ denotes the Kleiman–Mori cone of $X$ and each $R_j$ is called a $K_X$-negative extremal ray of $\overline{NE}(X)$. By the original proof of the above cone theorem, which is based on Mori’s bend and break technique to create rational curves, we know that for each $K_X$-negative extremal ray $R$ there exists a (possibly singular) rational curve $C$ on $X$ such that the numerical equivalence class of $C$ spans $R$ and

$$0 < -K_X \cdot C \leq \dim X + 1$$

holds.

Let $X$ be a $\mathbb{Q}$-Gorenstein projective algebraic variety for which the cone theorem holds. Then for a $K_X$-negative extremal ray $R$ of $\overline{NE}(X)$, we put

$$l(R) := \min_{C \in R} (-K_X \cdot C)$$

and call it the length of $R$. We have already known that $l(R)$ is an important invariant and that some conditions on $l(R)$ determine the structure of the associated extremal contraction.

In this paper, we are interested in the case where $X$ is a toric variety. We note that $NE(X) = \overline{NE}(X)$ holds when $X$ is a projective toric variety. This is because $NE(X)$ is a rational polyhedral cone. We also note that the cone theorem holds for $\mathbb{Q}$-Gorenstein projective toric varieties without any extra assumptions.

Date: 2018/4/15, version 0.21.
2010 Mathematics Subject Classification. Primary 14M25; Secondary 14E30.
Key words and phrases. toric Mori theory, lengths of extremal rays, Fano contractions, projective bundles.
From now on, we will only treat \( \mathbb{Q} \)-factorial projective toric varieties defined over an algebraically closed field \( k \) of arbitrary characteristic for simplicity.

For a \( \mathbb{Q} \)-factorial projective toric \( n \)-fold \( X \) of Picard number \( \rho(X) = 1 \), there exists the unique extremal ray of \( \text{NE}(X) \). In this case, the following statement holds.

**Theorem 1.1** ([F1, Proposition 2.9] and [F2, Proposition 2.1]). Let \( X \) be a \( \mathbb{Q} \)-factorial projective toric \( n \)-fold of Picard number \( \rho(X) = 1 \) with \( R = \text{NE}(X) \). Then, the following statements hold.

1. If \( l(R) > n \), then \( X \cong \mathbb{P}^n \).
2. If \( l(R) \geq n \) and \( X \not\cong \mathbb{P}^n \), then \( X \cong \mathbb{P}(1,1,2,\ldots,2) \).

For the case where the associated extremal contraction is birational, we have the following estimates which are special cases of [FS2, Theorem 3.2.1].

**Theorem 1.2.** Let \( X \) be a \( \mathbb{Q} \)-factorial projective toric \( n \)-fold, and let \( R \) be a \( K_X \)-negative extremal ray of \( \text{NE}(X) \). Suppose that the contraction morphism \( \varphi_R : X \to W \) associated to \( R \) is birational. Then, we obtain

\[
l(R) < d + 1,
\]

where

\[
d = \max_{w \in W} \dim \varphi_R^{-1}(w) \leq n - 1.
\]

When \( d = n - 1 \), we have a sharper inequality

\[
l(R) \leq d = n - 1.
\]

In particular, if \( l(R) = n - 1 \) holds, then \( \varphi_R : X \to W \) can be described as follows. There exists a torus invariant smooth point \( P \in W \) such that \( \varphi_R : X \to W \) is a weighted blow-up at \( P \) with the weight \((1,a,\cdots,a)\) for some positive integer \( a \). In this case, the exceptional locus \( E \) of \( \varphi_R \) is a torus invariant prime divisor and is isomorphic to \( \mathbb{P}^{n-1} \).

This estimate shows that the extremal ray \( R \) with \( l(R) > n - 1 \) must be of fiber type. In this case, we can determine the structure of the associated contraction \( \varphi_R \) as follows.

**Theorem 1.3.** Let \( X \) be a \( \mathbb{Q} \)-factorial projective toric \( n \)-fold with \( \rho(X) \geq 2 \), and let \( R \) be a \( K_X \)-negative extremal ray of \( \text{NE}(X) \). If \( l(R) > n - 1 \), then the extremal contraction \( \varphi_R : X \to W \) associated to \( R \) is a \( \mathbb{P}^{n-1} \)-bundle over \( \mathbb{P}^1 \).

**Remark 1.4.** Theorem 1.3 holds for projective \( \mathbb{Q} \)-Gorenstein toric varieties (without the assumption that \( X \) is \( \mathbb{Q} \)-factorial). For the details, please see [FS2, Proposition 3.2.9].

As a generalization of Theorem 1.3, we prove the following theorem about the structure of extremal contractions of fiber type (see Corollary 3.3).

**Theorem 1.5** (Main theorem). Let \( X \) be a \( \mathbb{Q} \)-factorial projective toric \( n \)-fold. Let \( \varphi_R : X \to W \) be a Fano contraction associated to a \( K_X \)-negative extremal ray \( R \subset \text{NE}(X) \) such that the dimension of a fiber of \( \varphi_R \) is \( d \), equivalently, \( d = \dim X - \dim W \). If \( l(R) > d \), then \( \varphi_R \) is a \( \mathbb{P}^{d-1} \)-bundle over \( W \).

We show that this result is sharp by Examples 3.2 and 3.5. We note that Theorem 1.5 is nothing but Theorem 1.1 (1) if \( \dim W = 0 \). Therefore, we can see Theorem 1.5 as a generalization of Theorem 1.1 (1).

**Acknowledgments.** The first author was partially supported by JSPS KAKENHI Grant Numbers JP16H03925, JP16H06337. The second author was partially supported by JSPS KAKENHI Grant Number JP18K03262.
2. Preliminaries

In this section, we introduce some basic results and notation of the toric geometry in order to prove the main theorem. For the details, please see [CLS], [Fu] and [O]. See also [FS1], [Ma, Chapter 14] and [R] for the toric Mori theory.

Let \( X = X_\Sigma \) be the toric \( n \)-fold associated to a fan \( \Sigma \) in \( N = \mathbb{Z}^n \) over an algebraically closed field \( k \) of arbitrary characteristic. We will use the notation \( \Sigma = \Sigma_X \) to denote the fan associated to a toric variety \( X \). It is well known that there exists a one-to-one correspondence between the \( r \)-dimensional cones in \( \Sigma \) and the torus invariant subvarieties of dimension \( n - r \) in \( X \). Let \( G(\Sigma) \) be the set of primitive generators for 1-dimensional cones in \( \Sigma \). Thus, for \( v \in G(\Sigma) \), we have a torus invariant prime divisor corresponding to \( v \).

For an \( r \)-dimensional simplicial cone \( \sigma \subseteq \Sigma \), let \( N_\sigma \subseteq N \) be the sublattice generated by \( \sigma \cap N \) and let \( \sigma \cap G(\Sigma) = \{ v_1, \ldots, v_r \} \), that is, \( \sigma = \langle v_1, \ldots, v_r \rangle \), where \( \langle v_1, \ldots, v_r \rangle \) is the \( r \)-dimensional strongly convex cone generated by \( \{ v_1, \ldots, v_r \} \). Put

\[
\text{mult}(\sigma) := [N_\sigma : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_r]
\]

which is the index of the subgroup \( \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_r \) in \( N_\sigma \). The following property is fundamental.

**Proposition 2.1.** Let \( X \) be a \( \mathbb{Q} \)-factorial toric \( n \)-fold, and let \( \tau \in \Sigma \) be an \( (n - 1) \)-dimensional cone and \( v \in G(\Sigma) \). If \( v \) and \( \tau \) generate a maximal cone \( \sigma \) in \( \Sigma \), then

\[
D \cdot C = \frac{\text{mult}(\tau)}{\text{mult}(\sigma)},
\]

where \( D \) is the torus invariant prime divisor corresponding to \( v \), while \( C \) is the torus invariant curve corresponding to \( \tau \).

Let \( X \) be a projective toric variety. We put

\[
Z_1(X) := \{1\text{-cycles of } X\},
\]

and

\[
Z_1(X)_\mathbb{R} := Z_1(X) \otimes \mathbb{R}.
\]

Let

\[
\text{Pic}(X) \times Z_1(X) \to \mathbb{Z}
\]

be a pairing defined by \( (\mathcal{L}, C) \mapsto \deg_{\mathcal{L}} \mathcal{L} \). By extending it by bilinearity, we have a pairing

\[
(\text{Pic}(X) \otimes \mathbb{R}) \times Z_1(X)_\mathbb{R} \to \mathbb{R}.
\]

We define

\[
N^1(X) := (\text{Pic}(X) \otimes \mathbb{R}) / \equiv
\]

and

\[
N_1(X) := Z_1(X)_\mathbb{R} / \equiv,
\]

where the \textit{numerical equivalence} \( \equiv \) is by definition the smallest equivalence relation which makes \( N^1 \) and \( N_1 \) into dual spaces.

Inside \( N_1(X) \) there is a distinguished cone of effective 1-cycles of \( X \),

\[
\text{NE}(X) = \left\{ Z \left| Z \equiv \sum a_i C_i \text{ with } a_i \in \mathbb{R}_{\geq 0} \right. \right\} \subset N_1(X),
\]

which is usually called the \textit{Kleiman–Mori cone} of \( X \). It is known that \( \text{NE}(X) \) is a rational polyhedral cone. A face \( F \subset \text{NE}(X) \) is called an \textit{extremal face} in this case. A one-dimensional extremal face is called an \textit{extremal ray}. 

Next, we introduce a combinatorial description of toric Fano contractions which are main objects of this paper. Let $X = X_{\Sigma}$ be a $\mathbb{Q}$-factorial projective toric $n$-fold and $\varphi_R : X \to W$ be the extremal contraction associated to an extremal ray $R \subset \text{NE}(X)$ of fiber type. Put

$$d := \dim X - \dim W.$$

Up to automorphisms of $N$, $\Sigma$ is constructed as follows:

For the standard basis $\{e_1, \ldots, e_n\} \subset N = \mathbb{Z}^n$, put $N' := Ze_1 + \cdots + Ze_d$, while $N'' := Ze_{d+1} + \cdots + Ze_{n-d}$, that is, $N = N' \oplus N''$. Then, there exist $\{v_1, \ldots, v_{d+1}\} \subset G(\Sigma) \cap N'$ such that $\{v_1, \ldots, v_{d+1}\} \setminus \{v_i\}$ generates a $d$-dimensional cone $\sigma_i \in \Sigma$ for any $1 \leq i \leq d+1$, and $\sigma_1 \cup \cdots \cup \sigma_{d+1} = N' \oplus \mathbb{R}$. Namely, we obtain the complete fan $\Sigma_R$ in $N'$ whose maximal cones are $\sigma_1, \ldots, \sigma_{d+1}$. $\Sigma_R$ is associated to a general fiber $F$ of $\varphi_R$, and the Picard number $\rho(F)$ is $1$. Moreover, for any $\{y_1, \ldots, y_{n-d}\} \subset G(\Sigma) \setminus \{v_1, \ldots, v_{d+1}\}$ which generates a $(n-d)$-dimensional cone in $\Sigma$, $\{v_1, \ldots, v_{d+1}, y_1, \ldots, y_{n-d}\} \setminus \{v_i\}$ generates a maximal cone in $\Sigma$ for any $1 \leq i \leq d+1$. Thus, the projection $N = N' \oplus N'' \to N''$ induces $\varphi_R$.

**Remark 2.2.** This description shows that for a toric Fano contraction $\varphi_R : X \to W$, the dimension of any fiber is constant. As we saw above, the general fiber $F$ of $\varphi_R$ is a projective $\mathbb{Q}$-factorial toric variety of Picard number $\rho(F) = 1$. Moreover, it is known that the fiber $\varphi_R^{-1}(w)_{\text{red}}$ with the reduced structure is isomorphic to $F$ for every closed point $w \in W$ (see [CLS, Proposition 15.4.5] and [Ma, Corollary 14-2-2]).

A toric projective bundle is a special case of toric Fano contractions. We will use the following well-known lemma in the proof of Theorem 1.5. We prove it here for the reader’s convenience since we can not find it explicitly in the literature.

**Lemma 2.3** (Toric projective bundles). Let $f : X \to Y$ be a projective surjective toric morphism between toric varieties with $\dim X - \dim Y = d$. Then the following two conditions are equivalent.

(a) $f : X \to Y$ is a $\mathbb{P}^d$-bundle over $Y$.

(b) Let $U$ be any affine toric open subset of $Y$. Then $f^{-1}(U) \simeq \mathbb{P}^d \times U$ and the restriction of $f : X \to Y$ to $f^{-1}(U)$ is nothing but the second projection $f^{-1}(U) \simeq \mathbb{P}^d \times U$.

**Proof.** First, we assume (a), that is, $f : X \to Y$ is a $\mathbb{P}^d$-bundle over $Y$. Then it is well known that $X \simeq \mathbb{P}_Y(L_1 \oplus \cdots \oplus L_{d+1})$ for some line bundles $L_1, \ldots, L_{d+1}$. Since $\text{Pic}(U) = 0$ (see, for example, [CLS, Proposition 4.2.2]), we obtain (b).

Next, we assume that (b) holds. Then it is obvious that $f$ is a smooth morphism. Moreover, we can take a torus invariant Cartier divisor $H$ on $X$ such that $\mathcal{O}_{f^{-1}(y)}(H|_{f^{-1}(y)}) \simeq \mathcal{O}_{\mathbb{P}^d}(1)$ for any closed point $y \in Y$. In this situation, we see that $\mathcal{E} := f_* \mathcal{O}_X(H)$ is a locally free sheaf of rank $d + 1$ on $Y$ such that $X$ is isomorphic to $\mathbb{P}_Y(\mathcal{E})$ over $Y$. Thus (a) holds.

Anyway, (a) and (b) are equivalent.

\begin{proof}
\end{proof}

### 3. Fano contractions

The following result is the main theorem of this paper.

**Theorem 3.1.** Let $X = X_{\Sigma}$ be a $\mathbb{Q}$-factorial projective toric $n$-fold. Let $\varphi_R : X \to W$ be a Fano contraction associated to a $K_X$-negative extremal ray $R \subset \text{NE}(X)$, and $d = n - \dim W$ be the dimension of a fiber of $\varphi_R$. If a general fiber of $\varphi_R$ is isomorphic to $\mathbb{P}^d$ and

$$-K_X \cdot C > \frac{d + 1}{2}$$

holds for any curve $C$ on $X$ contracted by $\varphi_R$, then $\varphi_R$ is a $\mathbb{P}^d$-bundle over $W$. 

Proof. We may assume that \( \varphi : X \to W \) is induced by the following projection:

\[
\begin{align*}
N &= \mathbb{Z}^n \\
\cup & \to \mathbb{Z}^{n-d} \\
(x_1, \ldots, x_n) & \mapsto (x_{d+1}, \ldots, x_n)
\end{align*}
\]

Let \( \{e_1, \ldots, e_n\} \) be the standard basis for \( N = \mathbb{Z}^n \). We put

\[
v_1 := e_1, \ldots, v_d := e_d, \text{ and } v_{d+1} := -(e_1 + \cdots + e_d).
\]

Then \( \Sigma \) contains the \( d \)-dimensional subfan \( \Sigma_F \) corresponding to a general fiber \( F \simeq \mathbb{P}^d \) whose maximal cones are

\[
\langle \{v_1, \ldots, v_{d+1}\} \ \setminus \ \{v_i\} \rangle \quad (1 \leq i \leq d + 1).
\]

Let \( V_\sigma \subset N \otimes_\mathbb{Z} \mathbb{R} \) be the linear subspace spanned by \( \sigma \) for any \( (n-d) \)-dimensional cone \( \sigma \) in \( \Sigma \) such that \( (\sigma \cap G(\Sigma)) \cap \{v_1, \ldots, v_{d+1}\} = \emptyset \). Then it is sufficient to show that

\[
V_\sigma \cap \mathbb{Z}^n \overset{p}{\longrightarrow} \mathbb{Z}^{n-d}
\]

is bijective by Lemma 2.3. The injectivity of (3.1) is trivial. Therefore, we will show the surjectivity of (3.1).

Let \( y_1, \ldots, y_{n-d} \in G(\Sigma) \setminus \{v_1, \ldots, v_{d+1}\} \) be the primitive generators for any \( (n-d) \)-dimensional cone in \( \Sigma \) such that \( p((y_1, \ldots, y_{n-d})) \) is also \( (n-d) \)-dimensional. Put

\[
y_1 = (b_{1,1}, \ldots, b_{d,1}; a_{1,1}, \ldots, a_{n-d,1}), \\
\vdots \\
y_{n-d} = (b_{1,n-d}, \ldots, b_{d,n-d}; a_{1,n-d}, \ldots, a_{n-d,n-d}).
\]

For any \( (z_1, \ldots, z_{n-d}) \in \mathbb{Z}^{n-d} \), we can take \( (c_1, \ldots, c_{n-d}) \in \mathbb{R}^{n-d} \) satisfying

\[
p(c_1y_1 + \cdots + c_{n-d}y_{n-d}) = c_1p(y_1) + \cdots + c_{n-d}p(y_{n-d}) = (z_1, \ldots, z_{n-d}).
\]

We note that the matrix

\[
A := \begin{pmatrix}
a_{1,1} & \cdots & a_{1,n-d} \\
\vdots & \ddots & \vdots \\
a_{n-d,1} & \cdots & a_{n-d,n-d}
\end{pmatrix}
\]

is regular because \( p(y_1), \ldots, p(y_{n-d}) \) generates an \( (n-d) \)-dimensional cone. Therefore, \( (c_1, \ldots, c_{n-d}) \) is uniquely determined by

\[
\begin{pmatrix}
c_1 \\
\vdots \\
c_{n-d}
\end{pmatrix} = A^{-1} \begin{pmatrix}
z_1 \\
\vdots \\
z_{n-d}
\end{pmatrix} \in \mathbb{Q}^{n-d}.
\]

Thus, all we have to do is to show that

\[
c_1b_{r,1} + \cdots + c_{n-d}b_{r,n-d} \in \mathbb{Z}
\]

for any \( 1 \leq r \leq d \).

By considering the principal Cartier divisors of the dual basis of \( \{e_1, \ldots, e_n\} \), we obtain the relations

\[
\begin{align*}
D_1 - D_{d+1} + b_{1,1}E_1 + \cdots + b_{1,n-d}E_{n-d} + H_1 &= 0, \\
\vdots \\
D_d - D_{d+1} + b_{d,1}E_1 + \cdots + b_{d,n-d}E_{n-d} + H_d &= 0, \\
a_{1,1}E_1 + \cdots + a_{1,n-d}E_{n-d} + H_{d+1} &= 0, \\
\vdots \\
a_{n-d,1}E_1 + \cdots + a_{n-d,n-d}E_{n-d} + H_n &= 0
\end{align*}
\]
in \( N^1(X) \), where \( D_1, \ldots, D_{d+1}, E_1, \ldots, E_{n-d} \) are the torus invariant prime divisors corresponding to \( v_1, \ldots, v_{d+1}, y_1, \ldots, y_{n-d} \), respectively, and \( H_1, \ldots, H_n \) are some linear combinations of torus invariant prime divisors other than \( D_1, \ldots, D_{d+1}, E_1, \ldots, E_{n-d} \). Let \( C = C_r \) \((1 \leq r \leq d)\) be the torus invariant curve corresponding to the \((n-1)\)-dimensional cone

\[
\langle \{v_1, \ldots, v_d, y_1, \ldots, y_{n-d}\} \setminus \{v_r\} \rangle.
\]

Since \( H_i \cdot C = 0 \) for any \( 1 \leq i \leq n \), we may ignore \( H_1, \ldots, H_n \) in the following calculation. Since the matrix \( A \) is regular, we have

\[
E_1 \cdot C = \ldots = E_{n-d} \cdot C = 0,
\]

and

\[
D_1 \cdot C = D_2 \cdot C = \ldots = D_{d+1} \cdot C
\]

by the above equalities. Thus, we obtain

\[
-K_X \cdot C = (d + 1)D_i \cdot C
\]

for any \( 1 \leq i \leq d + 1 \).

Put

\[
\alpha := \text{mult} \left( \langle \{v_1, \ldots, v_d, y_1, \ldots, y_{n-d}\} \setminus \{v_r\} \rangle \right)
\]

and

\[
\beta := \text{mult} \left( \langle \{v_1, \ldots, v_d, y_1, \ldots, y_{n-d}\} \rangle \right).
\]

Then we get

\[
D_r \cdot C = \frac{\alpha}{\beta}
\]

by Proposition 2.1. We note that \( \alpha \mid \beta \) always holds. Obviously, \( \beta = \left| \det A \right| \). On the other hand, \( \alpha \) is the product of the elementary divisors of the \( n \times (n-1) \) matrix

\[
\left( v_1, \ldots, v_r, \ldots, v_d, y_1, \ldots, y_{n-d} \right).
\]

One can easily check that \( \alpha \) is also the product of the elementary divisors of the \((n-d+1) \times (n-d)\) matrix

\[
\overline{A} = \begin{pmatrix}
  b_{r,1} & \cdots & b_{r,n-d} \\
  a_{1,1} & \cdots & a_{1,n-d} \\
  \vdots & \ddots & \vdots \\
  a_{n-d,1} & \cdots & a_{n-d,n-d}
\end{pmatrix}.
\]

Suppose that \( D_r \cdot C < 1 \) holds. Then, more strongly, we obtain the inequality

\[
D_r \cdot C \leq \frac{d + 1}{2}
\]

by the relation \( \alpha \mid \beta \). Thus, the following inequality

\[
-K_X \cdot C = (d + 1)D_r \cdot C \leq \frac{d + 1}{2}
\]

holds. However, this contradicts the assumption that \( \frac{d + 1}{2} < -K_X \cdot C \). Therefore, the equality

\[
\frac{\alpha}{\beta} = D_r \cdot C = 1
\]

must always hold. Since the general theory of elementary divisors says that \( \alpha \) is the greatest common divisor of the \((n-d) \times (n-d)\) minor determinants of \( \overline{A} \), the \((n-d) \times (n-d)\)
Example 3.2. Let $d < n$ be the dimension of a fiber of $\varphi_R$. From this noncomplete variety, one can easily construct a projective toric manifold $X$ associated to a $(n-d)$-fold, where $X$ is isomorphic to the projective space $\mathbb{P}^{n-d}$ automatically holds as expected. However, $X$ is not a $\mathbb{Q}$-factorial projective toric $n$-fold. Let $\varphi_R : X \to W$ be a Fano contraction associated to a $K_X$-negative extremal ray $R \in \text{NE}(X)$, and $d = n - \dim W$ be the dimension of a fiber of $\varphi_R$. If $-K_X \cdot C > d$ holds for any curve $C$ on $X$ contracted by $\varphi_R$, then $\varphi_R$ is a $\mathbb{P}^d$-bundle over $W$. 

**Corollary 3.3.** Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric $n$-fold. Let $\varphi_R : X \to W$ be a Fano contraction associated to a $K_X$-negative extremal ray $R \in \text{NE}(X)$, and $d = n - \dim W$ be the dimension of a fiber of $\varphi_R$. If $-K_X \cdot C > d$ holds for any curve $C$ on $X$ contracted by $\varphi_R$, then $\varphi_R$ is a $\mathbb{P}^d$-bundle over $W$. 

The following example shows that Theorem 3.1 is sharp. 

**Example 3.2.** Let $\{e_1, \ldots, e_n\}$ be the standard basis for $N = \mathbb{Z}^n$ and $p : N \to \mathbb{Z}^{n-d}$ be the projection

$$(x_1, \ldots, x_d, x_{d+1}, \ldots, x_n) \mapsto (x_{d+1}, \ldots, x_n)$$

for $1 \leq d < n$. Put $v_1 := e_1, \ldots, v_d := e_d$, $v_{d+1} := -(e_1 + \cdots + e_d)$,

$$y_1 := e_{d+1}, \ldots, y_{n-d-1} := e_{n-1}, y_{n-d} := e_1 + e_{d+1} + \cdots + e_{n-1} + 2e_n.$$ 

Let $\Sigma$ be the fan in $N$ whose maximal cones are generated by $\{v_i, \ldots, v_{d+1}, y_1, \ldots, y_{n-d}\}$ \setminus \{v_i\}$ for $1 \leq i \leq d + 1$. In this case, $X = X_\Sigma$ has a Fano contraction whose general fiber is isomorphic to $\mathbb{P}^d$. Moreover, every fiber with the reduced structure is isomorphic to $\mathbb{P}^d$ (see Remark 2.2). However, $X$ does not decompose into $\mathbb{P}^d$ and a toric affine $(n-d)$-fold, because

$$\frac{p(y_1) + \cdots + p(y_{n-d})}{2} = e_{d+1} + \cdots + e_n \in \mathbb{Z}^{n-d},$$

while

$$\frac{y_1 + \cdots + y_{n-d}}{2} = \frac{1}{2}(e_1 + e_{d+1} + \cdots + e_n) \notin N.$$ 

From this noncomplete variety, one can easily construct a projective toric manifold $X$ which has a Fano contraction associated to an extremal ray of length $\frac{d+1}{2}$ (for example, add the generator $y_{n-d+1} := -(e_{d+1} + \cdots + e_n)$ and compactify $\Sigma$).

If we make the inequality in Theorem 3.1 stronger, then the assumption that a general fiber of a Fano contraction is isomorphic to the projective space automatically holds as follows.
Proof. Let $F$ be a general fiber of $\varphi_R$ and let $C$ be any curve on $F$. Then, by adjunction, we have
\[ d < -K_X \cdot C = -K_F \cdot C. \]
Therefore, by Theorem 1.1 (1), $F \simeq \mathbb{P}^d$ holds. Since $\frac{d+1}{2} \leq d$, we can apply Theorem 3.1. □

As an easy consequence of Corollary 3.3, we obtain:

**Corollary 3.4.** Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric $n$-fold and let $\Delta$ be any effective (not necessarily torus invariant) $\mathbb{R}$-divisor on $X$. Let $\varphi_R : X \to W$ be a Fano contraction associated to a $(K_X + \Delta)$-negative extremal ray $R \subset \text{NE}(X)$ with $d = n - \dim W$. If $-(K_X + \Delta) \cdot C > d$ for any curve $C$ on $X$ contracted by $\varphi_R$, then $\varphi_R$ is a $\mathbb{P}^d$-bundle over $W$.

**Proof.** We can easily see that $D \cdot C \geq 0$ for any effective Weil divisor $D$ on $X$ and any curve $C$ on $X$ contracted by $\varphi_R$ since $\varphi_R : X \to W$ is a toric Fano contraction of a $\mathbb{Q}$-factorial projective toric variety $X$. Therefore, we get
\[ d < -(K_X + \Delta) \cdot C \leq -K_X \cdot C \]
for any curve $C$ on $X$ contracted by $\varphi_R$. Thus, we see that $\varphi_R : X \to W$ is a $\mathbb{P}^d$-bundle over $W$ by Corollary 3.3. □

The following example shows that Corollary 3.3 is sharp.

**Example 3.5.** Let $F := \mathbb{P}(1,1,2,\ldots,2)$ be the $d$-dimensional weighted projective space and $W$ a $\mathbb{Q}$-factorial projective toric $(n-d)$-fold. Then, the length of the extremal ray corresponding to the first projection $\varphi : X = W \times F \to W$ is $d$ (see [F2, Proposition 2.1] and [FS2, Proposition 3.1.6]).

**References**


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